I. OVERVIEW

In this lecture, we will investigate geodesics. These can be defined in two ways – as curves of zero acceleration, and as curves of stationary length. We will show these to be equivalent. We will then define the Riemann curvature tensor and relate it to the behavior of neighboring geodesics.

The recommended reading for this lecture is:

- MTW §8.7.

II. GEODESICS

Geodesics play a central role in GR: they are the trajectories followed by freely falling particles. So today we are finally ready to define and use them. We will define them first as curves of zero acceleration. We will follow this by showing that the same curves can be derived from an action principle: they are curves of stationary length.

A. Geodesics as zero acceleration curves

In flat spacetime, we defined the acceleration as $a = d^2x/dt^2$. The existence of the covariant derivative has made it possible to define the derivative of the velocity in curved spacetime. To do this in the most general way possible, consider any trajectory $\mathcal{P}(\lambda)$, parameterized by $\lambda$ (which may be the proper time $\tau$, but we won’t impose this since we want to be able to treat photons as well), and with a tangent vector $v = d\mathcal{P}/d\lambda$. If $S$ is a general tensor field, we define its derivative along the curve as

$$\frac{DS}{d\lambda} = \nabla_{d\mathcal{P}/d\lambda} S = \nabla v S,$$  

or in component language for a $(1,1)$ tensor,

$$\frac{DS^\alpha_\beta}{d\lambda} = v^\gamma S^\alpha_\beta,_{\gamma}.$$  

The capital $D$ is used here to remind us that we are using the covariant derivative (the lower case $d$ would denote the derivative of the components $S^\alpha_\beta$ – usually not what we want). Expansion of the covariant derivative in terms of partial derivatives tells us that

$$\frac{DS^\alpha_\beta}{d\lambda} = v^\gamma S^\alpha_\beta,_{\gamma} + v^\gamma \Gamma^\alpha_{\mu\gamma} S^\mu_\beta - v^\gamma \Gamma^\mu_{\beta\gamma} S^\alpha_\mu = \frac{dS^\alpha_\beta}{d\lambda} + v^\gamma \Gamma^\alpha_{\mu\gamma} S^\mu_\beta - v^\gamma \Gamma^\mu_{\beta\gamma} S^\alpha_\mu.$$  

In this form, one can see that $DS/d\lambda$ depends only on the value of $S$ along the trajectory. So it is perfectly valid to speak of the derivative of the tangent vector, $Dv/d\lambda$. In the special case of a curve parameterized by proper time, $v = d\mathcal{P}/d\tau$ is the 4-velocity and $a = Dv/d\tau$ is the 4-acceleration. In this case we have

$$a \cdot v = \frac{Dv}{d\tau} \cdot v = \frac{1}{2} \frac{d}{d\tau}(v \cdot v) = 0.$$  

*Electronic address: chirata@tapir.caltech.edu
We now consider curves of zero acceleration, i.e. for which $Dv/d\lambda = 0$. Such a curve will be called a **geodesic**. Writing $v^\alpha = dx^\alpha/d\lambda$, we see that

$$\frac{Dv^\alpha}{d\lambda} = \frac{dv^\alpha}{d\lambda} + v^\gamma \Gamma^\alpha_{\mu\gamma} v^\mu = \frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\gamma} \frac{dx^\mu}{d\lambda} \frac{dx^\gamma}{d\lambda}.$$  \hspace{1cm} (5)

If we set this equal to zero, we get a 2nd order system of ODEs for $x^\alpha(\lambda)$:

$$\frac{d^2 x^\alpha}{d\lambda^2} = -\Gamma^\alpha_{\mu\gamma} \frac{dx^\mu}{d\lambda} \frac{dx^\gamma}{d\lambda}.$$  \hspace{1cm} (6)

Note that in flat space ($\mathbb{R}^n$) or spacetime ($\mathcal{M}^4$) we find that in the standard coordinate systems all $\Gamma$s are zero and $x^\alpha$ are linear functions of $\lambda$.

**General definition**: A tensor $S$ is said to be **parallel-transported** along a curve if $DS/d\lambda = 0$. As an example, the velocity for a geodesic is parallel-transported.

