Black holes on supercomputers: Numerical relativity applications to astrophysics and high-energy physics

U. Sperhake

DAMTP, University of Cambridge

53 Cracow School of Theoretical Physics
1^{st} July 2013

Overview

- Introduction, motivation
- Foundations of numerical relativity
  - Formulations of Einstein's eqs.: 3+1, BSSN, GHG, characteristic
  - Beyond $4D$: Reducing dimensionality
  - Initial data, Gauge, Boundaries
  - Technical ingredients: Discretization, mesh refinement,...
  - Diagnostics: Horizons, Momenta, GWs
- Applications and Results of NR
  - Astrophysics
  - Gravitational wave physics
  - High-energy physics
  - Fundamental properties of BHs

1. Introduction, motivation

The Schwarzschild solution

- Einstein 1915

  General relativity: geometric theory of gravity

- Schwarzschild 1916

  $$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

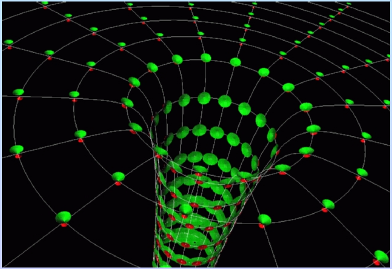
- Singularities:

  $r = 0$: physical

  $r = 2M$: coordinate

- Newtonian escape velocity

  $$v = \sqrt{\frac{2M}{r}}$$



Evidence for astrophysical black holes

- X-ray binaries

  e. g. Cygnus X-1 (1964)

  MS star + compact star

  $\Rightarrow$ Stellar Mass BHs
  $\sim 5 \dots 50\ M_\odot$

- Stellar dynamics

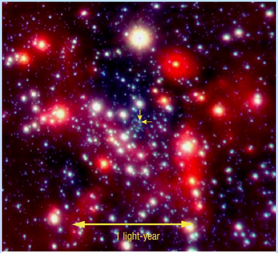
  near galactic centers,

  iron emission line profiles

  $\Rightarrow$ Supermassive BHs

  $\sim 10^6 \dots 10^9\ M_\odot$

  AGN engines



The Centre of the Milky Way
(VLT YEPUN + NACO)
ESO PR Photo 23a/02 (9 October 2002) ©European Southern Observatory

Conjectured BHs

- Intermediate mass BHs

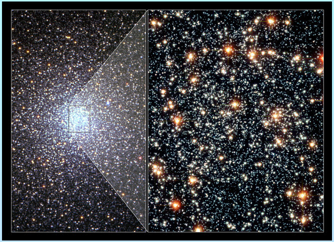
  $\sim 10^2 \ldots 10^5\ M_\odot$

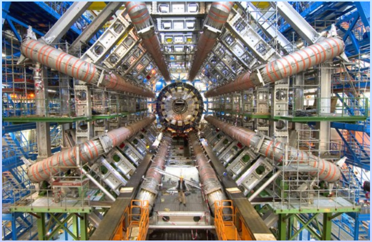
- Primordial BHs

  $\leq M_{Earth}$

- Mini BHs, LHC

  $\sim TeV$



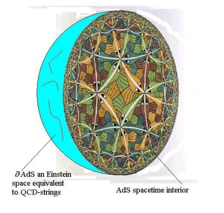


Note: BH solution is scale invariant!

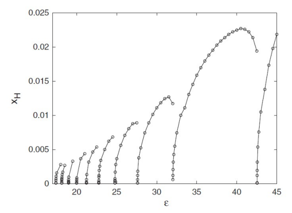
Research areas: Black holes have come a long way!

Astrophysics

Gauge-gravity duality

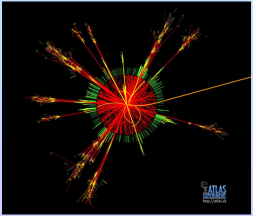


Fundamental studies

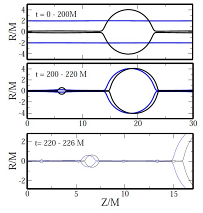


GW physics

High-energy physics



Fluid analogies



General Relativity: Curvature

- Curvature generates acceleration

  "geodesic deviation"

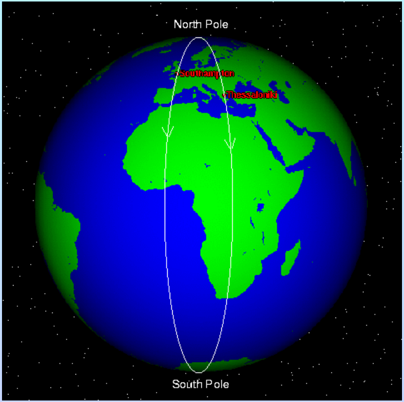
  No "force"!!

- Description of geometry

  Metric $g_{\alpha\beta}$

  Connection $\Gamma^{\alpha}_{\beta\gamma}$

  Riemann Tensor $R^{\alpha}{}_{\beta\gamma\delta}$



The metric defines everything

- Christoffel connection

  $\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu} \left(\partial_\beta g_{\gamma\mu} + \partial_\gamma g_{\mu\beta} - \partial_\mu g_{\beta\gamma} \right)$

- Covariant derivative

  $\nabla_\alpha T^{\beta}{}_{\gamma} = \partial_\alpha T^{\beta}{}_{\gamma} + \Gamma^{\beta}_{\mu\alpha} T^{\mu}{}_{\gamma} - \Gamma^{\mu}_{\gamma\alpha} T^{\beta}{}_{\mu}$

- Riemann Tensor

  $R^{\alpha}{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}_{\beta\delta} - \partial_\delta \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\mu\gamma} \Gamma^{\mu}_{\beta\delta} - \Gamma^{\alpha}_{\mu\delta} \Gamma^{\mu}_{\beta\gamma}$

- $\Rightarrow$ Geodesic deviation,

  Parallel transport,

  ...

How to get the metric?

Train cemetery
Uyuni, Bolivia

- Solve for the metric $g_{\alpha\beta}$

How to get the metric?

- The metric must obey the Einstein Equations
- Ricci-Tensor, Einstein Tensor, Matter Tensor

  $R_{\alpha\beta} \equiv R^{\mu}{}_{\alpha\mu\beta}$

  $G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R^{\mu}{}_{\mu}$ "Trace reversed" Ricci

  $T_{\alpha\beta}$ "Matter"

- Einstein Equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$
- Solutions: Easy! Take metric

  $\Rightarrow$ Calculate $G_{\alpha\beta}$

  $\Rightarrow$ Use that as matter tensor

- Physically meaningful solutions: Difficult!

Solving Einstein's equations: Different methods

- Analytic solutions
  - Symmetry assumptions
  - Schwarzschild, Kerr, FLRW, Myers-Perry, Emparan-Reall,...
- Perturbation theory
  - Assume solution is close to known solution $g_{\alpha\beta}$
  - Expand $\hat{g}_{\alpha\beta} = g_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \ldots \;\Rightarrow$ linear system
  - Regge-Wheeler-Zerilli-Moncrief, Teukolsky, QNMs, EOB,...
- Post-Newtonian Theory
  - Assume small velocities $\Rightarrow$ expansion in $\frac{v}{c}$
  - N^{th} order expressions for GWs, momenta, orbits,...
  - Blanchet, Buonanno, Damour, Kidder, Will,...
- Numerical Relativity

2. Foundations of numerical relativity

A list of tasks

- Target: Predict time evolution of BBH in GR
- Einstein equations: 1) Cast as evolution system
  2) Choose specific formulation
  3) Discretize for computer
- Choose coordinate conditions: Gauge
- Fix technical aspects: 1) Mesh refinement / spectral domains
  2) Singularity handling / excision
  3) Parallelization
- Construct realistic initial data
- Start evolution and waaaaiiiiit...
- Extract physics from the data

2.1 Formulations of Einstein's equations

The Einstein equations

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

$$\Leftrightarrow \quad R_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \tfrac{1}{D-2} T g_{\mu\nu} \right) + \tfrac{2}{D-2} \Lambda g_{\mu\nu}$$

- In this form no well-defined mathematical character
  hyperbolic, elliptic, parabolic?
- Coordinate x^α on equal footing; time only through signature of $g_{\alpha\beta}$
- Well-posedness of the equations? Suitable for numerics?
- Several ways to identify character and coordinates
  $\rightarrow$ Formulations

2.1.1 ADM like $D-1+1$ formulations

3+1 Decomposition

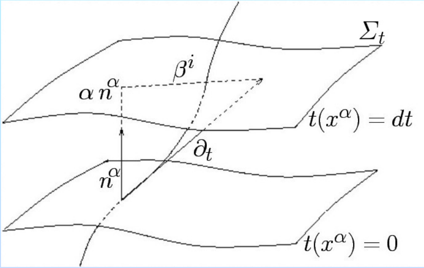
- NR: ADM 3+1 split   Arnowitt, Deser & Misner '62
  York '79, Choquet-Bruhat & York '80
  - Spacetime = Manifold $(\mathcal{M}, g)$
  - Hypersurfaces

    Scalar field $t : \mathcal{M} \to \mathbb{R}$

    such that $t = \text{const}$ defines Σ_t

    $\to$ 1 form $\boldsymbol{d}t$,   vector ∂_t

    $\langle \boldsymbol{d}t, \partial_t \rangle = 1$



- **Def.:** Timelike unit vector: $n_\mu \equiv -\alpha(\boldsymbol{d}t)_\mu$

  Lapse: $\alpha = 1/||\boldsymbol{d}t||$ Shift: $\beta^\mu = (\partial_t)^\mu - \alpha n^\mu$

  Adapted coordinate basis: $\partial_t = \alpha n + \beta, \quad \partial_i = \frac{\partial}{\partial_x^i}$

3+1 Decomposition

Def.: A vector v^α is tangent to Σ_t $\quad :\Leftrightarrow \quad \langle \mathbf{d}t, v\rangle = (\boldsymbol{d}t)_\mu v^\mu = 0$

Projector: $\perp^\alpha{}_\mu = \delta^\alpha{}_\mu + n^\alpha n_\mu$

For a vector tangent to Σ_t one easily shows

- $n_\mu v^\mu = 0$
- $\perp^\alpha{}_\mu v^\mu = v^\alpha$

Projection of the metric

- $\gamma_{\alpha\beta} := \perp^\mu{}_\alpha \perp^\nu{}_\beta g_{\mu\nu} = g_{\alpha\beta} + n_\alpha n_\beta \Rightarrow \gamma_{\alpha\beta} = \perp_{\alpha\beta}$
- For v^α tangent to Σ_t: $\quad g_{\mu\nu} v^\mu v^\nu = \gamma_{\mu\nu} v^\mu v^\nu$

Adapted coordinates: $x^\alpha = (t,\ x^i)$

$\Rightarrow$ we can ignore t components for tensors tangential to Σ_t

$\Rightarrow$ γ_{ij} is the metric on Σ_t First fundamental form

3+1 decomposition of the metric

In adapted coordinates, we write the spacetime metric

$$g_{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^2 + \beta_m \beta^m & \beta_j \\ \hline \beta_i & \gamma_{ij} \end{array}\right)$$

$$\Leftrightarrow \qquad g^{\alpha\beta} = \left(\begin{array}{c|c} -\alpha^{-2} & \alpha^{-2}\beta^j \\ \hline \alpha^{-2}\beta^i & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{array}\right)$$

$$\Leftrightarrow \; ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

Gauge variables: Lapse α, Shift vector β^i

For any tensor tangent in all components to Σ_t we raise and lower indices with γ_{ij}:

$S^{ij}{}_k = \gamma^{jm} S^i{}_{mk}$ etc.

