1. Basis Independence of Contractions

The tensor $S$ is a rank $(2_1)$ tensor, $S^\alpha\gamma$, and $T$ is a rank $(1_0)$ tensor, $T^\beta = S^\alpha_\alpha$. More explicitly,

$$T(\tilde{k}) = \sum_{\alpha=0}^{3} S(\tilde{\omega}^\alpha, \tilde{k}, e_\alpha). \quad (1)$$

Let’s consider an unprimed basis and a primed basis, where the two bases are related by a basis transformation $L$. The transformation acts on basis vectors and basis one-forms as

$$\tilde{e}_{\mu'} = L^\mu_\mu' \tilde{e}_\mu \quad \text{and} \quad \tilde{\omega}^{\mu'} = (L^{-1})^\mu_\mu' \tilde{\omega}^\mu. \quad (2)$$

We now consider our contracted tensor $T$ in the primed basis,

$$T^{\nu'} = S^\mu_\mu'^{\nu'} = (L^{-1})^\mu_\mu'^{\nu'} (L^{-1})^{\nu'}_\nu L^\rho_\mu' S^{\mu\nu}_\rho = \delta^\mu_\mu'(L^{-1})^{\nu'}_\nu S^{\mu\nu}_\rho \quad (3)$$

$$= (L^{-1})^{\nu'}_\nu S^{\mu\nu}_\mu = (L^{-1})^{\nu'}_\nu T^{\nu}. \quad (4)$$

Since the components of the contracted tensor $T^\beta$, transform as $T^{\nu'} = (L^{-1})^{\nu'}_\nu T^{\nu}$, just as we would expect any rank $(1_0)$ to transform, we may conclude that the contraction $T$ of $S$ is independent of the choice of basis.

2. Forms in 2 Dimensions

a) We are working in two-dimensional Euclidean space $\mathbb{R}^2$. First we want to find the Hodge dual of a scalar $f$,

$$\star f = (\star f)^{ij} = \epsilon^{ij} f. \quad (5)$$

Now we want to find the dual of a vector,

$$(\star v)^{ij} = \epsilon^{ij} v_i. \quad (6)$$
Explicitly, we see that for a vector \( v = (v^1, v^2) \), the dual vector is \( \star v = (-v^2, v^1) \). The geometric interpretation of this is that taking the dual of a vector corresponds to a \( \pi/2 \) rotation counterclockwise. Thus we may preempt the next question and guess that taking the dual of the dual vector will correspond to a rotation by \( \pi \), and gives us \( -\vec{v} \).

\[
(\star \star v)^i = \epsilon^{ij}(\star v)_j = \epsilon^{ij}\epsilon_{ki}v^k = -\delta^i_k v^k = -v^k;
\]
as expected. Now the dual of the dual of a scalar,

\[
(\star \star f) = \star(\epsilon^{ij}f) = \frac{1}{2}\epsilon_{ij}\epsilon^{ij}f = \frac{1}{2}2f = f.
\]

**A technical aside**

For fun let’s try and rederive the same results in a more general discussion of some of these concepts. First let’s review some general definitions which will be useful in our discussion, we write an \( r \)-form in a coordinate basis as

\[
\omega = \frac{1}{r!}\omega_{\mu_1...\mu_r}dx^{\mu_1} \wedge ... \wedge dx^{\mu_r}
\]

where the space of \( r \)-forms on an \( m \)-dimensional manifold \( \mathcal{M} \) as is \( \Omega^r(\mathcal{M}) \). Dropping the basis allows us to write \( \omega = \omega_{\mu_1...\mu_r}dx^{\mu_1} \), where the prefactor is absorbed in the antisymmetrization. The exterior derivative on an \( r \)-form is defined as the map \( d : \Omega^r(\mathcal{M}) \rightarrow \Omega^{r+1}(\mathcal{M}) \). Thus for a general \( r \)-form we find the exterior derivative to be

\[
d\omega = \frac{1}{r!}\partial_\alpha\omega_{\mu_1...\mu_r}dx^{\alpha} \wedge dx^{\mu_1} \wedge ... \wedge dx^{\mu_r}.
\]