**B. Geodesics as curves of stationary proper time**

There is an alternative viewpoint of a geodesic which leads to the same conclusion. Suppose we have a timelike curve $\mathcal{P}(\lambda)$. Let us suppose that the curve runs from $A = \mathcal{P}(\lambda_i)$ to $B = \mathcal{P}(\lambda_f)$. We want to find the conditions under which the proper time measured by an observer who follows the curve is stationary. To do this, write the total proper time along a trajectory

$$T = \int_{\lambda_i}^{\lambda_f} d\tau d\lambda.$$  \hspace{1cm} (7)

We want to determine the variation $\delta T$ of $T$ with respect to perturbations of the trajectory, $\delta x^\mu(\lambda)$. We will assume without loss of generality that the original curve is parameterized by proper time, $\lambda = \tau$ so that $|v|^2 = -1$, but we cannot assume this for the perturbed curve (since there is no guarantee that the perturbed curve will have proper time interval $\lambda_f - \lambda_i$).

We can see first that

$$\frac{d\tau}{d\lambda} = \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}},$$  \hspace{1cm} (8)

so the variation is

$$\delta T = \int_{\lambda_i}^{\lambda_f} \frac{\delta \left[ -g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right]}{2 \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}} d\lambda.$$  \hspace{1cm} (9)

If we perturb around a curve with $d\tau/d\lambda = 1$, then the square root in the denominator is unity, and so

$$\delta T = \int_{\lambda_i}^{\lambda_f} \left[ \frac{1}{2} g_{\mu\nu,\sigma} \delta x^\sigma \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} - \frac{1}{2} g_{\mu\nu} \frac{d\delta x^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} - \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{d\delta x^\nu}{d\lambda} \right] d\lambda.$$  \hspace{1cm} (10)

Now if we fix the endpoints $A$ and $B$, then $\delta x^\mu = 0$ at $\lambda_i$ and $\lambda_f$. Therefore we may use integration by parts on the last term,

$$\delta T = \int_{\lambda_i}^{\lambda_f} \left[ \frac{1}{2} g_{\mu\nu,\sigma} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} + \frac{d}{d\lambda} \left( g_{\sigma\nu} \frac{dx^\nu}{d\lambda} \right) \right] d\lambda.$$  \hspace{1cm} (11)

In order for the curve to have stationary proper time, we must have the quantity in $[]$ always equal to zero. This gives us a differential equation,

$$-\frac{1}{2} g_{\mu\nu,\sigma} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} + \frac{d}{d\lambda} \left( g_{\sigma\nu} \frac{dx^\nu}{d\lambda} \right) = 0,$$  \hspace{1cm} (12)
or
\[-\frac{1}{2}g_{\mu\nu,\sigma} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} + g_{\sigma\nu,\mu} \frac{dx^\nu}{d\lambda} \frac{dx^\mu}{d\lambda} + g_{\sigma\nu} \frac{d^2x^\nu}{d\lambda^2} = 0.\]  
(13)

We now raise indices (multiply through by $g^{\kappa\sigma}$ – since this is invertible the original and final equations have equivalent information) and find
\[\frac{d^2x^\kappa}{d\lambda^2} = \frac{1}{2}g^{\kappa\sigma}\left(-g_{\mu\nu,\sigma} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} + 2g_{\sigma\nu,\mu} \frac{dx^\nu}{d\lambda} \frac{dx^\mu}{d\lambda}\right).\]  
(14)

Using the Christoffel symbols, we can turn this into
\[\frac{d^2x^\kappa}{d\lambda^2} = -\Gamma^\kappa_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}.\]  
(15)

This is equivalent to Eq. (6) so we conclude that the timelike curves of extremal proper time are the timelike geodesics!

[Note: We have shown that the timelike curves are stationary, but have not said whether they are maxima, minima, or saddle points. In Minkowski space it is easy to show that they are maxima. Saddle points are possible in more general spacetimes (not even exotic ones – the Earth’s trajectory from its location in April 2010 to October 2011 was a saddle point). There are however no minima, since one can always add little wiggles to a trajectory to shorten its proper time.]

C. Example: Polar coordinates

In polar coordinates in $\mathbb{R}^2$, we showed that the nonzero Christoffel symbols were
\[\Gamma^r_{\theta\theta} = -r, \quad \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r}.\]  
(16)

It follows that the equation of motion for a geodesic is
\[\frac{d^2r}{d\lambda^2} = r \left(\frac{d\theta}{d\lambda}\right)^2 \quad \text{and} \quad \frac{d^2\theta}{d\lambda^2} = -\frac{2}{r} \frac{dr}{d\lambda} \frac{d\theta}{d\lambda}.\]  
(17)

This equation is the first example we have seen of inertial forces: the apparent bending of a trajectory because of the coordinate system. In particular, you can see that $d^2r/d\lambda^2 \geq 0$: particles with any motion in the $\theta$-direction appear to be repelled from the origin (“centrifugal force”). GR describes gravity entirely in such terms.