Projections and spatial covariant derivative

- For an arbitrary tensor S of type $\begin{pmatrix} p \\ q \end{pmatrix}$, its projection is

  $(\perp S)^{\alpha_1 \ldots \alpha_p}{}_{\beta_1 \ldots \beta_q} = \perp^{\alpha_1}{}_{\mu_1} \ldots \perp^{\alpha_p}{}_{\mu_p} \perp^{\nu_1}{}_{\beta_1} \ldots \perp^{\nu_q}{}_{\beta_q} S^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q}$

  "Project every free index"

- For a tensor S on Σ_t, its covariant derivative is $DS := \perp(\nabla S)$

  $D_\rho S^{\alpha_1 \ldots \alpha_p}{}_{\beta_1 \ldots \beta_q} = \perp^{\alpha_1}{}_{\mu_1} \ldots \perp^{\alpha_p}{}_{\mu_p} \perp^{\nu_1}{}_{\beta_1} \ldots \perp^{\nu_q}{}_{\beta_q} \perp^{\sigma}{}_{\rho} \nabla_\sigma S^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q}$

- One can show that
  - $D = \perp\nabla$ is torsion free on Σ_t if ∇ is on $\mathcal{M}$
  - $(\perp\nabla\gamma)_{ijk} = 0$ metric compatible
  - $\perp\nabla$ is unique in satisfying these properties

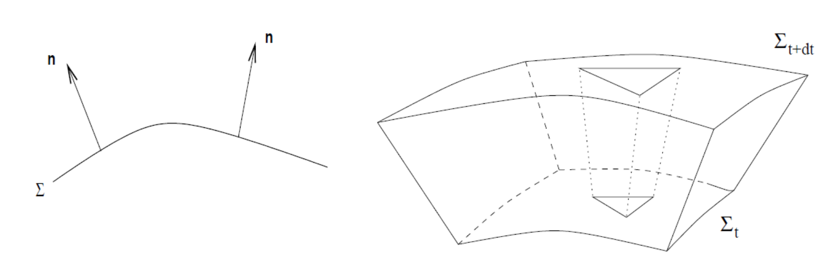
Extrinsic curvature

Def.: $K_{\alpha\beta} = -\perp\nabla_\beta n_\alpha$

- $\nabla_\beta n_\alpha$ is not symmetric, but $\perp\nabla_\beta n_\alpha$ and, thus, $K_{\alpha\beta}$ is!
- One can show that

  $\mathcal{L}_n\gamma_{\alpha\beta} = n^\mu\nabla_\mu\gamma_{\alpha\beta} + \gamma_{\mu\beta}\nabla_\alpha n^\mu + \gamma_{\alpha\mu}\nabla_\beta n^\mu = -2K_{\alpha\beta}$

  $K_{\alpha\beta} = -\frac{1}{2}\mathcal{L}_n\gamma_{\alpha\beta}$
- Two interpretations of $K_{\alpha\beta}$ → embedding of Σ_t in $\mathcal{M}$



The projections of the Riemann tensor

$\perp^{\mu}{}_{\alpha}\perp^{\nu}{}_{\beta}\perp^{\gamma}{}_{\rho}\perp^{\sigma}{}_{\delta}\ R^{\rho}{}_{\sigma\mu\nu} = \mathcal{R}^{\gamma}{}_{\delta\alpha\beta} + K^{\gamma}{}_{\alpha}K_{\delta\beta} - K^{\gamma}{}_{\beta}K_{\delta\alpha}$ Gauss Eq.

$\perp^{\mu}{}_{\alpha}\perp^{\nu}{}_{\beta}\ R_{\mu\nu} + \perp_{\mu\alpha}\perp^{\nu}{}_{\beta}n^{\rho}n^{\sigma}\ R^{\mu}{}_{\rho\nu\sigma} = \mathcal{R}_{\alpha\beta} + KK_{\alpha\beta} - K^{\mu}{}_{\beta}K_{\alpha\mu}$ contracted

$R + 2\ R_{\mu\nu}n^{\mu}n^{\nu} = \mathcal{R} + K^2 - K^{\mu\nu}K_{\mu\nu}$ scalar Gauss eq.

$\perp^{\gamma}{}_{\rho}n^{\sigma}\perp^{\mu}{}_{\alpha}\perp^{\nu}{}_{\beta}\ R^{\rho}{}_{\sigma\mu\nu} = D_{\beta}K^{\gamma}{}_{\alpha} - D_{\alpha}K^{\gamma}{}_{\beta}$ Codazzi eq.

$n^{\sigma}\perp^{\nu}{}_{\beta}\ R_{\sigma\nu} = D_{\beta}K - D_{\mu}K^{\mu}{}_{\beta}$ contracted

$\perp_{\alpha\mu}\perp^{\nu}{}_{\beta}n^{\sigma}n^{\rho}\ R^{\mu}{}_{\rho\nu\sigma} = \frac{1}{\alpha}\mathcal{L}_m K_{\alpha\beta} + K_{\alpha\mu}K^{\mu}{}_{\beta} + \frac{1}{\alpha}D_{\alpha}D_{\beta}\alpha$

$\perp^{\mu}{}_{\alpha}\perp^{\nu}{}_{\beta}\ R_{\mu\nu} = -\frac{1}{\alpha}\mathcal{L}_m K_{\alpha\beta} - 2K_{\alpha\mu}K^{\mu}{}_{\beta} - \frac{1}{\alpha}D_{\alpha}D_{\beta}\alpha + \mathcal{R}_{\alpha\beta} + KK_{\alpha\beta}$

$R = -\frac{2}{\alpha}\mathcal{L}_m K - \frac{2}{\alpha}\gamma^{\mu\nu}D_{\mu}D_{\nu}\alpha + \mathcal{R} + K^2 + K^{\mu\nu}K_{\mu\nu}$

- Here $\mathcal{L}$ is the Lie derivative and $m^{\mu} = \alpha n^{\mu} = (\partial_t)^{\mu} + \beta^{\mu}$
- Summation of spatial tensors: ignore time indices; $\mu,\ \nu, \ldots \to m,\ n, \ldots$

Decomposition of the Einstein equations

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}$$

$$\Leftrightarrow R_{\alpha\beta} = 8\pi \left(T_{\alpha\beta} - \frac{1}{D-2} g_{\alpha\beta} T \right) + \frac{2}{D-2} \Lambda g_{\alpha\beta}$$

Energy momentum tensor

- $\rho = T_{\mu\nu} n^{\mu} n^{\nu}$ energy density

  $j_{\alpha} = -T_{\mu\nu} n^{\mu} \perp^{\nu}{}_{\alpha}$ momentum density

  $S_{\alpha\beta} = \perp^{\mu}{}_{\alpha} \perp^{\nu}{}_{\beta} T_{\mu\nu}, \quad S = \gamma^{\mu\nu} S_{\mu\nu}$ stress tensor

- $T_{\alpha\beta} = S_{\alpha\beta} + n_{\alpha} j_{\beta} + n_{\beta} j_{\alpha} + \rho n_{\alpha} n_{\beta}, \quad T = S - \rho$

Lie derivative $\mathcal{L}_m = \mathcal{L}_{(\partial_t - \beta)}$

- $\mathcal{L}_m K_{ij} = \partial_t K_{ij} - \beta^m \partial_m K_{ij} - K_{mj} \partial_i \beta^m - K_{im} \partial_j \beta^m$

  $\mathcal{L}_m \gamma_{ij} = \partial_t \gamma_{ij} - \beta^m \partial_m \gamma_{ij} - \gamma_{mj} \partial_i \beta^m - \gamma_{im} \partial_j \beta^m$

Decomposition of the Einstein equations

Definition:

$\mathcal{L}_m \gamma_{ij} = -2\alpha K_{ij}$

$\perp^\mu{}_\alpha \perp^\nu{}_\beta$ projection:

$\mathcal{L}_m K_{ij} = -D_i D_j \alpha + \alpha(\mathcal{R}_{ij} + K K_{ij} - 2 K_{im} K^m{}_j) + 8\pi\alpha \left[\frac{S-\rho}{D-2}\gamma_{ij} - S_{ij}\right] - \frac{2}{D-2}\Lambda\gamma_{ij}$

Evolution equations

$n^\mu n^\nu$ projection

$\mathcal{R} + K^2 - K^{mn} K_{mn} = 2\Lambda + 16\pi\rho$ Hamiltonian constraint

$\perp^\mu{}_\alpha n^\nu$ projection

$D_i K - D_m K^m{}_i = -8\pi j_i$ Momentum constraint

Well-posedness

- Consider a field ϕ evolved with a first-order system of PDEs
- The system has a well posed initial value formulation

  $\Leftrightarrow$ There exists some norm and a smooth function

  $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\quad ||\phi(t)|| \leq F(||\phi(0)||, t)\ ||\phi(0)||$
- Well-posed systems have unique solutions for given initial data
- There can still be fast growth, e.g. exponential
- Strong hyperbolicity is necessary for well-posedness
- The general ADM equations are only weakly hyperbolic
- Details depend on: gauge, constraints, discretization

Sarbach & Tiglio, Living Reviews Relativity **15** (2012) 9; Gundlach & Martín-García, PRD **74** (2006) 024016; Reula, gr-qc/0403007