Dropping the coordinate basis allows us to recover the familiar \( d\omega = (r+1)\partial_\alpha\omega_{\mu_1...\mu_r} \), where \( r \) is the rank of the form. The \( r + 1 \) prefactor comes from the antisymmetrization of all \( r + 1 \) indices in the basis. Since \( \dim \Omega^r(\mathcal{M}) = \dim \Omega^{m-r}(\mathcal{M}) \) on an \( m \)-dimensional manifold \( \mathcal{M} \), it is natural to define the Hodge dual action as the map \( \star : \Omega^r(\mathcal{M}) \rightarrow \Omega^{m-r}(\mathcal{M}) \). Thus for a general \( r \)-form we find the Hodge dual to be

\[
\star \omega = \left(\frac{1}{|g|}\right)^{\frac{r}{2}}\omega_{\mu_1...\mu_r}\epsilon^{\mu_1...\mu_r}_{\mu_{r+1}...\mu_m}dx^{\mu_{r+1}} \wedge ... \wedge dx^{\mu_m}.
\]

Applying this operation twice we find that and using the identity,

\[
\epsilon^{\alpha_1...\alpha_r\alpha_{r+1}...\alpha_m}\epsilon_{\alpha_1...\alpha_r\beta_{r+1}...\beta_m} = r!(m-r)! (\det g)^{-1} \delta^{[\alpha_1}_{\beta_{r+1}} \delta^{\alpha_{r+2}}_{\beta_{r+2}} ... \delta^{\alpha_m]}_{\beta_m}
\]

For an \( r \)-form \( \omega \in \Omega^r(\mathcal{M}) \), we find that for Euclidean metric signature \((0,m)\) and for Lorentzian metric signature \((1,m-1)\), that

\[
(0,m) : \quad \star \star \omega = +(-1)^{r(m-r)}\omega
\]

\[
(1,m-1) : \quad \star \star \omega = -(-1)^{r(m-r)}\omega
\]

Now we focus our general discussion to \( \mathbb{R}^2 \), where the signature is \((0,2)\). Now for an \( r \)-form \( \omega \in \Omega^r(\mathbb{R}^2) \), the expression for \( \star \star \omega \) becomes

\[
\star \star \omega = +(-1)^{r(2-r)}\omega.
\]
Formally, thinking about a scalar $f$ in the language of forms means we think about it as a 0-form, which means we must make the definition $f \in \Omega^0 = \mathbb{R}$. Proceeding we see that

$$**f = +f,$$

whereas for a 1-form $v \in \Omega^1(\mathbb{R}^2)$, we find that

$$**v = -v. \tag{15}$$

b) We now consider a vector field $v(x)$, or equivalently a 1-form field $\omega$. The two ways we may construct scalar fields are $f = \star d\omega$ and $h = \star d\star\omega$. We should see that these are both scalars as $d\omega$ is a 2-form and $\star d\omega$ is a $(2 - 2 = 0)$-form. Similarly, $\star d\omega$ is a $(2 - (1 + 2 - 1) = 0)$-form. Let’s evaluate these explicitly, we start with

$$(d\omega)_{ij} = 2\partial_i\omega_j - \partial_j\omega_i. \tag{16}$$

This inherent antisymmetry is clear from the graded commutativity in the coordinate basis of wedge products,

$$d\omega = \partial_\mu\omega_\nu dx^\mu \wedge dx^\nu \tag{17}$$

Continuing,

$$\star d\omega = \frac{1}{2}\epsilon^{ij}(\partial_i\omega_j - \partial_j\omega_i) = \epsilon^{ij}\partial_i\omega_j \tag{18}$$