III. RIEMANN CURVATURE TENSOR

Thus far we have described how to compute geodesics (and, by implication, the trajectories of freely falling particles in any spacetime of known metric tensor). But we also said that in GR, matter causes spacetime “curvature.” We have a way to describe how much matter there is – this is the stress-energy tensor $T_{\mu\nu}$ – but we don’t yet have a way of quantifying curvature. This is our job now. We will approach the problem of curvature in two ways: one by considering the (non)commutativity of the covariant derivative, and one by considering the behavior of neighboring geodesics. The former lends itself to mathematical analysis, but the latter is most closely associated with our intuitive notion of how the curvature of spacetime is measured.

A. Switching the order of covariant derivatives

In ordinary calculus, you learned about the commutativity of partial derivatives for any smooth function,
\[\frac{\partial^2 f}{\partial x \, \partial y} = \frac{\partial^2 f}{\partial y \, \partial x}.\]  
(18)

In special relativity, this carried over to the commutativity of partial derivatives of an arbitrary tensor,
\[S^\alpha_{\beta,\gamma\delta} = S^\alpha_{\beta,\delta\gamma}.\]  
(19)
An obvious question now is the behavior of covariant derivatives – in this case does the order matter? For scalars, the answer is no:

\[ f_{;\alpha \beta} \equiv \nabla_\alpha \nabla_\beta f = \nabla_\alpha f_{,\beta} = f_{,\beta} - \Gamma^\mu_{\beta \alpha} \partial_\mu f, \quad (20) \]

and on account of the symmetry of the Christoffel symbol this is symmetric under interchange of \( \alpha \) and \( \beta \).

But for vectors, the story is different: if we have a vector field \( w^\gamma \), then

\[
w^\gamma_{;\beta \alpha} \equiv \nabla_\alpha \nabla_\beta w^\gamma = \nabla_\alpha (w^\gamma_{,\beta} + \Gamma^\gamma_{\mu \beta} w^\mu) \\
= (w^\gamma_{,\beta} + \Gamma^\gamma_{\mu \beta} w^\mu)_{,\alpha} + \Gamma^\gamma_{\nu \alpha} (w^\nu_{,\beta} + \Gamma^\nu_{\mu \beta} w^\mu) - \Gamma^\gamma_{\beta \alpha} (w^\gamma_{,\pi} + \Gamma^\gamma_{\mu \pi} w^\mu) \\
= w^\gamma_{,\beta \alpha} + \Gamma^\gamma_{\mu \beta, \alpha} w^\mu + \Gamma^\gamma_{\mu \alpha, \beta} w^\mu + \Gamma^\gamma_{\nu \alpha} w^\nu_{,\beta} + \Gamma^\gamma_{\nu \beta} w^\nu_{,\alpha} - \Gamma^\gamma_{\beta \alpha} w^\gamma_{,\pi} - \Gamma^\gamma_{\beta \alpha} \Gamma^\gamma_{\mu \pi} w^\mu. \quad (21)\]

There are 7 terms here. Of these, terms \#1, \#6, and \#7 are symmetric in \( \alpha \) and \( \beta \), and switching \( \alpha \) and \( \beta \) swaps terms \#3 and \#4. However, terms \#2 and \#5 are not symmetric. We therefore have

\[
w^\gamma_{;\beta \alpha} - w^\gamma_{;\alpha \beta} = (\Gamma^\gamma_{\mu \beta, \alpha} + \Gamma^\gamma_{\nu \alpha} \Gamma^\nu_{\mu \beta} - \Gamma^\gamma_{\mu \alpha, \beta} - \Gamma^\gamma_{\nu \beta} \Gamma^\nu_{\mu \alpha}) w^\mu. \quad (22)\]

Thus we see that in general covariant derivatives of vectors need not commute. The amount by which they do not commute depends only on the value of the vector field at the point in question (and not on its derivatives). The object in parentheses is called the Riemann curvature tensor (or “Riemann”). It is a tensor because the covariant derivatives were defined in such a way as to transform appropriately (i.e., according to the Jacobian) under changes of coordinate system.