The BSSN system

- Goal: modify ADM to get a strongly hyperbolic system

  Baumgarte & Shapiro, PRD **59** (1998) 024007, Shibata & Nakamura, PRD **52** (1995) 5428

- Conformal decomposition, trace split, auxiliary variable

  $\phi = \frac{1}{4(D-1)} \ln \gamma, \quad K = \gamma^{ij} K_{ij}$

  $\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij} \quad \Leftrightarrow \quad \tilde{\gamma}^{ij} = e^{4\phi}\gamma^{ij}$

  $\tilde{A}_{ij} = e^{-4\phi}\left(K_{ij} - \frac{1}{D-1}\gamma_{ij}K\right) \quad \Leftrightarrow \quad K_{ij} = e^{4\phi}\left(\tilde{A}_{ij} + \frac{1}{D-1}\tilde{\gamma}_{ij}K\right)$

  $\tilde{\Gamma}^i = \tilde{\gamma}^{mn}\tilde{\Gamma}^i_{mn}$

- Auxiliary constraints

  $\tilde{\gamma} = \det \tilde{\gamma}_{ij} = 1, \quad \tilde{\gamma}^{mn}\tilde{A}_{mn} = 0$

The BSSN equations

$$\partial_t \phi = \beta^m \partial_m \phi + \frac{1}{2(D-1)}(\partial_m \beta^m - \alpha K)$$

$$\partial_t \tilde{\gamma}_{ij} = \beta^m \partial_m \tilde{\gamma}_{ij} + 2\tilde{\gamma}_{m(i}\partial_{j)}\beta^m - \frac{2}{D-1}\tilde{\gamma}_{ij}\partial_m \beta^m - 2\alpha \tilde{A}_{ij}$$

$$\partial_t K = \beta^m \partial_m K - e^{-4\phi}\tilde{\gamma}^{mn} D_m D_n \alpha + \alpha \tilde{A}^{mn}\tilde{A}_{mn} + \frac{1}{D-1}\alpha K^2$$
$$+ \frac{8\pi}{D-2}\alpha[S + (D-3)\rho] - \frac{2}{D-2}\alpha\Lambda$$

$$\partial_t \tilde{A}_{ij} = \beta^m \partial_m \tilde{A}_{ij} + 2\tilde{A}_{m(i}\partial_{j)}\beta^m - \frac{2}{D-1}\tilde{A}_{ij}\partial_m \beta^m + \alpha K \tilde{A}_{ij} - 2\alpha \tilde{A}_{im}\tilde{A}^m{}_j$$
$$+ e^{-4\phi}\left(\alpha \mathcal{R}_{ij} - D_i D_j \alpha - 8\pi\alpha S_{ij}\right)^{\mathrm{TF}}$$

$$\partial_t \tilde{\Gamma}^i = \beta^m \partial_m \tilde{\Gamma}^i + \frac{2}{D-1}\tilde{\Gamma}^i \partial_m \beta^m + \tilde{\gamma}^{mn}\partial_m \partial_n \beta^i + \frac{D-3}{D-1}\tilde{\gamma}^{im}\partial_m \partial_n \beta^n$$
$$+ 2\tilde{A}^{im}[2(D-1)\alpha \partial_m \phi - \partial_m \alpha] + 2\alpha \tilde{\Gamma}^i{}_{mn}\tilde{A}^{mn} - 2\frac{D-2}{D-1}\alpha \tilde{\gamma}^{im}\partial_m K - 16\pi\alpha j^i$$

Note: There are alternative versions using $\chi = e^{-4\phi}$ or $W = e^{-2\phi}$

The BSSN equations

In the BSSN equations we use

$$\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk} + 2(\delta^i{}_k \partial_j \phi + \delta^i{}_j \partial_k \phi - \tilde{\gamma}_{jk} \tilde{\gamma}^{im} \partial_m \phi)$$

$$\mathcal{R}_{ij} = \tilde{\mathcal{R}}_{ij} + \mathcal{R}^{\phi}_{ij}$$

$$\mathcal{R}^{\phi}_{ij} = 2(3-D) \tilde{D}_i \tilde{D}_j \phi - 2 \tilde{\gamma}_{ij} \tilde{\gamma}^{mn} \tilde{D}_m \tilde{D}_n \phi + 4(D-3)(\partial_i \phi \, \partial_j \phi - \tilde{\gamma}_{ij} \tilde{\gamma}^{mn} \partial_m \phi \, \partial_n \phi)$$

$$\tilde{\mathcal{R}}_{ij} = -\frac{1}{2} \tilde{\gamma}^{mn} \partial_m \partial_n \tilde{\gamma}_{ij} + \tilde{\gamma}_{m(i} \partial_{j)} \tilde{\Gamma}^m + \tilde{\Gamma}^m \tilde{\Gamma}_{(ij)m} + \tilde{\gamma}^{mn} [2 \tilde{\Gamma}^k_{m(i} \tilde{\Gamma}_{j)kn} + \tilde{\Gamma}^k_{im} \tilde{\Gamma}_{kjn}]$$

$$D_i D_j \alpha = \tilde{D}_i \tilde{D}_j \alpha - 2(\partial_i \phi \, \partial_j \alpha + \partial_j \phi \, \partial_i \alpha) + 2 \tilde{\gamma}_{ij} \tilde{\gamma}^{mn} \partial_m \phi \, \partial_n \alpha$$

The constraints are

$$\mathcal{H} = \mathcal{R} + \frac{D-2}{D-1} K^2 - \tilde{A}^{mn} \tilde{A}_{mn} - 16\pi\rho - 2\Lambda = 0$$

$$\mathcal{M}_i = \tilde{D}_m \tilde{A}^m{}_i - \frac{D-2}{D-1} \partial_i K + 2(D-1) \tilde{A}^m{}_i \partial_m \phi - 8\pi j_i = 0$$

2.1.2 Generalized Harmonic formulation

The Generalized Harmonic (GH) formulation

- Harmonic gauge: choose coordinates such that

  $\Box x^\alpha = \nabla_\mu \nabla^\mu x^\alpha = -g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = 0$

- 4-dim. version of Einstein equations

  $R_{\alpha\beta} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta} + \ldots$

  Principal part of wave equation $\Rightarrow$ Manifestly hyperbolic

- Problem: Start with spatial hypersurface $t = \mathrm{const.}$

  Does t remain timelike?

- Solution: Generalize harmonic gauge

  Garfinkle, APS Meeting (2002) 12004, Pretorius, CQG **22** (2005) 425, Lindblom et al, CQG **23** (2006) S447

  $\rightarrow$ Source functions $H^\alpha = \nabla_\mu \nabla^\mu x^\alpha = -g^{\mu\nu}\Gamma^\alpha_{\mu\nu}$

The Generalized harmonic formulation

- Any spacetime in any coordinates can be formulated in GH form!
  Problem: find the corresponding H^α
- Promote H^α to evolution variables
- Einstein field equations in GH form:

$$\tfrac{1}{2} g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} = -\partial_\nu g_{\mu(\alpha}\,\partial_{\beta)} g^{\mu\nu} - \partial_{(\alpha} H_{\beta)} + H_\mu \Gamma^\mu_{\alpha\beta}$$

$$-\Gamma^\mu_{\nu\alpha}\Gamma^\nu_{\mu\beta} - \tfrac{2}{D-2}\Lambda g_{\alpha\beta} - 8\pi\left(T_{\mu\nu} - \tfrac{1}{D-2} T g_{\alpha\beta}\right)$$

  with constraints

$$\mathcal{C}^\alpha = H^\alpha - \Box x^\alpha = 0$$

- Still principal part of wave equation !!! Manifestly hyperbolic

  Friedrich, Comm.Math.Phys. **100** (1985) 525, Garfinkle, PRD **65** (2002) 044029, Pretorius, CQG **22** (2005) 425

Constraint damping in the GH system

- One can show that

$$\mathcal{C}^{\alpha}|_{t=0} = 0, \ \ \partial_t \mathcal{C}^{\alpha}|_{t=0} = 0 \ \Leftrightarrow \ \text{The ADM } \mathcal{H} = 0, \ \ \mathcal{M}_i = 0$$

- Bianchi identities imply evolution of $\mathcal{C}^{\alpha}$:

$$\Box \mathcal{C}_{\alpha} = -\mathcal{C}^{\mu} \nabla_{(\mu} \mathcal{C}_{\alpha)} - \mathcal{C}^{\mu} \left[8\pi \left(T_{\mu\alpha} - \frac{1}{D-2} T g_{\mu\alpha} \right) + \frac{2}{D-2} \Lambda g_{\mu\alpha} \right]$$

- In practice: numerical violations of $\mathcal{C}^{\mu} = 0 \ \Rightarrow$ unstable modes

- Solution: add constraint damping

$$\frac{1}{2} g^{\mu\nu} \partial_{\mu} \partial_{\nu} g_{\alpha\beta} = -\partial_{\nu} g_{\mu(\alpha} \, \partial_{\beta)} g^{\mu\nu} - \partial_{(\alpha} H_{\beta)} + H_{\mu} \Gamma^{\mu}_{\alpha\beta} - \Gamma^{\mu}_{\nu\alpha} \Gamma^{\nu}_{\mu\beta}$$
$$- \frac{2}{D-2} \Lambda g_{\alpha\beta} - 8\pi \left(T_{\mu\nu} - \frac{1}{D-2} T g_{\alpha\beta} \right) - \kappa \left[2 n_{(\alpha} \mathcal{C}_{\beta)} - \lambda g_{\alpha\beta} n^{\mu} \mathcal{C}_{\mu} \right]$$

  Gundlach et al, CQG **22** (2005) 3767

- E.g. Pretorius, PRL **95** (2005) 121101: $\kappa = 1.25/m, \ \ \lambda = 1$

Summary GH formulation

- Specify initial data $g_{\alpha\beta}, \ \partial_t g_{\alpha\beta}$ at $t=0$

  which satisfy the constraints $\mathcal{C}^\mu = \partial_t \mathcal{C}^\mu = 0$
- Constraints preserved due to Bianchi identities
- Alternative first-order version of GH formulation

  Lindblom et al, CQG **23** (2006) S447

  - Auxiliary variables $\rightarrow$ First-order system
  - Symmetric hyperbolic system

    $\rightarrow$ constraint-preserving boundary conditions
  - Used for spectral BH code SpEC

