Lecture XXVII: Kerr black holes: II. Precession, circular orbits, and stability

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I. OVERVIEW

Having developed the formalism, we now turn to the behavior of orbits in Kerr.
Reading:

- MTW Ch. 33.

II. BASIC EQUATIONS

We recall here the basic equations for the particle behavior in the Kerr metric. Remembering that we parameterized the orbit in terms of $\lambda = \int d\tau/\Sigma$, we found that the inclination was related to the Carter constant via

$$\mathcal{Q} = a^2(1-\mathcal{E}^2)\sin^2 I + \mathcal{L}^2\tan^2 I \tag{1}$$

and that the period in units of λ, P_θ, is

$$P_\theta = 4\int_{\pi/2-I}^{\pi/2} \frac{d\theta}{\sqrt{-a^2(1-\mathcal{E}^2)\cos^2\theta - \mathcal{L}^2\cot^2\theta + \mathcal{Q}}}. \tag{2}$$

The radial motion was determined by an effective potential

$$V(r) = (1-\mathcal{E})^2 r^4 - 2Mr^3 + [a^2(1-\mathcal{E}^2) + \mathcal{L}^2 + \mathcal{Q}]r^2 - 2M[(a\mathcal{E}-\mathcal{L})^2 + \mathcal{Q}]r + a^2\mathcal{Q}, \tag{3}$$

and the particle bounces back and forth between two zeroes of this potential (it is trapped in regions where V is negative). The period, again in units of λ, is

$$P_r = 2\int_{r_{\rm min}}^{r_{\rm max}} \frac{dr}{\sqrt{V(r)}}. \tag{4}$$

The time and longitude increase at some rate

$$\frac{dt}{d\lambda} = \Sigma u^t = \left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\mathcal{E} - \frac{2aMr\mathcal{L}}{\Delta} \tag{5}$$

and

$$\frac{d\phi}{d\lambda} = \Sigma u^t = \left[\csc^2\theta - \frac{a^2}{\Delta}\right]\mathcal{L} + \frac{2aMr\mathcal{E}}{\Delta}. \tag{6}$$

Of interest to us is the mean rate of increase averaged over radial and angular cycles,

$$b_t = \left\langle \frac{dt}{d\lambda}\right\rangle_\lambda = \left\langle \frac{(r^2+a^2)^2}{\Delta}\right\rangle_\lambda \mathcal{E} - 2aM\left\langle\frac{r}{\Delta}\right\rangle_\lambda \mathcal{L} - a^2\mathcal{E}\left\langle \sin^2\theta\right\rangle_\lambda \tag{7}$$

and

$$b_\phi = \left\langle \frac{d\phi}{d\lambda}\right\rangle_\lambda = 2aM\left\langle\frac{r}{\Delta}\right\rangle_\lambda \mathcal{E} - a^2\left\langle\frac{1}{\Delta}\right\rangle_\lambda \mathcal{L} + \mathcal{L}\left\langle \csc^2\theta\right\rangle_\lambda. \tag{8}$$

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We recall that t is equal to $b_t\lambda$ plus terms whose "frequencies" (in λ) are integer multiples of $2\pi/P_r$ and $2\pi/P_\theta$. It then follows that

$$\lambda = \frac{t}{b_t} + \text{periodic terms}, \tag{9}$$

where the periodic terms have frequencies in t given by integer multiples of the fundamental frequencies Ω_r and Ω_θ. The particle's orbit has fundamental frequencies

$$\Omega_r = \frac{2\pi}{P_r b_t}, \quad \Omega_\theta = \frac{2\pi}{P_\theta b_t}, \quad \text{and} \quad \Omega_\phi = \frac{b_\phi}{b_t}. \tag{10}$$

III. THE EQUATORIAL ORBITS

We begin our discussion with the equatorial orbits. These orbits have $I = 0$ and hence $\mathcal{Q} = 0$. (It is more convenient to describe retrograde equatorial orbits with $a < 0$ than with $I = \pi$.) In this case, the constant term in the effective potential goes away:

$$V(r) = (1-\mathcal{E}^2)r^4 - 2Mr^3 + [a^2(1-\mathcal{E}^2) + \mathcal{L}^2]r^2 - 2M(a\mathcal{E} - \mathcal{L})^2 r. \tag{11}$$

The minimum and maximum radial coordinates of the orbit are then determined by the zeroes of $V(r)$, i.e. by a cubic equation. We know from the last lecture that $V(r_+) < 0$ and $V(+\infty) = +\infty$, and the existence or not of stable bound orbits depends on whether there are 1 or 3 positive real zeroes of $V(r)/r$.