So the Riemann tensor is a rank \((1,3)\) tensor, whose components are given by

\[
R^\gamma_{\mu \alpha \beta} = \Gamma^\gamma_{\mu \beta, \alpha} + \Gamma^\gamma_{\nu \alpha} \Gamma^\nu_{\mu \beta} - \Gamma^\gamma_{\mu \alpha, \beta} - \Gamma^\gamma_{\nu \beta} \Gamma^\nu_{\mu \alpha}. \quad (23)\]

Thus

\[
w^\gamma_{;\beta \alpha} - w^\gamma_{;\alpha \beta} = R^\gamma_{\mu \alpha \beta} w^\mu. \quad (24)\]

Since the Christoffel symbols depend on the metric and its 1st derivative, the Riemann tensor depends on the metric and its 2nd derivative. It has \( n^4 \) components (256 in 4-dimensional spacetime!) but most are zero or related by symmetries. For example, it is obvious that the Riemann tensor is antisymmetric on the last two indices,

\[
R^\gamma_{\mu \alpha \beta} = -R^\gamma_{\mu \beta \alpha}. \quad (25)\]

The non-commutation of covariant derivatives is a phenomenon that does not occur for flat spacetime. It does not even occur for flat spacetime with curved coordinates, since the Riemann tensor is a tensor and hence could be computed to be 0 in Cartesian coordinates and then converted to any other system. Therefore it is appropriate to use Riemann as a measure of spacetime curvature.

A common alternative way to write the Riemann tensor in terms of directional derivatives is as follows –

\[
w^\mu \nabla_\mu (w^\nu \nabla_\nu w^\beta) - w^\nu \nabla_\nu (w^\mu \nabla_\mu w^\beta) = (w^\mu \nabla_\mu w^\nu) \nabla_\nu w^\beta + w^\mu w^\nu \nabla_\nu w^\beta - (w^\mu \nabla_\mu w^\nu) \nabla_\nu w^\beta + w^\mu w^\nu \nabla_\nu w^\beta \\
= (w^\mu \nabla_\mu w^\nu - w^\nu \nabla_\mu w^\nu) \nabla_\nu w^\beta + w^\mu w^\nu R^\gamma_{\gamma \mu \nu} w^\gamma. \quad (26)\]

If we define the commutator of two vector fields \( u \) and \( v \) as the vector field given by

\[
[u, v] = \nabla_u v - \nabla_v u, \quad (27)\]

then we may write in abstract notation:

\[
\nabla_u \nabla_v w - \nabla_v \nabla_u w = [u,v] w + \text{Riemann}(\cdot, w, u, v). \quad (28)\]

In this equation, the non-commutation of the directional derivative appears is directly related to curvature (the Riemann tensor). Note that the commutator term has nothing at all to do with spacetime curvature and is merely a property of the vector fields involved. In fact, because of the symmetry of the Christoffel symbol, we see that

\[
[u, v]^\alpha = w^\beta w_{\beta , \alpha} = w_{\alpha , \beta} - w_{\beta , \alpha} \quad (29)\]

(the terms with Christoffel symbols cancel) – there is just differential geometry involved, no metric. In fact (homework exercise), the commutator of the vector fields describes the extent to which following a trajectory with velocity given by \( u \) and then a trajectory with velocity given by \( v \) leads to a different path than a trajectory with velocity given by \( v \) and then one with velocity given by \( u \). The coordinate basis vectors, whose components are always \((0...1...0)\), always commute:

\[
[e_\alpha, e_\beta] = 0. \quad (30)\]
B. Riemann tensor and parallel transport

An alternative description of the Riemann tensor that more directly illustrates that it is a “local measure of curvature” is to consider the parallel transport of a vector $\zeta$ around a small closed loop.