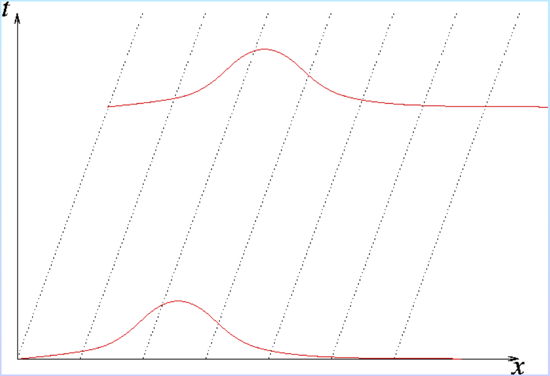
    Caltech, Cornell, CITA

2.1.3 Characteristic formulation

Characteristic coordinates

- Consider advection equation $\partial_t f + a \partial_x f = 0$
- Characteristics: curves $\mathcal{C} : x \to at + x_0 \;\Leftrightarrow\; \frac{dx}{dt} = a$

$\frac{df}{dt}|_{\mathcal{C}} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt}|_{\mathcal{C}} = \frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} = 0 \;\Rightarrow\;$ f constant along $\mathcal{C}$



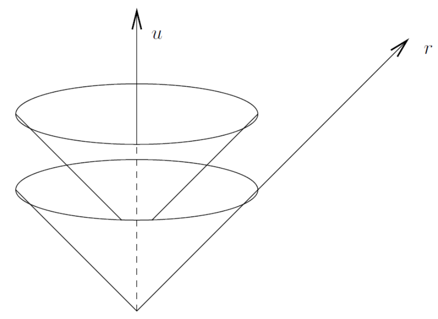
Characteristic “Bondi-Sachs” formulation

Here: $D = 4, \quad \Lambda = 0$

- Foliate spacetime using characteristic surfaces; light cones

  Bondi, Proc.Roy.Soc.A **269** (1962), 21; Sachs, Proc.Roy.Soc.A **270** (1962), 103

- “$u = t - r, \quad v = t + r$” $\to$ double null, ingoing or outgoing



outgoing null timelike foliation

Characteristic “Bondi-Sachs” formulation

- Write metric as

$$ds^2 = V\frac{e^{2\beta}}{r}du^2 - 2e^{2\beta}dudr + r^2 h_{AB}(dx^A - U^A du)(dx^B - U^B du)$$

$$2h_{AB}dx^A dx^B = (e^{2\gamma} + e^{2\delta})d\theta^2 + 4\sin\theta\ \sinh(\gamma-\delta)d\theta d\phi + \sin^2\theta(e^{-2\gamma} + e^{-2\delta})d\phi^2$$

- Introduce tetrad $k,\ \ell,\ m,\ \bar{m}$ such that

  $g(k,\ell) = 1,\ \ g(m,\bar{m}) = 1$ and all other products vanish

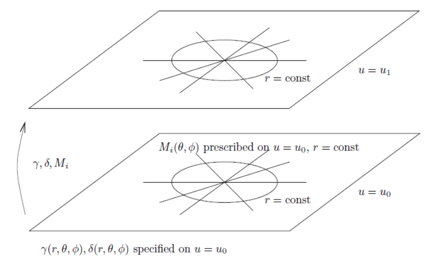
- The Einstein equations become
  - 4 hypersurface eqs.: $R_{\mu\nu}k^\mu k^\nu = R_{\mu\nu}k^\mu m^\nu = R_{\mu\nu}m^\mu \bar{m}^\nu = 0$
  - 2 evolution eqs.: $R_{\mu\nu}m^\mu m^\nu = 0$
  - 1 trivial eq.: $R_{\mu\nu}k^\mu \ell^\nu = 0$
  - 3 supplementary eqs.: $R_{\mu\nu}\ell^\mu m^\nu = R_{\mu\nu}\ell^\mu \ell^\nu = 0$

Integration of the characteristic equations

- Provide initial data for $\gamma,\ \delta$ on hypersurface $u = \text{const}$
- Integrate hypersurface eqs. along $r \ \rightarrow\ \beta,\ V, U^A$ at u

  $\rightarrow$ 3 "constants" of integration $M_i(\theta,\phi)$
- Evolve $\gamma,\ \delta$ using evolution eqs.

  $\rightarrow$ 2 "constants" of integration $\rightarrow$ complex news $\partial_u c(u,\theta,\phi)$
- Evolve the M_i through the supplementary eqs.



Summary characteristic formulation

- Naturally adapted to the causal structure of GR
- Clear hierarchy of equations $\rightarrow$ isolated degrees of freedom
- Problem: caustics $\rightarrow$ breakdown of coordinates
- Well suited for symmetric spacetimes, planar BHs
- Solution for binary problem?

  Recent investigation: Babiuc, Kreiss & Winicour, arXiv:1305.7179 [gr-qc]
- Application to characteristic GW extraction

  Babiuc, Winicour & Zlochower, CQG **28** (2011) 134006

  Reisswig et al, CQG **27** (2010) 075014

Direct methods

- Use symmetry to write line element, e.g.

  $ds^2 = -a^2(\mu, t)dt^2 + b^2(\mu, t)d\mu^2 - R^2(\mu, t)d\Omega^2$

  May & White, PR **141** (1966) 1232

- Energy momentum tensor

  $T^0{}_0 = -\rho(1+\epsilon), \; T^1{}_1 = T^2{}_2 = T^3{}_3 = 0$ Lagrangian coords.

- GRTENSOR, MATHEMATICA,...

  $\Rightarrow$ Field equations:

  $a' = \ldots$

  $b' = \ldots$

  $\ddot{R} = \ldots$

Further reading

- 3+1 formalism

  Gourgoulhon, gr-qc/0703035

- Characteristic formalism

  Winicour, Liv. Rev. Rel. **15** 2012 2

- Numerical relativity in general

  Alcubierre, "*Introduction to 3+1 Numerical Relativity*", Oxford University Press

  Baumgarte & Shapiro, "*Numerical Relativity*", Cambridge University Press

- Well-posedness, Einstein eqs. as an Initial-Boundary-Value problem

  Sarbach & Tiglio, Liv. Rev. Rel. **15** (2012) 9

2.2. NR beyond 4D

A list of tasks

- NR in 3+1 dimensions: $\mathcal{O}(100)$ cores, Gb of memory
- Each extra dimension can introduce a factor of $\mathcal{O}(100)$

  $\Rightarrow$ reduce D to $3+1$ dimensions; symmetries
- Three approaches:
  - Dimensional reduction to $3+1$ GR $+$ quasi matter
  - CARTOON type methods
  - Simplify line element using symmetry
- Outer boundary conditions: regularization, background subtraction

2.2.1 Dimensional reduction

Conventions

- Reduce D dimensions to d; typically $d = 3 + 1 = 4$
- Indices
  - $A,\ B,\ C, \ldots = 0\ \ldots\ D-1$: D dimensional spacetime
  - $\alpha,\ \beta,\ \gamma, \ldots = 0\ \ldots\ d-1$: d dimensional base spacetime
  - $a,\ b,\ c, \ldots = d\ \ldots\ D-1$: $D-d$ dimensional fibre
- Symmetry: $SO(D-d+1)$

  $\Rightarrow$ rotations in $D-d+1$ space dimensions $\Leftrightarrow$ on S^{D-d} sphere

  Typically: $SO(D-3)$ symmetry, S^{D-4} sphere

General formalism

Cho, Phys. Lett. B **186** (1987) 38
Cho & Kim, J. Math. Phys. **30** (1987) 1570
Zilhão, arXiv:1301.1509 [gr-qc]

The general D metric can be written

$$
\begin{aligned}
ds^2 &= g_{AB} dx^A dx^B \\
&= \left(g_{\mu\nu} + e^2\kappa^2 g_{ab} B^a{}_\mu B^b{}_\nu\right) dx^\mu dx^\nu + 2e\kappa B^a{}_\mu g_{ab} dx^\mu dx^b + g_{ab} dx^a dx^b
\end{aligned}
$$

Comments

- e, κ are coupling and scale parameters; they'll eventually drop out
- This metric is completely general!
- We used a special case of this: ADM 3+1 decomposition!

General formalism

- Assumption: g_{AB} admits m Killing vectors $\xi_{(i)} = \xi^a_{(i)} \partial_a$

  $\Rightarrow \mathcal{L}_{\xi_{(i)}} g_{AB} = 0$

- Def.: dual form $\xi^{(j)} = \xi^{(j)}_a \boldsymbol{d}x^a$ such that $\xi^{(j)}_a \xi^a_{(i)} = \delta^i_j$

- Def.: $F^a{}_{bc} \equiv -\xi^{(i)}_b \partial_c \xi^a_{(i)}$

- Then $\Rightarrow \mathcal{L}_{\xi_{(i)}} g_{AB} = 0$ implies

  $\partial_a g_{bc} = F^d{}_{ab} g_{dc} + F^d{}_{ac} g_{db}$

  $\partial_a B^b{}_{\mu} = -F^b{}_{ad} B^d_{\mu}$

  $\partial_a g_{\mu\nu} = 0$

General formalism

Def.: $D_\mu \equiv \partial_\mu - e\kappa B^d_\mu \partial_d$

$$\mathcal{F}^a{}_{\mu b} \equiv e\kappa \partial_b B^a_\mu = -e\kappa F^a{}_{bc} B^c_\mu$$

$$\mathcal{F}^a{}_{\mu\nu} \equiv -e\kappa \left[\partial_\mu B^a_\nu - \partial_\nu B^a_\mu + e\kappa (F^a{}_{bc} - F^a{}_{cb})\right]$$

The covariant derivatives are defined as

$$\nabla_\sigma T^{a\mu}{}_{b\nu} \equiv D_\sigma T^{a\mu}{}_{b\nu} + \mathcal{F}^a{}_{\sigma c} T^{c\mu}{}_{b\nu} - \mathcal{F}^c{}_{\sigma b} T^{a\mu}{}_{c\nu} + \Gamma^\mu_{\lambda\sigma} T^{a\lambda}{}_{b\nu} - \Gamma^\lambda_{\nu\sigma} T^{a\mu}{}_{b\lambda}$$

$$\nabla_c T^{a\mu}{}_{b\nu} \equiv \partial_c T^{a\mu}{}_{b\nu} + \Gamma^a_{dc} T^{d\mu}{}_{b\nu} - \Gamma^d_{bc} T^{a\mu}{}_{d\nu}$$

Here, $\Gamma^\mu_{\lambda\sigma}$ and Γ^a_{dc} are the connections associated with $g_{\mu\nu}$ and g_{ab}.