To investigate this, it is possible to take particular values of $\mathcal{E}$ and $\mathcal{L}$ and check the behavior of $V(r)$. However, it is easier to map out the boundary in the $(\mathcal{L},\mathcal{E})$-plane by noting that at the transition between 1 and 3 positive roots we have a point where $V(r) = V'(r) = 0$. At such points, we have one of three behaviors:

- $V''(r) > 0$: A particle is on a *stable circular orbit*, constrained to remain exactly at that value of r.
- $V''(r) < 0$: The particle straddles the boundary between two regions with $V(r) < 0$; if perturbed one way it falls into the hole, if perturbed the other way it flies outward (and, after reaching the outermost zero of V) turns back inward. In this case, the particle is on an *unstable circular orbit*.
- $V''(r) = 0$: This is the junction between the two cases: the *marginally stable circular orbit*.

To identify these cases, we begin by taking the derivative of Eq. (11):

$$V'(r) = 4(1-\mathcal{E}^2)r^3 - 6Mr^2 + 2[a^2(1-\mathcal{E}^2) + \mathcal{L}^2]r - 2M(a\mathcal{E} - \mathcal{L})^2. \tag{12}$$

We take the following two linear combinations of $V = 0$ and $V' = 0$:

$$\begin{aligned} r^{-1}V'(r) - r^{-2}V(r) &= (3r^2 + a^2)(1-\mathcal{E}^2) - 4Mr + \mathcal{L}^2 = 0 \quad \text{and} \\ V'(r) - 2r^{-1}V(r) &= 2(1-\mathcal{E}^2)r^3 - 2Mr^2 + 2M(a\mathcal{E} - \mathcal{L})^2 = 0. \end{aligned} \tag{13}$$

Equation (13) can be written as a quadratic equation for $\mathcal{E}^2$ by using the first equation to eliminate $\mathcal{L}$ from the second. The solution is tedious (see Chandrasekhar Ch. 7) but the solution is

$$\mathcal{E} = \frac{1 - 2M/r + aM^{1/2}/r^{3/2}}{\sqrt{1 - 3M/r + 2aM^{1/2}/r^{3/2}}} \tag{14}$$

and

$$\mathcal{L} = \frac{M^{1/2}r^{1/2} - 2aM/r + M^{1/2}a^2/r^{3/2}}{\sqrt{1 - 3M/r + 2aM^{1/2}/r^{3/2}}}. \tag{15}$$

As $a \rightarrow 0$ these approach the usual Schwarzschild values. However, in general, we can see that the curve traced out in the $(\mathcal{L},\mathcal{E})$-plane is quite complex. The energy is slightly less than 1 at large radii (and $\mathcal{L} \approx M^{1/2}r^{1/2}$ is large), but at some point the energy reaches a minimum. This is the point where $d\mathcal{E}/dr = 0$, or

$$0 = \frac{(1 - 3M/r + 2aM^{1/2}/r^{3/2})(2M/r^2 - 3aM^{1/2}/2r^{5/2}) - (1 - 2M/r + aM^{1/2}/r^{3/2})(3M/r^2 - 3aM^{1/2}/r^{5/2})/2}{(1 - 3M/r + 2aM^{1/2}/r^{3/2})^{3/2}}. \tag{16}$$

Simplifying the numerator gives

$$0 = \frac{M}{2r^2} - \frac{3M^2}{r^3} + \frac{4aM^{3/2}}{r^{7/2}} - \frac{3a^2M}{2r^4}, \tag{17}$$

implying that $r^{-1/2}$ obeys the quartic equation:

$$0 = 1 - 6\frac{M}{r} + 8\frac{aM^{1/2}}{r^{3/2}} - 3\frac{a^2}{r^2}. \tag{18}$$

The value of r satisfying Eq. (18) represents the minimum energy stable orbit around the hole. A similar calculation for the angular momentum shows that it is also the minimum angular momentum orbit. The allowed region in the $(\mathcal{L}, \mathcal{E})$-plane thus has a spike pointed toward the lower-left. The spike corresponds to the *innermost stable circular orbit.*

A. Properties of the ISCO

For $a \to 0$ it is easily seen that the solution to Eq. (18) is $r = 6M$ Moreover, as a becomes slightly positive, the right-hand side as a function increases (at least for small a where the order-a term dominates over order-a^2) and so the ISCO moves inward. In the limit of a maximally spinning hole, $\chi = a/M = 1$, we see that the ISCO lies at $r = M$, i.e. at the same radial coordinate as the horizon itself.