Let us work in a local Lorentz (or Cartesian) frame at some point $P$, so that $P$ is associated with the coordinates $(0, ..., 0)$ and $g_{\alpha\beta} = \eta_{\alpha\beta}$ and $g_{\alpha\beta,\gamma} = 0$ at $P$. Now imagine a small loop $\zeta(\lambda)$, $0 \leq \lambda \leq 1$, with $C(0) = C(1) = P$. By “small” we mean that we will take the size of the loop (linear dimension) to be $\sim \epsilon$, and we will work only to the lowest nontrivial order in $\epsilon$. In particular, in the local Lorentz frame the Christoffel symbols vanish at $P$. Then for any vector field $\zeta$:

$$\frac{D\zeta^\mu}{d\lambda} = \zeta^\mu,\nu \frac{d\zeta^\nu}{d\lambda} + \Gamma^\mu_{\beta\nu} \zeta^\beta \frac{d\zeta^\nu}{d\lambda} = \frac{d\zeta^\mu}{d\lambda} + \Gamma^\mu_{\beta\nu,\rho} \epsilon^\beta \frac{d\zeta^\nu}{d\lambda} \rho + O(\epsilon^3).$$

[The error made in the last approximation is $O(\epsilon^3)$, since the 2nd order term in the Taylor expansion of the Christoffel symbol is $O(\epsilon^2)$ and $d\zeta^\nu/d\lambda = O(\epsilon)$. Also note that $d\zeta^\nu/d\lambda$ is the derivative of a component of a vector, whereas $D\zeta^\nu/d\lambda$ is the component of the covariant derivative.] Now if $\zeta$ is to be parallel-transported around the loop, the left-hand side is zero. If we integrate around the loop, we then find

$$0 = \zeta^\mu|^{1}_{\lambda=0} + \int_{0}^{1} \Gamma^\mu_{\beta\nu,\rho} \epsilon^\beta \frac{d\zeta^\nu}{d\lambda} \rho d\lambda + O(\epsilon^3).$$

We now define the area bivector of an infinitesimal loop to be

$$\Omega^{\rho\nu} = \int_{0}^{1} \frac{d\zeta^\nu}{d\lambda} \rho d\lambda = \oint P x^\rho d\nu.$$

(I’ve called this a bivector: you can easily prove its antisymmetry using integration by parts.) This bivector is meaningful with an error of order $\epsilon^3$ – i.e. the $O(\epsilon^2)$ term, with units of infinitesimal area, is meaningful without regards to the choice of coordinate system in the vicinity of $P$, but smooth changes in the coordinate system can change $\Omega^{\rho\nu}$ at $O(\epsilon^3)$. (I haven’t been fully rigorous here: if you care about mathematical rigor, you should define a sequence of curves parameterized by $\epsilon$, and take the limit: $\lim_{\epsilon \to 0} [\zeta^\nu(\lambda = 1) - \zeta^\nu(\lambda = 0)]/\epsilon^2$.) Then Eq. (32) says

$$\zeta^\mu|^{1}_{\lambda=0} = -\Gamma^\mu_{\beta\nu,\rho} \epsilon^\beta \Omega^{\rho\nu} + O(\epsilon^3).$$

Using the antisymmetry of $\Omega$ and the fact that in the local Lorentz frame the Christoffel symbols at $P$ vanish, we see that the change in the vector $\zeta$ as it is parallel-transported around a small loop is

$$\zeta^\mu|^{1}_{\lambda=0} = -\frac{1}{2} R^\mu_{\beta\nu,\rho} \epsilon^\beta \Omega^{\rho\nu} + O(\epsilon^3).$$

Equation (35) shows that the Riemann tensor describes how a vector transported around an infinitesimal loop is modified. If the Riemann tensor in a spacetime vanishes, then this act of transportation has no effect: the vector comes back to its original position. If the Riemann tensor is nonzero, then transportation around a loop can change the direction of the vector. This is the essence of curvature of spacetime!

C. Symmetries of the Riemann tensor

Equation (23) gives us a way to compute all 256 components of the Riemann tensor. Unfortunately, this is a rather tedious process. It also obscures several of the symmetries of the Riemann tensor, which are not just computationally useful but also conceptually enlightening.

The good news is that Riemann contains many symmetries that reduce the computational burden. One of these, the antisymmetry on the last two indices, Eq. (25):

$$R^\gamma_{\mu\alpha\beta} = -R^\gamma_{\mu\beta\alpha} \quad \text{or} \quad R^\gamma_{\mu(\alpha\beta)} = 0$$

has already been identified, and reduces the number of components to $\frac{1}{2} n^3 (n - 1) = 96$. It is also intuitive in light of Eq. (35) where the Riemann tensor acts on an area bivector.