Note: $\nabla_\sigma g_{\mu\nu} = \nabla_c g_{ab} = 0, \quad$ but $\quad \nabla_\sigma g_{ab} \neq 0!$

General formalism

A tedious but straightforward calculation gives us the Ricci tensor as

$$R_{ab} = \mathcal{R}_{ab} - \frac{1}{4} g^{cd} \nabla_\mu g_{cd}\ \nabla^\mu g_{ab} + \frac{1}{2} g^{cd} \nabla_\mu g_{ac}\ \nabla^\mu g_{bd} + \frac{1}{4} g^{\mu\nu} g^{\rho\sigma} g_{bc} g_{ad} \mathcal{F}^c{}_{\rho\nu} \mathcal{F}^d_{\sigma\mu} - \frac{1}{2} \nabla^\mu \nabla_\mu g_{ab}$$

$$R_{\mu a} = e\kappa R_{ac} B^c_\mu + \frac{1}{2} g^{\rho\sigma} \nabla_\rho \left(g_{ac} \mathcal{F}^c{}_{\sigma\mu} \right) + \frac{1}{4} g^{cd} \nabla_\rho g_{cd}\ g^{\rho\sigma} \mathcal{F}^e{}_{\sigma\mu} g_{ae} + \frac{1}{2} \nabla_c \left(g^{cd} \nabla_\mu g_{da} \right)$$

$$R_{\mu\nu} = \mathcal{R}_{\mu\nu} + 2e\kappa B^c_{(\mu} R_{\nu)c} - e^2\kappa^2 R_{cd} B^c_\mu B^d_\nu - \frac{1}{2} g^{\rho\sigma} g_{cd} \mathcal{F}^c{}_{\sigma\mu} \mathcal{F}^d{}_{\rho\nu} - \frac{1}{2} \nabla_\nu \left(g^{cd} \nabla_\mu g_{cd} \right) - \frac{1}{4} g^{cd} g^{ab} \nabla_\mu g_{ca}\ \nabla_\nu g_{db} - \frac{1}{2} \nabla_c \mathcal{F}^c{}_{\mu\nu}$$

$$R = \mathcal{R}(g_{\mu\nu}) + \mathcal{R}(g_{ab}) - \frac{1}{4} g_{cd} g^{\rho\sigma} g^{\mu\nu} \mathcal{F}^d{}_{\sigma\mu} \mathcal{F}^c{}_{\rho\nu} - \nabla^\mu \left(g^{cd} \nabla_\mu g_{cd} \right) - \frac{1}{4} g^{ca} g^{bd} \nabla^\mu g_{cd}\ \nabla_\mu g_{ab} - \frac{1}{4} g^{cd} g^{ab} \nabla^\mu g_{cd}\ \nabla_\mu g_{ab}$$

Case: $SO(D-d+1)$ symmetry, $\Leftrightarrow S^{D-d}$ sphere

In practice: $SO(D-3)$, S^{D-4} sphere (e.g. $D=6 \;\Rightarrow\; SO(3),\; S^2$)

- S^n sphere: $(n+1)n/2$ Killing vectors $\xi_{(i)}$
- E.g. S^2 sphere: $\partial_\phi,\;\; \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi,\;\; \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi$

  Rotations around x, y, z axes
- Killing's equation $\mathcal{L}_{\xi_{(i)}} g_{AB} = 0$ implies

  $\mathcal{L}_{\xi_{(i)}} g_{ab} = 0, \quad \mathcal{L}_{\xi_{(i)}} B^a_\mu = 0, \quad \mathcal{L}_{\xi_{(i)}} g_{\mu\nu} = 0$
- Consequences:
  - $g_{ab} = e^{2\psi(x^\mu)} h_{ab}$ with h_{ab} = metric on S^{D-d} with unit radius
  - $g_{\mu\nu} = g_{\mu\nu}(x^\mu)$ in adapted coordinates
  - $[\xi_{(i)}, B_\mu] = 0$ for $n \geq 2$ (only vector field commuting with all KVs: 0)

    $\Rightarrow B^a_\mu = 0$

Case: $SO(D-d+1)$ symmetry, $\Leftrightarrow S^{D-d}$ sphere

With these consequences we get

- $R_{ab} = \left\{(D-d-1) - e^{2\psi}\left[(D-d)\partial^\mu\psi\,\partial_\mu\psi + \nabla^\mu\partial_\mu\psi\right]\right\} h_{ab}$

  $R_{\mu a} = 0$

  $R_{\mu\nu} = \mathcal{R}_{\mu\nu} - (D-d)(\nabla_\nu\partial_\mu\psi - \partial_\mu\psi\,\partial_\nu\psi)$

  $$R = \mathcal{R} + (D-d)\left[(D-d-1)e^{-2\psi} - 2\nabla^\mu\partial_\mu\psi - (D-d+1)\partial^\mu\psi\,\partial_\mu\psi\right]$$

- The D dimensional vacuum Einstein equations $R_{\mu\nu}$ thus become

  $e^{2\psi}\left[(D-d)\partial^\mu\psi\,\partial_\mu\psi + \nabla^\mu\partial_\mu\psi\right] = (D-d-1)$

  $\mathcal{R}_{\mu\nu} = (D-d)(\nabla_\nu\partial_\mu\psi - \partial_\mu\psi\,\partial_\nu\psi)$

  i.e. the d dimensional Einstein equations plus quasi-matter terms

Regularity of the variables

- Note: One of the $d-1$ spatial coordinates x, y, $z, \ldots$ is a radius
  Without loss of generality we choose y
  $\Rightarrow$ Computational domain: $x,\ z, \ldots \in \mathbb{R}$, $y \geq 0$
- Analysing our equations for some analytically known data,
  e.g. Brill-Lindquist, shows that $e^{2\psi} = 0$ at $y = 0$
- Solution: Use instead $\zeta = \frac{e^{-4\phi}}{y^2} e^{2\psi}$
  where $e^{-4\phi}$ is the BSSN conformal factor.
- With that we get the BSSN equations with matter terms...

BSSN source terms for d=4

Note: We set $d=4$, $i,\ j,\ldots=1,\ 2,\ 3$ and use $\chi\equiv e^{-4\phi}$

$$
\begin{aligned}
\frac{4\pi(\rho+S)}{D-4} =\ & (D-5)\frac{\chi}{\zeta}\frac{\tilde{\gamma}^{yy}\zeta-1}{y^2}-\frac{2D-7}{4\zeta}\tilde{\gamma}^{mn}\partial_m\eta\,\partial_n\chi-\chi\frac{\tilde{\Gamma}^y}{y}+\frac{D-6}{4}\frac{\chi}{\zeta^2}\tilde{\gamma}^{mn}\partial_m\zeta\,\partial_n\zeta \\
& +\frac{1}{2\zeta}\tilde{\gamma}^{mn}(\chi\tilde{D}_m\partial_n\zeta-\zeta\tilde{D}_m\partial_n\chi)+(D-4)\frac{\tilde{\gamma}^{ym}}{y}\left(\frac{\chi}{\zeta}\partial_m\zeta-\partial_m\chi\right) \\
& -\frac{KK_\zeta}{\zeta}-\frac{K^2}{3}-\frac{1}{2}\frac{\tilde{\gamma}^{ym}}{y}\partial_m\chi+\frac{D-1}{4}\tilde{\gamma}^{mn}\frac{\partial_m\chi\,\partial_n\chi}{\chi}-(D-5)\left(\frac{K_\zeta}{\zeta}+\frac{K}{3}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{8\pi\chi S^{\mathrm{TF}}_{ij}}{D-4} =\ & \frac{1}{2}\Bigg[\frac{2\chi}{y\zeta}(\delta^y{}_{(j}\partial_{i)}\zeta-\zeta\tilde{\Gamma}^y_{ij})+\frac{1}{2\chi}\partial_i\chi\,\partial_j\chi-\tilde{D}_i\partial_j\chi+\frac{\chi}{\zeta}\tilde{D}_i\partial_j\zeta \\
& +\frac{1}{2\chi}\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\partial_n\chi\left(\partial_m\chi-\frac{\chi}{\zeta}\partial_m\zeta\right)-\tilde{\gamma}_{ij}\frac{\tilde{\gamma}^{ym}}{y}\partial_m\chi-\frac{\chi}{2\zeta^2}\partial_i\zeta\,\partial_j\zeta\Bigg]^{\mathrm{TF}} \\
& -\left(\frac{K_\zeta}{\zeta}+\frac{K}{3}\right)\tilde{A}_{ij}
\end{aligned}
$$

BSSN source terms for d=4

$$\frac{16\pi j_i}{D-4} = \frac{2}{y}\left(\delta^y{}_i \frac{K_\zeta}{\zeta} - \tilde{\gamma}^{ym}\tilde{A}_{mi}\right) + \frac{2}{\zeta}\partial_i K_\zeta - \frac{K_\zeta}{\zeta}\left(\frac{1}{\chi}\partial_i\chi + \frac{1}{\zeta}\partial_i\zeta\right) + \frac{2}{3}\partial_i K$$
$$-\tilde{\gamma}^{nm}\tilde{A}_{mi}\left(\frac{1}{\zeta}\partial_n\zeta - \frac{1}{\chi}\partial_n\chi\right)$$

The matter evolution is given by

$$\partial_t\zeta = \beta^m\partial_m\zeta - 2\alpha K_\zeta - \frac{2}{3}\zeta\partial_m\beta^m + 2\zeta\frac{\beta^y}{y}$$

$$\partial_t K_\zeta = \beta^m\partial_m K_\zeta - \frac{2}{3}K_\zeta\partial_m\beta^m + 2\frac{\beta^y}{y}K_\zeta - \frac{1}{3}\zeta(\partial_t - \mathcal{L}_\beta)K - \chi\zeta\frac{\tilde{\gamma}^{ym}}{y}\partial_m\alpha$$
$$-\frac{1}{2}\tilde{\gamma}^{mn}\partial_m\alpha\left(\chi\partial_n\zeta - \zeta\partial_n\chi\right) + \alpha\left[(5-D)\chi\frac{\zeta\tilde{\gamma}^{yy}-1}{y^2} + (4-D)\chi\frac{\tilde{\gamma}^{ym}}{y}\partial_m\zeta\right.$$
$$+\frac{2D-7}{2}\zeta\frac{\tilde{\gamma}^{ym}}{y}\partial_m\chi + \frac{6-D}{4}\frac{\chi}{\zeta}\tilde{\gamma}^{mn}\partial_m\zeta\,\partial_n\zeta + \frac{2D-7}{4}\tilde{\gamma}^{mn}\partial_m\zeta\,\partial_n\chi$$
$$+\frac{1-D}{4}\frac{\zeta}{\chi}\tilde{\gamma}^{mn}\partial_m\chi\,\partial_n\chi + (D-6)\frac{K_\zeta^2}{\zeta} + \frac{2D-5}{3}KK_\zeta + \frac{D-1}{9}\zeta K^2$$
$$\left.+\frac{1}{2}\tilde{\gamma}^{mn}(\zeta\tilde{D}_m\partial_n\chi - \chi\tilde{D}_m\partial_n\zeta) + \chi\zeta\frac{\tilde{\Gamma}^y}{y}\right]$$