where we have used the antisymmetry of $\epsilon$. We can identify this as the curl of $\omega$. We recall that in three dimensions the curl of a one-form (or equivalently, a vector) gives us a vector, but in two dimensions the curl of a vector gives a scalar as we see above. Now we consider the second way to construct a scalar $\star d\star\omega$,

$$\star d\star\omega = \star d(\epsilon^i_j\omega_i dx^j) = \star d(g^{ik}\epsilon_{kj}\omega_i dx^j)$$

$$= 2\star(g^{ik}\epsilon_{kj}\partial_i\omega_i dx^j \wedge dx^\ell)$$

$$= \epsilon^{ijg}g^{ik}\epsilon_{kj}\partial_j\omega_i = \delta^{ijg}g^{ik}\partial_j\omega_i = g^{ij}\partial_j\omega_i = \partial_\ell\omega^\ell.$$

We identify this as the divergence of $\omega$. We also note that since our space is Euclidean and obviously has a well-defined metric, we have not concerned ourselves in the above discussion with the distinction between one-forms and vectors.

3. Existence of 1-form Potentials

a) Consider a closed 2-form $H$, where the last row vanishes $H_{ni} = 0$ for all values of $i$, and $H_{ij}(x^1, \ldots, x^{n-1}, 0) = 0$, i.e. $H_{ij} = 0$ on $\mathbb{R}^{n-1}$. We start with the fact that $H$ is closed,

$$(dH)_{ijk} = 3\partial_i H_{jk} = \frac{3}{3!}(\partial_i H_{jk} + \partial_j H_{ki} + \partial_k H_{ij} - \partial_j H_{ki} - \partial_k H_{ij} - \partial_k H_{ij}) = 0 \tag{19}$$

where $1 \geq i, j, k \leq n$. Using the antisymmetry of $H$ we find that

$$(dH)_{ijk} = \partial_i H_{jk} + \partial_j H_{ki} + \partial_k H_{ij} = 0. \tag{20}$$
We note that as the entire object is antisymmetric in all three indices there can be not repeated index, i.e. \( i \neq j \neq k \). We split the above equation into components and take \( k \) to run over just \( n \), and \( i, j \) to run over the other coordinates, \( 1 \geq i, j < n \). We find that

\[
\partial_i H_{jn} + \partial_j H_{ni} + \partial_n H_{ij} = 0. \tag{21}
\]

The first two terms vanish as \( H_{ni} = 0 \) on all of \( \mathbb{R}^n \), which leaves us with the statement that \( \partial_n H_{ij} = 0 \). This means that \( H = H(x^1, \ldots, x^{n-1}) \) and is only a function of coordinates on the \( \mathbb{R}^{n-1} \) subspace. Since we already know that \( H_{ij} \) on \( \mathbb{R}^{n-1} \) we may conclude that \( H = 0 \) everywhere on \( \mathbb{R}^n \).

b) We now take the dimension of our space to be \( n = 2 \) and claim that there exists a two-form \( F \) that is closed. In two dimensions any two-form is automatically closed. In case this isn’t immediately obvious consider taking the exterior derivative of a two-form in a coordinate basis \( F \) that is closed. In two dimensions any two-form is automatically closed. In case this isn’t.

We now take the dimension of our space to be \( n = 2 \) and claim that there exists a two-form \( F \) that is closed. In two dimensions any two-form is automatically closed. In case this isn’t immediately obvious consider taking the exterior derivative of a two-form in a coordinate basis of forms, \( d\omega = \partial_i \omega_{jk} dx^i \wedge dx^j \wedge dx^k \), where \( i \neq j \neq k \). Thus as the indices only run over two dimensions, \( d\omega = 0 \) by the antisymmetry of the indices. Now we define a one-form \( A = A_i dx^i \) and take the exterior derivative

\[
dA = (\partial_i A_j - \partial_j A_i) dx^i \wedge dx^j. \tag{22}\]

