O.D.E.s: Boundary Value Problems

BVP - problem of finding a solution to an ODE on \([a,b]\) with constraints on both ends.

Typical example:

\[ g'' = f(x, g, g') \quad g(a) = A \quad g(b) = B \quad x \in [a,b] \]

(but note that we'll also need \(g'(a)\), which is not given!)

(1) Shooting Method

Idea: reduce BVP to an IVP by guessing (an) unknown inner boundary conditions and then iterating until (a) modified inner boundary conditions lead to the correct outer boundary value

In (1): \(g'(a)\) not known, but needed.

\[ \text{guess } g'(a) = \ell \quad \text{then know } f(a, g(a), g'(a)) \]

and can reduce second-order problem into two first-order problems. Easy just integrate to b.

N.B.: Reduction to first order: example: \(g''(x) = r(x) - q(x) g'(x)\)

\[ \begin{align*}
(1) \quad g'(x) & = v(x) \\
(2) \quad v'(x) & = r(x) - q(x) v(x)
\end{align*} \]
- Now, since we have chosen $z$, we have now solved $y = y(x, z)$, but goal: $y(b, z) = B$

  New function: $\phi(z) = y(b, z) - B$
  and search for $z$ so that $\phi(z) = 0$ — looking for root of $\phi(z)$! — use root-finding method!

  Full shooting algorithm for $y'' = f(x, y, y')$:

  1. Guess starting value $z_0 = y'(a)$, set counter $i = 0$

  2. Compute $y = y(x, z_i)$ by integrating IVP

  3. Compute $\phi(z_i) = y(b, z_i) - B$.

    If $\phi(z_i)$ not sufficiently close to zero, increment $i = i + 1$, find new guess $z_{i+1}$, go back to (2).

Note: since $\frac{dy}{dx}$ must known in most cases, need two initial guesses + secant method.
(2) Finite-Difference method

If the ODE is linear in $g$ and $g'$, we can write $f(x, g, g')$ as

$$ g''(x) = \frac{g(x) - p(x) g'(x) - q(x) g(x)}{f(x, g, g')} $$

with $g(x), p(x), q(x)$ functions of $x$ only and $sign$ convention is arbitrary.

We can now discretize $g'$ and $g''$ on an evenly spaced grid with step size $h$

$$ g'(x_i) = \frac{g(x_{i+1}) - g(x_{i-1})}{2h} \quad \text{(central diff.)} $$

$$ g''(x_i) = \frac{g(x_{i+1}) + g(x_{i-1}) - 2g(x_i)}{h^2} $$

where $x_i = a + ih$, $i = 0, ..., n+1$ and $h = (b-a)/(n+1)$

get a system of $n+2$ linear algebraic equations:

$$ g_0 = A \quad \text{inner BC} $$

$$ g_{i-1} (1 - \frac{h^2}{2} p_i) - g_i (2 - h^2 q_i) + g_{i+1} (1 + \frac{h^2}{2} p_i) = h^2 g_i $$

$$ g_{n+1} = B \quad \text{outer BC} $$

where $g_i = g(x_i), p_i = p(x_i), g_i = g(x_i), \text{and } g_i = g(x_i)$
This can be nicely combined into an $n \times n$ LSE with a tridiagonal coefficient matrix:

$$
\begin{pmatrix}
-2 + h^2 q_i & 1 + \frac{h}{2} p_i \\
1 - \frac{h}{2} p_2 & & 1 + \frac{h}{2} p_{n-1} \\
& \ddots & \ddots & \ddots \\
& & 1 - \frac{h}{2} p_n & -2 + h^2 q_n
\end{pmatrix}
\begin{pmatrix}
y_1 \\
\vdots \\
y_n
\end{pmatrix}
= \begin{pmatrix}
h^2 q_1 - A(1 - \frac{h}{2} p_1) \\
h^2 q_2 \\
\vdots \\
h^2 q_{n-1} \\
h^2 q_n - B(1 + \frac{h}{2} p_n)
\end{pmatrix}
$$

→ solve with tridiagonal solver!

NB: what did we do in the above?
- just taken centric line of (44+) and plugged in boundary values (appearing on RHS).