Regularization at $y = 0$

- Note that the previous equations contain divisions by y,
  E.g. : $\frac{\beta^y}{y}$, $\frac{\tilde{\gamma}^{ym}}{y}\partial_m\zeta, \ldots$
- These can all be regularized!
- E.g. : Symmetry of a vector across $y = 0$ implies
  $\beta^y(-y) = -\beta^y(y)$
  We can therefore Taylor expand β^y around $y = 0$ as
  $\beta^y(y) = b_1 y + \mathcal{O}(y^2)$
  $\Rightarrow \lim_{y\to 0} \frac{\beta^y}{y} = b_1 = \partial_y \beta^y$
- Similar tricks work for all such terms
  see Zilhão et al, PRD **81** (2010) 084052

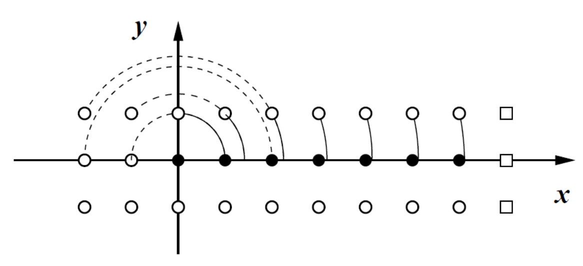
2.2.2 CARTOON methods

Dimensional reduction through CARTOON

- Originally developed for axisymmetry around z in 3+1 dimensions

  Alcubierre et al, IJMPD **10** (2001) 273

- Coordinates $(z, x, y) \leftrightarrow (z, \rho, \theta)$ where $x = \rho \cos\phi, \;\; y = \rho \sin\phi$
- Killing vector $\partial_\phi = x\partial_y - y\partial_x$
- Extend 2D grid by ghostzones for derivatives; rotate, interpolate



Dimensional reduction through CARTOON

Yoshino & Shibata. PRD **80** (2009) 084025

- Scalar: $\psi(z,x,y) = \psi(z,\rho,0)$
- Vector: Express $v^x(z,x,y)$, $v^y(z,x,y)$ through $v^\rho(z,\rho)$, $v^\phi(z,\rho)$ and replace those through the relation on the xy plane

$$v^x(z,\rho,0) = v^\rho(z,\rho), \quad v^y(z,\rho,0) = \rho v^\phi(z,\rho)$$

$$\Rightarrow \; v^x(z,x,y) = \frac{x}{\rho} v^x(z,\rho,0) - \frac{y}{\rho} v^y(z,\rho,0)$$

$$v^y(z,x,y) = \frac{y}{\rho} v^x(z,\rho,0) + \frac{x}{\rho} v^y(z,\rho,0)$$

- Likewise for tensors: T_{zz} like scalar, T_{zx}, T_{zy} like vector

$$T_{xx}(z,x,y) = \left(\frac{x}{\rho}\right)^2 T_{xx}(z,\rho,0) + \left(\frac{y}{\rho}\right)^2 T_{yy}(z,\rho,0) - \frac{2xy}{\rho^2} T_{xy}(z,\rho,0)$$

$$T_{yy}(z,x,y) = \left(\frac{y}{\rho}\right)^2 T_{xx}(z,\rho,0) + \left(\frac{x}{\rho}\right)^2 T_{yy}(z,\rho,0) + \frac{2xy}{\rho^2} T_{xy}(z,\rho,0)$$

$$T_{xy}(z,x,y) = \frac{xy}{\rho^2} \left[T_{xx}(z,\rho,0) - T_{yy}(z,\rho,0)\right] + \frac{x^2-y^2}{\rho^2} T_{xy}(z,\rho,0)$$

CARTOON in D=5 with $SO(3)$ symmetry

- Cartesian coordinates: $(w,\ x,\ y,\ z)$

  each hypersurface $w = \mathrm{const}$ is spher. symmetric
- 3 Killing vectors

  $\xi_1 = y\partial_z - z\partial_y,\ \xi_2 = z\partial_z - x\partial_z,\ \ \xi_3 = x\partial_y - y\partial_x$
- Use data in xw plane, set $r = \sqrt{x^2 + y^2 + z^2}$
- Scalar: $\psi(w, x, y, z) = \psi(w, r, 0, 0)$
- Vector, Tensor fields: ...

  cf. Yoshino & Shibata, PRD **80** (2009) 084025
- $\Rightarrow$ effective 2+1 Cartesian code

A modified CARTOON method

Shibata & Yoshino, PRD **81** (2010) 104035

Yoshino & Shibata, PTPS **189** (2011) 269

- For larger D, CARTOON ghostzones require considerable memory
- Solution: Trade derivatives
- Coordinates: $(x,\ y,\ z,\ w_i),\ \ i = 1 \ldots D-4,\ \ \rho = \sqrt{z^2 + \sum_i w_i^2}$
- Symmetry: $SO(D-3)$, i.e. Rotations in w_i
- Scalar: $\psi(x, y, z, w_i) = \psi(x, y, \rho, 0)$

  $\Rightarrow\ \partial_{w_i}\psi = \partial_{(x,y)}\partial_{w_i}\psi = \partial_z\partial_{w_i} = 0,\ \ \partial_{w_i}\partial_{w_j}\psi = \frac{\partial_z\psi}{z}\delta_{ij}$

  where (x, y) stands for either x or y

A modified CARTOON method

- Vector: $\beta^z(x,y,z,w_i) = \frac{z}{\rho}\beta^z(x,y,\rho,0)$

  $\beta^{w_i}(x,y,z,w_i) = \frac{w_i}{\rho}\beta^z(x,y,\rho,0)$

$$\Rightarrow \partial_{w_i}\beta^z = \partial_{(x,y)}\beta^{w_i} = \partial_z\beta^{w_i} = \partial_{(x,y)}\partial_{w_i}\beta^z = \partial_z\partial_{w_i}\beta^z$$

$$= \partial_{(x,y)}\partial_{(x,y)}\beta^{w_i} = \partial_{(x,y)}\partial_z\beta^{w_i} = \partial_{w_j}\partial_{w_k}\beta^{w_i} = 0$$

$$\partial_{w_j}\beta^{w_i} = \frac{\beta^z}{z}\delta_{ij}, \quad \partial_{(x,y)}\partial_{w_j}\beta^{w_i} = \frac{\partial_{(x,y)}\beta^z}{z}\delta_{ij},$$

$$\partial_{w_i}\partial_{w_j}\beta^z = \partial_z\partial_{w_j}\beta^{w_i} = \left(\frac{\partial_z\beta^z}{z} - \frac{\beta^z}{z^2}\right)\delta_{ij}$$

A modified CARTOON method

- Tensors:

$$T_{zz}(x,y,z,w_i) = \frac{z^2}{\rho^2} T_{zz}(x,y,\rho,0) + \left(1 - \frac{z^2}{\rho^2}\right) T_{ww}(x,y,\rho,0)$$

$$T_{w_i w_i}(x,y,z,w_i) = \frac{w_i^2}{\rho^2} T_{zz}(z,y,\rho,0) + \left(1 - \frac{w_i^2}{\rho^2}\right) T_{ww}(x,y,\rho,0)$$

$$T_{zw_i}(x,y,z,w_i) = \frac{zw_i}{\rho^2} \left[T_{zz}(x,y,\rho,0) - T_{ww}(x,y,\rho,0)\right]$$

$$T_{w_i w_j}(x,y,z,w_i) = \frac{w_i w_j}{\rho^2} \left[T_{zz}(x,y,\rho,0) - T_{ww}(x,y,\rho,0)\right]$$

where $T_{ww} \equiv T_{w_1 w_1} = T_{w_2 w_2} = \ldots$ which are all equal

$$\begin{aligned} \Rightarrow \partial_{w_i} T_{zz} &= \partial_{w_j} T_{w_i w_i} = \partial_{(x,y)} T_{zw_i} = \partial_z T_{zw_i} = \partial_{(x,y)} \partial_{w_i} T_{zz} \\ &= \partial_z \partial_{w_i} T_{zz} = \partial_{(x,y)} \partial_{w_j} T_{w_i w_i} = \partial_z \partial_{w_j} = \partial_{(x,y)} \partial_{(x,y)} T_{zw_i} \\ &= \partial_{(x,y)} \partial_z T_{zw_i} = \partial_z \partial_{w_i} T_{zz} = 0 \end{aligned}$$

A modified CARTOON method

- Tensors continued: for $i \neq j$ we also have

$$\partial_{(x,y)} T_{w_i w_j} = \partial_z T_{w_i w_j} = \partial_{w_k} T_{w_i w_j} = \partial_{(x,y)} \partial_{(x,y)} T_{w_i w_j} = \partial_{(x,y)} \partial_z T_{w_i w_j}$$
$$= \partial_z \partial_z T_{w_i w_j} = \partial_{(x,y)} \partial_{w_k} T_{w_i w_j} = \partial_z \partial_{w_k} T_{w_i w_j} = 0$$

- The non-zero derivatives appearing in the BSSN eqs. are

$$\partial_{w_j} T_{z w_i} = \frac{T_{zz} - T_{ww}}{z} \delta_{ij}, \quad \partial_{(x,y)} \partial_{w_j} T_{z w_i} = \frac{\partial_{(x,y)} T_{zz} - \partial_{(x,y)} T_{ww}}{z} \delta_{ij},$$

$$\partial_{w_i} \partial_{w_j} \frac{1}{z} \left[\partial_z T_{zz} + \frac{2(T_{ww} - T_{zz}}{z} \right] \delta_{ij},$$

$$\partial_{w_j w_k} T_{w_i w_i} = \frac{2(T_{zz} - T_{ww})}{z^2} \delta_{ik} \delta_{ij} + \frac{\partial_z T_{ww}}{z} \delta_{jk},$$

$$\partial_z \partial_{w_j} T_{z w_i} = \frac{1}{z} \left[\partial_z T_{zz} - \partial_z T_{ww} - \frac{T_{zz} - T_{ww}}{z} \right] \delta_{ij},$$

$$\partial_{w_k} \partial_{w_l} T_{w_i w_j} = \frac{T_{zz} - T_{ww}}{z^2} (\delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl}) \text{ for } i \neq j$$