A second property, which follows from Eq. (35), is related to the fact that parallel transport does not change the square norm of a vector. [Try explicitly computing $D(g_{\alpha\beta} \zeta^\alpha \zeta^\beta) / d\lambda$.] Now let us suppose that in transporting $\zeta$...
around a small loop, it changes by \( \Delta \zeta \equiv \zeta |_{\lambda=0} \propto \epsilon^2 \). Then to order \( \epsilon^2 \), we have that there is a change in the square norm given by

\[
\Delta(\zeta \cdot \zeta) = 2\zeta \cdot \Delta \zeta = -\xi^\mu R_{\mu\beta\nu\rho} \zeta^\beta \Omega^\nu. \tag{37}
\]

This must be zero for any loop – i.e. for any bivector \( \Omega^{\nu\rho} \) – and for any vector \( \zeta \). It follows that \textbf{Riemann} is antisymmetric in the first two indices:

\[
R_{(\mu\beta)\rho\nu} = 0. \tag{38}
\]

This reduces the number of components to \( n(n-1)/2 = 36 \). It also highlights the geometrical meaning of \textbf{Riemann}, which says: if we go around a loop that has area \( \Omega \), then a vector parallel-transported around that loop is rotated by \( R_{\mu\beta\rho\nu} \Omega^{\rho\nu} \) in the \( x^\mu - x^\beta \) plane; infinitesimal rotations are described by antisymmetric rank 2 tensors.

A third property relates to the fact that the exterior derivative of a \( p \)-form does not depend on the metric structure. This is because \( (d\sigma)_{i_1 \ldots i_{p+1}} = (p+1)\nabla_{[i_1} \sigma_{i_2 \ldots i_{p+1}]} = (p+1)\partial_{[i_1} \sigma_{i_2 \ldots i_{p+1}]} \), \( \tag{39} \)

where all of the Christoffel symbol terms cancel due to their symmetry. Thus the exterior derivative is a metric-independent object! (This is really a statement that when one changes coordinate systems, the antisymmetrization in the exterior derivative gets rid of the dependences on derivatives of the Jacobian and causes the antisymmetrized partial derivative to transform as a tensor.) Then the familiar flat-spacetime rule that \( dd\sigma = 0 \) for any \( p \)-form \( \sigma \) still holds. If we write this in component language for a 1-form:

\[
0 = \nabla_{[\alpha} \nabla_{\beta}\sigma_{\gamma]} = 2R_{[\gamma|\mu|\alpha\beta]}\sigma^\mu. \tag{40}
\]

(Here the vertical bars indicate that the \( \mu \) index is exempt from the antisymmetrization.) This is true for any \( \sigma \), so we find that \( R_{[\gamma|\mu|\alpha\beta]} = 0 \) or – using Eq. (38) –

\[
R_{\mu[\gamma\alpha\beta]} = 0. \tag{41}
\]

Before we consider how many independent components are left, it is useful to find another symmetry that is a consequence of Eqs. (36,38,41). Let’s consider the tensor defined by \( K_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - R_{\gamma\delta\alpha\beta} \). Then

\[
K_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - R_{\gamma\delta\alpha\beta} = R_{\alpha\beta\gamma\delta} + R_{\gamma\delta\alpha\beta} = -R_{\alpha\beta\gamma\delta} - R_{\alpha\gamma\delta\beta} + R_{\gamma\delta\alpha\beta} = -R_{\alpha\beta\gamma\delta} - R_{\alpha\gamma\delta\beta} + R_{\gamma\delta\alpha\beta} = -K_{\alpha\beta\gamma\delta}. \tag{42}
\]

If we repeat this argument 3 times, we find

\[
K_{\alpha\beta\gamma\delta} = -K_{\alpha\beta\delta\gamma} = K_{\alpha\gamma\delta\beta} = -K_{\alpha\beta\gamma\delta}, \tag{43}
\]

so we must have \( K = 0 \). This then implies that the Riemann tensor has the symmetry:

\[
R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}. \tag{44}
\]

This symmetry has a simple geometrical interpretation if one thinks in a local Lorentz frame. The \( \alpha\beta \) rotation of a basis transported around a loop in the \( \gamma\delta \) plane is equal to the \( \gamma\delta \) rotation of a basis transported around a loop in the \( \alpha\beta \) plane.