In order to understand the true behavior near maximal spin, however, we must be more careful. Let us suppose that $\chi = 1 - \epsilon$ for some small ϵ. Let us also suppose that $r = M(1 + \zeta)$, with ζ small. Then expanding to order ϵ or ζ^3, Eq. (18) becomes

$$0 = 1 - 6(1 - \zeta + \zeta^2 - \zeta^3) + 8\left(1 - \epsilon - \frac{3}{2}\zeta + \frac{15}{8}\zeta^2 - \frac{35}{16}\zeta^3\right) - 3(1 - 2\epsilon - 2\zeta + 3\zeta^2 - 4\zeta^3). \tag{19}$$

Simplifying gives $0 = -2\epsilon + \frac{1}{2}\zeta^3$, so $\zeta = \sqrt[3]{4\epsilon}$ or

$$r_{\rm ISCO} \approx M\left(1 + \sqrt[3]{4\epsilon}\right). \tag{20}$$

This is to be compared to the horizon radius,

$$r_+ = M + \sqrt{M^2 - M^2(1-\epsilon)^2} \approx M\left(1 + \sqrt{2\epsilon}\right), \tag{21}$$

so we see that always $r_{\rm ISCO} > r_+$.

A more subtle question is whether, as we approach maximal spin, the ISCO actually touches the horizon or whether this is an artifact of the coordinate system. In fact it is the latter. The radial distance element in the equatorial plane is $ds = \sqrt{\Sigma/\Delta}\, dr$, so if one integrates along a radial arc from a particle at the ISCO down to the horizon, we find that if $r = M(1 + \varsigma)$, then for $\varsigma \gg \sqrt{\epsilon}$ we have

$$ds = \frac{\sqrt{\Sigma}}{\sqrt{\Delta}}\, dr = \frac{r}{r\varsigma}\, r\, d\varsigma = r\, d\ln\varsigma, \tag{22}$$

and hence as $\chi \to 1$ the physical distance from the ISCO to the horizon actually grows as $\sim r\ln(\epsilon^{1/3}/\epsilon^{1/2}) \sim \frac{1}{6}r\ln(\epsilon^{-1})$.

Going the other way, as $a \to -1$ (retrograde orbit around a maximally spinning hole) we find $r \to 9M$.

IV. PRECESSION OF AN INCLINED ORBIT

Before we close, we investigate the problem of precession of an inclined circular orbit around Kerr. This has critical implications for the behavior of accretion disks – in particular it causes the inner portion of the disk to align with the hole's spin, since each orbit precesses around the hole at a different rate. We evaluate this precession, but only to leading order in $1/r$.

A "circular" orbit at a given value of r and at some inclination I will precess at a rate equal to the rate of the particle orbiting around the hole in the longitude direction minus the rate at which it bobs up and down in latitude. That is,

$$\Omega_{\rm prec} = \Omega_\phi - \Omega_\theta = \frac{1}{b_t}\left(b_\phi - \frac{2\pi}{P_\theta}\right). \tag{23}$$

This relation is general. However, in the particular case of a circular orbit at large r, we may note that $\mathcal{E} \sim 1$ and $\mathcal{L} \sim \sqrt{Mr}\cos I$. Then:

$$b_t = \frac{(r^2+a^2)^2}{\Delta}\mathcal{E} - 2aM\frac{r}{\Delta}\mathcal{L} - a^2\mathcal{E}\langle\sin^2\theta\rangle_\lambda \to r^2 - \mathcal{O}(aM^{3/2}r^{-1/2}) - \mathcal{O}(a^2) = r^2. \tag{24}$$

The treatment of b_ϕ and P_θ^{-1} is more complicated because the leading order in r (i.e. the $\sim r^{1/2}$) terms cancel. It turns out that we will need to compute b_ϕ and P_θ^{-1} through the $\sim r^{-1}$ terms. To this order, we may write

$$b_\phi - \frac{2\pi}{P_\theta} = \frac{2aMr\mathcal{E}}{\Delta} - \frac{a^2\mathcal{L}}{\Delta} + \mathcal{L}\langle\csc^2\theta\rangle_\lambda - \frac{2\pi}{P_\theta}. \tag{25}$$

Here the four terms scale with radius as $\sim r^{-1}$, $\sim r^{-3/2}$, $\sim r^{1/2}$, and $\sim r^{1/2}$. The second term is therefore irrelevant, and the first term can be approximated by its leading-order expansion $2aM/r$. Expanding $\langle\csc^2\theta\rangle_\lambda$ as an integral then gives

$$b_\phi - \frac{2\pi}{P_\theta} \approx \frac{2aM}{r} + \frac{1}{P_\theta}\left(\mathcal{L}\int_0^{P_\theta}\csc^2\theta\, d\lambda - 2\pi\right), \tag{26}$$

and hence

$$\Omega_{\rm prec} \approx \frac{2aM}{r^3} + \frac{1}{r^2 P_\theta}\left(\mathcal{L}\int_0^{P_\theta}\csc^2\theta\, d\lambda - 2\pi\right). \tag{27}$$