A modified CARTOON method

- Plugging these into the D dim. equations enables us to work on a genuine d dim. hypersurface with no ghost zones
- Note: There are additional fields, e.g. T_{zw_i}, but these are only required on the (xyz) hyper plane!
- Note: There are divisions by z

  $\Rightarrow$ regularization at $z = 0$ required!

  cf. $y = 0$ in the dimensional reduction

Further reading

- Dimensional reduction

  Zilhão, arXiv:1301.1509

- Modified Cartoon

  Yoshino & Shibata, PTPS **189** (2011) 269

2.3. Initial data, Gauge, Boundaries

2.3.1. Initial data

Analytic initial data

- Schwarzschild, Kerr, Tangherlini, Myers Perry,...

  e.g. Schwarzschild in isotropic coordinates:

  $$ds^2 = -\left(\frac{M-2r}{M+2r}\right)^2 dt^2 + \left(1+\frac{M}{2r}\right)^4 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

- Time symmetric N BH initial data: Brill-Lindquist, Misner 1960s
- Problem: Finding initial data for dynamic systems
- Goals
  - 1) Solve constraints
  - 2) Realistic snapshot of physical system
- This is mostly done using the ADM 3+1 split

The York-Lichnerowicz split

- We work in $D = 4$
- Conformal metric: $\gamma_{ij} = \psi^4 \bar{\gamma}_{ij}$

  Lichnerowicz, J.Math.Pures Appl. **23** (1944) 37

  York, PRL **26** (1971) 1656, PRL **28** (1972) 1082
- Note: in contrast to BSSN we do not set $\bar{\gamma} = 1$
- Conformal traceless split of the extrinsic curvature

  $K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij} K$

  $A^{ij} = \psi^{-10} \bar{A}_{ij} \;\; \Leftrightarrow \;\; A_{ij} = \psi^{-2} \bar{A}_{ij}$

Bowen-York data

- By further splitting $\bar{A}_{ij}$ into a longitudinal and a transverse traceless part, the momentum constraint simplifies significantly
  Cook, Living Review Relativity (2000) 05
- Further assumptions: vacuum, $K = 0, \quad \bar{\gamma}_{ij} = f_{ij}, \quad \psi|_\infty = 1$
  where f_{ij} is the flat metric in arbitrary coordinates.
  Conformal flatness, asymptotic flatness, traceless
- Then there exists an anlytic solution to the momentum constraint

$$\bar{A}_{ij} = \frac{3}{2r^2}\left[P_i n_j + P_j n_i - (f_{ij} - n_i n_j)P^k n_k\right] + \frac{3}{r^3}\left(\epsilon_{kil} S^l n^k n_j + \epsilon_{kjl} S^l n^k n_i\right)$$

  where r is a coordinate radius and $n^i = \frac{x^i}{r}$

Bowen & York, PRD **21** (1980) 2047

Properties of the Bowen York solution

- The momentum in an asymptotically flat hypersurface associated with the asymptotic translational and rotational Killing vectors $\xi^i_{(a)}$ is

  $\Pi^i = \frac{1}{8\pi} \oint_\infty \left(K^j{}_i - \delta^j{}_i K \right) \xi^i_{(a)} d^2 A_j$

  $\Rightarrow \ldots \Rightarrow$ P^i and S^i are the physical linear and angular momentum of the spacetime

- The momentum constraint is linear

  $\Rightarrow$ we can superpose Bowen-York data.

  The momenta then simply add up

- Bowen-York data generalizes (analytically!) to higher D

  Yoshino, Shiromizu & Shibata, PRD **74** (2006) 124022

Puncture data

Brandt & Brügmann, PRL **78** (1997) 3606

- The Hamiltonian constraint is now given by
  $\bar{\nabla}^2\psi + \frac{1}{8}\psi^{-7}\bar{A}_{mn}\bar{A}^{mn} = 0$
- Ansatz for conformal factor: $\psi = \psi_{\rm BL} + u$,
  where $\psi_{\rm BL} = \sum_{i=1}^{N} \frac{m_i}{2|\vec{r}-\vec{r}_i|}$ is the Brill-Lindquist conformal factor,
  i.e. the solution for $\bar{A}_{ij} = 0$.
- There then exist unique C^2 solutions u to the Hamiltonian constraints
- The Hamiltonian constraint in this form is further suitable for numerical solution

  e.g. Ansorg, Brügmann & Tichy, PRD **70** (2004) 064011

Properties of the puncture solutions

- m_i and $\vec{r}_i$ are bare mass and position of the i^{th} BH.
- In the limit of vanishing Bowen York parameters $P^i = S^i = 0$, the puncture solution reduces to Brill Lindquist data

  $$\gamma_{ij} dx^i dx^j = \left(1 + \sum_i \frac{m_i}{2|\vec{r}-\vec{r}_i|}\right)^4 (dx^2 + dy^2 + dz^2)$$

- The numerical solution of the Hamiltonian constraint generalizes rather straightforwardly to higher D

  Yoshino, Shiromizu & Shibata, PRD **74** (2006) 124022

  Zilhão et al, PRD **84** (2011) 084039

- Punctures generalize to asymptotically de-Sitter BHs

  Zilhão et al, PRD **85** (2012) 104039

  using McVittie coordinates McVittie, MNRAS **93** (1933) 325

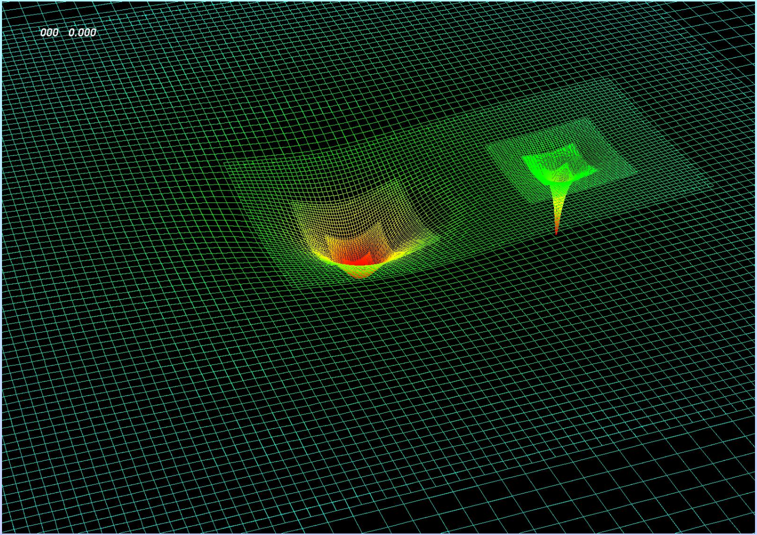
2.3.2. Gauge

The gauge freedom

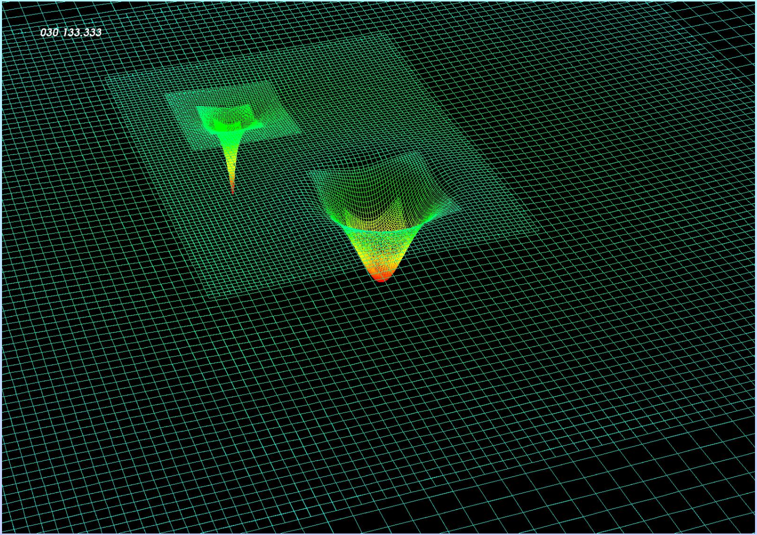
- Remember: Einstein equations say nothing about $\alpha, \quad \beta^i$
- Any choice of lapse and shift gives a solution
- This represents the coordinate freedom of GR
- Physics do not depend on $\alpha, \quad \beta^i$

  So why bother?
- The performance of the numerics DO depend strongly on the gauge!

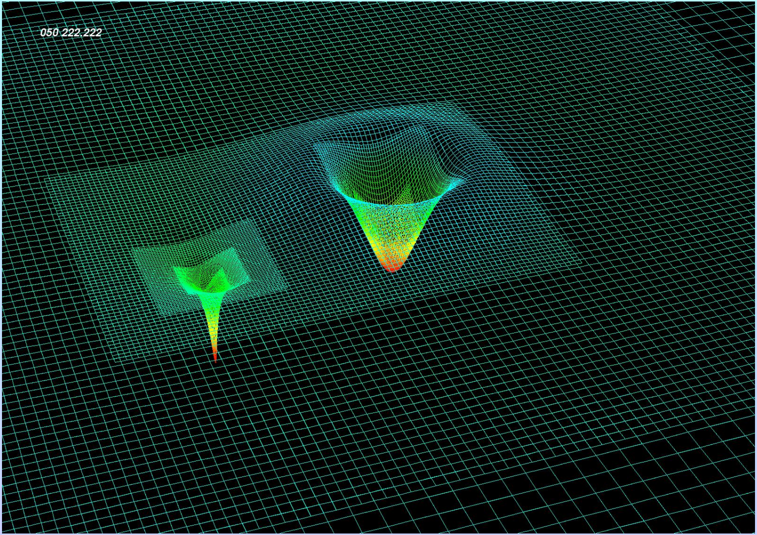
What goes wrong with bad gauge?