In two dimensions \( F \) one has one degree of freedom, \( F_{12} = -F_{21} \), as \( F_{11} = F_{22} = 0 \). Let that one degree of freedom be given by some function \( f(x^1, x^2) \), i.e. \( F_{12} = -F_{21} = f(x^1, x^2) \). Then as \( dA \) is a two-form, we may write

\[
f(x^1, x^2) = \partial_1 A_2 - \partial_2 A_1. \tag{23}\]

There exist many choices of \( A^1(x^1, x^2) \) and \( A^2(x^1, x^2) \) that satisfy our conditions for \( F \). But we may make the choice that \( A^1(x^1, x^2) = 0 \) and

\[
A^2(x^1, x^2) = \int_0^{x^1} f(y^1, x^2) dy^1. \tag{24}\]

Thus for any 2-form \( F \) in two dimensions, there exists a 1-form \( A \), where we may write \( F = dA \).

c) We now want to extend our argument in part b, to \( n > 2 \) dimensions. All we need to do is show that if in \( d = n - 1 \) there exists a 2-form \( F \), such that \( F = dA \), then this also holds for \( d = n \). Let \( F = F(x^1, \ldots, x^n) \) be a closed 2-form in \( n \) dimensions and let there be another closed 2-form in \( n - 1 \) dimensions \( G(x^1, \ldots, x^{n-1}) = F(x^1, \ldots, x^{n-1}, 0) \), where \( G = G_{ij} \) for \( 1 \leq i, j \leq n - 1 \). From our argument in part b, there then exists a 1-form field \( \sigma_i(x^1, \ldots, x^{n-1}) \) defined on \( \mathbb{R}^{n-1} \), the \( n - 1 \) dimensional plane, so \( G = d\sigma \) on \( x^n = 0 \). Thus we may define

\[
A_i(x^1, \ldots, x^n) = \sigma_i(x^1, \ldots, x^{n-1}) + \int_0^{x^n} F_{ni}(x^1, \ldots, y^n) dy^n \tag{25}\]

where the index \( i \) runs over \( i = 1, \ldots, n - 1 \). Just as in part b, we fix the degrees of freedom of \( A \) and take \( A_n = 0 \). Now consider a 2-form \( H \) defined as \( H = F - dA \), which is clearly a closed form. We now want to show that all the components of \( H \) vanish. Clearly by antisymmetry \( H_{mn} \) vanishes. Now consider \( H_{in} \) where \( 1 \geq i \leq n - 1 \),

\[
H_{in} = F_{in} - \partial_i A_n + \partial_n A_i = F_{in} + \partial_n \sigma_i - \partial_i \left( \int_0^{x^n} F_{in}(x^1, \ldots, y^n) dy^n \right) = F_{in} - F_{in} = 0 \tag{26}\]
using the definition of $A$ above and the fact that $\sigma$ vanishes in the $x^n$ direction. Now we want to consider $H_{ij}$,  
\begin{align}
H_{ij} &= F_{ij} - \partial_i A_j + \partial_j A_i \\
&= F_{ij} - (d\sigma)_{ij} - \frac{\partial}{\partial x^i} \int_0^{x^n} F_{nj}(x^1, \ldots, y^n) dy^n + \frac{\partial}{\partial x^j} \int_0^{x^n} F_{ni}(x^1, \ldots, y^n) dy^n \\
&= F_{ij} - (d\sigma)_{ij} - \int_0^{x^n} (\partial_i F_{nj} - \partial_j F_{ni}) dy^n \quad (27)
\end{align}

Since $1 \geq i, j \leq n - 1$, the integral will clearly vanish in the integration over the $n$-th direction and since we defined $F_{ij} = (d\sigma)_{ij}$, we find that $H_{ij} = 0$. We have found that $H$ aligns with the statement in part a, that the last row vanishes $H_{in} = 0$ on $\mathbb{R}^x$, and $H_{ij} = 0$ on the $\mathbb{R}^{n-1}$ subspace, and thus $H$ vanishes everywhere. We have then found that everywhere in $\mathbb{R}^x$ $F$ can be written as $F = dA$, meaning we have explicitly constructed the 2-form field $F$ as the exterior derivative of a 1-form $A$ in $n$ dimensions.