This set of symmetries can be shown to reduce the Riemann tensor to only 20 independent components.

\[\text{D. Covariant derivatives of the Riemann tensor}\]

The covariant derivative of \textbf{Riemann} is a rank 5 tensor with 1024 components. Nevertheless it plays a central role in gravitation theory.
Let us consider some point \( P \); what can be said about \( \nabla R \) without any fancy calculations? If we work in a local Lorentz frame, where the Christoffel symbols vanish at \( P \), we see that

\[
R^\gamma_{\mu\alpha\beta\zeta} = \Gamma^\gamma_{\mu\beta,\alpha\zeta} - \Gamma^\gamma_{\mu\alpha,\beta\zeta}.
\] (45)

It follows immediately that if we add up all 3 even permutations of \( \alpha\beta\zeta \) this vanishes due to a cancellation on the right-hand side. So:

\[
R^\gamma_{\mu[\alpha\beta\zeta]} = 0.
\] (46)

This result applies to a tensor, and hence is valid in all coordinate systems. It is known as the **Bianchi identity**.

What geometric sense shall we make of the Bianchi identity? MTW gives a geometrical picture. Remember that \( \epsilon^2 R^\gamma_{\mu\alpha\beta\zeta} \) represents the amount of rotation in the \( \gamma\mu \)-plane generated by circumnavigating a loop of area \( \epsilon^2 \) in the \( \alpha\beta\zeta \) hyperplane. We perform a loop around each of the 6 faces of the cube; this set of loops traverses each of the 12 edges of the cube twice, in opposite directions. Then \( \epsilon^3 R^\gamma_{\mu[\alpha\beta\zeta]} \) represents the amount of rotation in the \( \gamma\mu \)-plane resulting from traversing all of these loops. Since each edge is traversed a net number of zero times, we would expect this to give zero – and it does. The rigorous proof by this method, however, is much longer than the above.

**IV. GEODESIC DEVIATION**

We are now in a position to ask what happens to neighboring geodesics. If we take two freely falling test particles, place them next to each other, and let them go, do they fall together? Do they fly apart? This question is intimately related to the subject of tidal fields, familiar from Newtonian gravity.

To begin, let us consider a geodesic \( P(\lambda) \), and consider a neighboring geodesic separated from it by an infinitesimal displacement vector \( \xi(\lambda) \). We want to know how \( \xi \) behaves as we move along the geodesic. To make the notion of “infinitesimal displacement” more precise, we consider a family of geodesics parameterized by some parameter \( n \in \mathbb{R} \), \( P(\lambda, n) \), such that our fiducial geodesic is \( P(\lambda, 0) \) and the infinitesimal displacement is

\[
\xi = \frac{\partial P(\lambda, n)}{\partial n}.
\] (47)

The fact that these curves are geodesics is captured by taking their 4-velocity \( v = \partial P(\lambda, n)/\partial \lambda \) and writing

\[
\frac{Dv}{\partial \lambda} = \nabla_v v = 0.
\] (48)

We will need one more fact about the vector fields \( v \) and \( \xi \): that they commute. This is easy to show since once one can build a coordinate system in which the trajectories of the particles lie in the \((x^0, x^1)\)-plane, and we assign coordinates \((\lambda, n, 0, \ldots, 0)\) to the point \( P(\lambda, n) \). Then in this system \( v^\mu = (1, 0, \ldots, 0) \) and \( \xi^\mu = (0, 1, 0, \ldots, 0) \), so trivially \([v, \xi] = 0\).

Our ambition is to learn something about how \( \xi \) varies with \( \lambda \). A natural thing to do is to take its derivative \( D\xi/\partial \lambda \). But if Newtonian gravity is a guide, the motion of infinitesimally separated test particles should be described by a 2nd order ODE, so let’s take another derivative:

\[
\frac{D^2 \xi^\mu}{\partial \lambda^2} = \nabla_v \nabla_v \xi^\gamma = \nabla_v \nabla_v \xi^\gamma = \nabla_v \nabla_v \xi^\gamma + \nabla_{[v, \xi]} v^\gamma + R^\gamma_{\mu\alpha\beta\zeta} v^\alpha \xi^\beta v^\zeta = R^\gamma_{\mu\alpha\beta\zeta} v^\alpha \xi^\beta v^\zeta.
\] (49)

Here the second line arises from commutativity, the third from the switching of order of covariant derivatives, and the fourth again from commutativity and the geodesic relation \( \nabla_v v = 0 \).

The upshot is that two nearby test particles appear to move relative to each other according to the Riemann tensor. Therefore it is really a description of tidal fields.