Now for the P_θ outside the parentheses, we may take the leading-order expansion: $\mathcal{Q} \sim \mathcal{L}^2\tan^2 I$ and then

$$\begin{aligned} P_\theta &\approx \frac{4}{\mathcal{L}}\int_{\pi/2-I}^{\pi/2}\frac{d\theta}{\sqrt{\cot^2\theta - \tan^2 I}} \\ &= -\frac{4}{\mathcal{L}}\int_{\sin I}^{0}\frac{dz}{\sqrt{z^2-(1-z^2)\tan^2 I}} \\ &= -\frac{4\cos I}{\mathcal{L}}\int_{\sin I}^{0}\frac{dz}{\sqrt{z^2-\sin^2 I}} \\ &= -\frac{4\cos I}{\mathcal{L}}\int_{\pi/2}^{0} d\alpha = \frac{2\pi\cos I}{\mathcal{L}} \approx 2\pi r^{-1/2}, \end{aligned} \tag{28}$$

where we have set $\cos\theta = z$, then $z = \sin I\sin\alpha$. We thus find:

$$\Omega_{\rm prec} \approx \frac{2aM}{r^3} + \frac{1}{2\pi r^{3/2}}\left(\mathcal{L}\int_0^{P_\theta}\csc^2\theta\, d\lambda - 2\pi\right). \tag{29}$$

Our task is thus to find the term β in parentheses through order $r^{-3/2}$. Using that

$$\frac{d\lambda}{d\theta} = \frac{1}{\sqrt{-a^2(1-\mathcal{E}^2)\cos^2\theta - \mathcal{L}^2\cot^2\theta + \mathcal{Q}}}, \tag{30}$$

we see that

$$\begin{aligned} \beta &= 4\int_{\pi/2-I}^{\pi/2}\frac{\mathcal{L}\csc^2\theta\, d\theta}{\sqrt{-a^2(1-\mathcal{E}^2)\cos^2\theta - \mathcal{L}^2\cot^2\theta + \mathcal{Q}}} - 2\pi \\ &= 4\int_{\pi/2-I}^{\pi/2}\frac{\csc^2\theta\, d\theta}{\sqrt{a^2\mathcal{L}^{-2}(1-\mathcal{E}^2)(\sin^2 I - \cos^2\theta) + \tan^2 I - \cot^2\theta}} - 2\pi. \end{aligned} \tag{31}$$

Now $a^2\mathcal{L}^{-2}(1-\mathcal{E}^2) \sim a^2/r^2$, so to order $r^{-3/2}$ this term may be dropped. We then have

$$\beta \approx 4\int_{\pi/2-I}^{\pi/2} \frac{\csc^2\theta\, d\theta}{\sqrt{\tan^2 I - \cot^2\theta}} - 2\pi. \tag{32}$$

Setting $y = \cot\theta/\tan I$, we reduce this to

$$\beta \approx -4\int_{1}^{0} \frac{dy}{\sqrt{1-y^2}} - 2\pi = 0. \tag{33}$$

We are thus left with the leading-order precession rate,

$$\Omega_{\rm prec} \approx \frac{2aM}{r^3} = \frac{2J}{r^3} = \frac{2GJ}{c^2r^3}. \tag{34}$$

The precession rate of a circular orbit around a rotating black hole is thus $\propto r^{-3}$ and is prograde (in the direction of the hole's rotation).

Precession as determined by Eq. (34) occurs around any object with angular momentum. Under ordinary circumstances, it is negligible – for a satellite orbiting the Earth at 300 km altitude, with $r = 6.7\times10^6\,$m and $J = 6\times10^{33}\,$kg m^2 s^{-1}, we find

$$\Omega_{\rm prec} = 3\times10^{-14}\,{\rm rad/s} = 0.2\,{\rm arcsec/yr}. \tag{35}$$

In practice, the precession of satellites is dominated by the Earth's quadrupole moment (equatorial bulge) and proceeds at the rate of 1 cycle per 2 months for a low-inclination satellite. The relativistic precession rate of a gyroscope orbiting the Earth in polar orbit (a factor of order unity smaller) has been measured by the Gravity Probe B mission, and agrees with the GR prediction of 40 mas/yr.

But near a black hole, the precession rate can become of order a few tenths of a cycle per orbit, or even more. For low-inclination prograde orbits around a Kerr hole, we can even precess through multiple cycles per orbit – e.g. $\Omega_\phi/\Omega_\theta$ can reach 2 if $\chi > 0.9678$.