This result is useful in differential geometry and is often discussed in a more general context. The more general statement of this goes by the name Poincaré lemma and states that for any contractible open set on a manifold, or as is more applicable in our case, any contractible domain in $\mathbb{R}^n$, $\omega \in \Omega^r$ such that $d\omega = 0$, for $r > 0$ there exists $\alpha \in \Omega^{r-1}$ such that $\omega = d\alpha$. An $r$-form that may be written as $\omega = d\alpha$, where $\alpha \in \Omega^{r-1}$, is called an exact form. In other words, for any contractible domain in $\mathbb{R}^n$ a closed $r$-form is also locally an exact form. The measure of which closed $r$-forms are exact forms on some manifold $\mathcal{M}$ is called the $r$-th de Rham cohomology $H^r(\mathcal{M})$. Thus if all closed 2-forms are exact forms on $\mathbb{R}^2$ then the second de Rham cohomology class is trivial, $H^2(\mathbb{R}^2) = 0$.

d) From the nilpotency of the exterior derivative $d^2 \omega = 0$, we see that for an exact 2-form $F = dA$, the 1-form can be rewritten as $A + d\alpha$, where $\alpha \in \Omega^0$ (i.e. a scalar), without changing our definition of $F$. Thus the 1-form potential is not unique. This is equivalent to saying that the 1-form $A$ is only unique up to an exact 1-form $\beta = d\alpha$. This is just a mathematical restatement of the gauge invariance of $F_{\mu\nu}$, which is the field strength of the gauge field $A_{\mu}$, where a general gauge transformation is given by $A_{\mu} \rightarrow A_{\mu} + \partial_{\mu} \beta$.

4. Electromagnetic Potential

We know that $F$ is an exact 2-form, given in terms of a 1-form $A$ as $F = dA$. In the language of forms in a coordinate basis, where a general $r$-form $\omega \in \Omega^r$ can be written as $\omega = \omega_{\mu_1...\mu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}$, the field strength $F$ and potential $A$ are written as

\begin{align}
A &= A_\mu dx^\mu \\
F &= F_{\mu\nu} dx^\mu \wedge dx^\nu \quad (31)
\end{align}

Thus we have

\begin{align}
dA &= 2\partial_\mu A_{\nu} dx^\mu \wedge dx^\nu = (\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu \\
&= \partial_\mu A_\nu - \partial_\nu A_\mu \quad (32)
\end{align}

using the graded commutivity of the wedge product (in which the antisymmetry of forms is encoded). Thus we arrive at the familiar expression

\begin{align}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (33)
\end{align}
Splitting the potential $A$ into its time component $\Phi$ and spatial components, $A_\mu = (-\Phi, A_i)$ and recalling that the electromagnetic field strength encodes the electric and magnetic fields as

$$F_{\mu\nu} = \begin{pmatrix}
0 & -E_1 & -E_2 & -E_3 \\
-E_1 & 0 & B_3 & -B_2 \\
-E_2 & -B_3 & 0 & B_1 \\
-E_3 & B_2 & -B_1 & 0
\end{pmatrix}$$

or equivalently, $F_{i0} = E_i$ and $F_{ij} = \epsilon_{ijk}B_k$. Thus, we find that

$$F_{i0} = \partial_i A_0 - \partial_0 A_i = E_i \quad \rightarrow \quad \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \Phi$$

and that

$$F_{ij} = \partial_i A_j - \partial_j A_i = \epsilon_{ijk}B_k$$

contracting with $\epsilon_{ijm}$ we find that

$$B_m = \epsilon_{ijm}\partial_i A_j \quad \rightarrow \quad \vec{B} = \nabla \times \vec{A}.$$