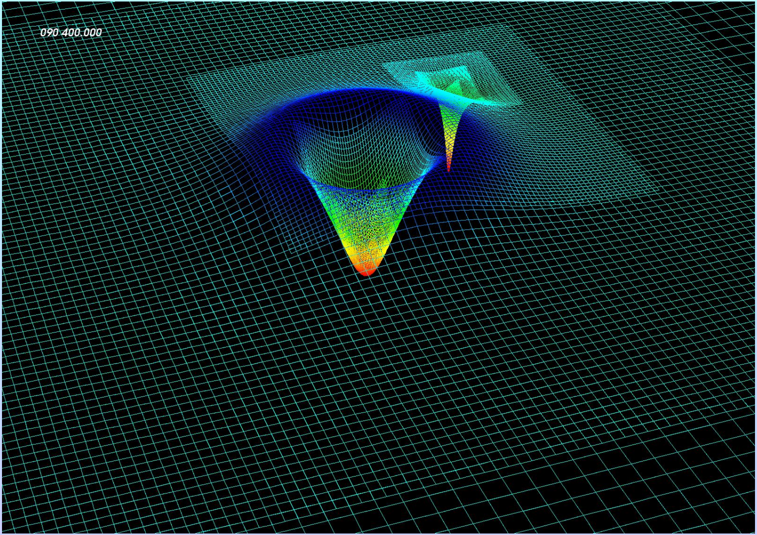
What goes wrong with bad gauge?



What goes wrong with bad gauge?



What goes wrong with bad gauge?



Ingredients for good gauge

- Singularity avoidance
- Avoid slice stretching
- Aim at stationarity in comoving frame
- Well posedness of system
- Generalize "good" gauge, e .g. harmonic
- Lots of good luck!

Bona et al, PRL **75** (1995) 600,

Alcubierre *et al.*, PRD **67** (2003) 084023,

Alcubierre, CQG **20** (2003) 607,

Garfinkle, PRD **65** (2001) 044029

Moving puncture gauge

- Gauge was a key ingredient in the Moving puncture breakthroughs

  Campanelli et al, PRL **96** (2006) 111101

  Baker et al, PRL **96** (2006) 111102

- Variant of $1 + \log$ slicing and Γ-driver shift

  Alcubierre et al, PRD **67** (2003) 084023

- Now in use as

  $$\partial_t \alpha = \beta^m \partial_m \alpha - 2\alpha K$$

  and

  $$\partial_t \beta^i = \beta^m \partial_m \beta^i + \frac{3}{4} B^i$$
  $$\partial_t B^i = \beta^m \partial_m B^i + \partial_t \tilde{\Gamma}^i - \beta^m \partial_m \tilde{\Gamma}^i - \eta B^i$$

  or

  $$\partial_t \beta^i = \beta^m \partial_m \beta^i + \frac{3}{4} \tilde{\Gamma}^i - \eta \beta^i$$

Moving puncture gauge continued

- Some people drop the advection derivatives $\beta^m \partial_m \ldots$
- η is a damping parameter or position-dependent function

  Alic et al, CQG **27** (2010) 245023, Schnetter, CQG **27** (2010) 167001, Müller et al, PRD **82** (2010) 064004

- Modifications in higher D:
  - Dimensional reduction Zilhão et al, PRD **81** (2010) 084052

    $\partial_t \alpha = \beta^m \partial_m \alpha - 2\alpha(\eta_K K + \eta_{K_\zeta} K_\zeta)$

  - CARTOON Yoshino & Shibata, PTPS **189** (2011) 269

    $\partial_t \beta^i = \frac{D-1}{2(D-2)} v_{\rm long}^2 B^i$

    $\partial_t B^i = \partial_t \tilde{\Gamma}^i - \eta B^i$

- Here η_K, η_{K_ζ}, $v_{\rm long}$ are parameters

Gauge conditions in the GH formulation

- How to choose H_μ? $\rightarrow$ some experimentation...
- Pretorius' breakthrough

  $\Box H_t = -\xi_1 \frac{\alpha - 1}{\alpha^\eta} + \xi_2 n^\mu \partial_\mu H_t$ with

  $\xi_1 = 19/m, \;\; \xi_2 = 2.5/m, \;\; \eta = 5$ where $m =$ mass of 1 BH
- Caltech-Cornell-CITA spectral code:

  Initialize H_α to minimize time derivatives of metric,

  adjust H_α to harmonic and damped harmonic gauge condition

  Lindblom & Szilágyi, PRD **80** (2009) 084019, with Scheel, PRD **80** (2009) 124010
- The H_α are related to lapse and shift: $n^\mu H_\mu = -K - n^\mu \partial_\mu \ln \alpha$

  $\gamma^{\mu i} H_\mu = -\gamma^{mn} \Gamma^i_{mn} + \gamma^{im} \partial_m (\ln \alpha) + \frac{1}{\alpha} n^\mu \partial_\mu \beta^i$

2.3.3. Boundaries

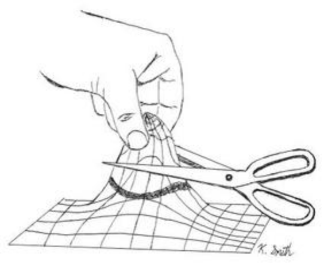
Inner boundary: Singularity treatment

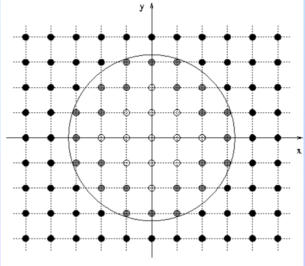
- Cosmic censorship ⇒ horizon protects outside
- We get away with it...

  Moving Punctures

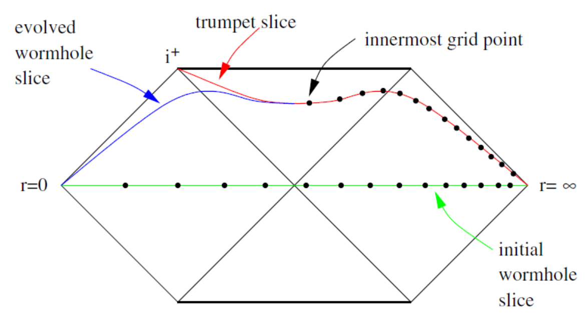
  UTB, NASA Goddard '05
- Excision: Cut out region around singularity

  Caltech-Cornell, Pretorius





Moving puncture slices: Schwarzschild



- Wormhole $\rightarrow$ Trumpet slice $=$ stationary 1+log slice

  Hannam et al, PRL **99** (2007) 241102, PRD **78** (2008) 064020

  Brown, PRD **77** (2008) 044018, CQG **25** (2008) 205004

- Gauge might propagate at $> c$, no pathologies

  Natural excision Brown, PRD **80** (2009) 084042

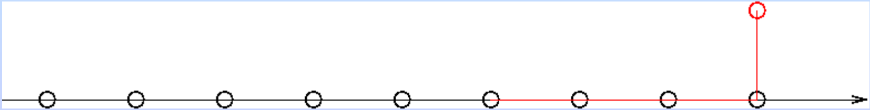
Outer boundary: Asymptotically flat case

- Computational domains often don't extend to ∞
- Outgoing Sommerfeld conditions

  Assume $f = f_0 + \frac{u(t-r)}{r^n}$ where $f_0 =$ asymptotic value

  $\partial_t u + \partial_r u = 0$

  $\partial_t f + n\frac{f-f_0}{r} + \frac{x^i}{r}\partial_i f = 0$
- Use upwinding, i.e. one-sided, derivatives!



Non-asymptotically flat case: de Sitter

- In McVittie coordinates:

  $r \to \infty \;\Rightarrow\; ds^2 = -dt^2 + a(t)^2(r^2 + r^2 d\Omega_2^2)$

  where $a(t) = e^{Ht}, \quad H = \sqrt{\Lambda/3}$

- Radial null geodesics: $dt = \pm a dr$

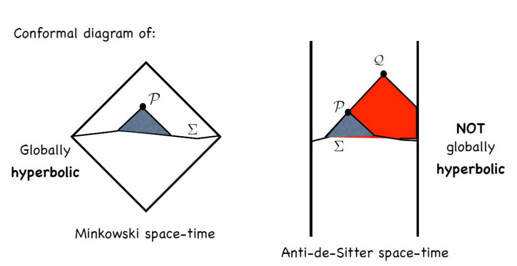
  We expect: $f = f_0 + \frac{a\,u(t - a\,r)}{r^n}$

  $\Rightarrow \partial_t f - \partial_t f_0 + \frac{1}{a(t)} \partial_r f + n \frac{f - f_0}{r\ a(t)} - H(f - f_0) = 0$

  Zilhão et al, PRD **85** (2012) 104039

Anti de Sitter

Much more complicated!



- Time-like outer boundary ⇒ affects interior
- AdS metric diverges at outer boundary

Anti de Sitter metric

- Maximally symmetric solution to Einstein eqs. with $\Lambda < 0$
- Hyperboloid $X_0^2 + X_D^2 - \sum_{i=1}^{D-1} X_i^2$
  embedded in $D+1$ dimensional flat spacetime of signature $--+\ldots+$
- Global AdS
  $X_0 = L\frac{\cos\tau}{\cos\rho}, \quad X_d = L\frac{\sin\tau}{\cos\rho}$
  $X_i = L\tan\rho\,\Omega_i$, for $i = 1\ldots D-1$, $\quad \Omega_i$ hyperspherical coords.
  $\Rightarrow ds^2 = \frac{L^2}{\cos^2\rho}(-d\tau^2 + d\rho^2 + \sin^2\rho\, d\Omega_{D-2}^2)$
  where $0 \le \rho < \pi/2, \quad -\pi < \tau \le \pi$
- Outer boundary at $\rho = \pi/2$

Anti de Sitter metric continued

- Poincaré coordinates

$X_0 = \frac{1}{2z}\left[z^2 + L^2 + \sum_{i=1}^{D-2}(x^i)^2 - t^2\right]$

$X_i = \frac{Lx^i}{z}$ for $i = 1 \ldots D-2$

$X_{D-1} = \frac{1}{2z}\left[z^2 - L^2 + \sum_{i=1}^{D-2}(x^i)^2 - t^2\right]$

$X_d = \frac{Lt}{z}$

$\Rightarrow ds^2 = \frac{L^2}{z^2}\left[-dt^2 + dz^2 + \sum_{i=1}^{D-2}(dx^i)^2\right]$

where $z > 0$, $t \in \mathbb{R}$

- Outer boundary at $z = 0$

e.g. Ballón Bayona & Braga, hep-th/0512182

AdS spacetimes: Outer boundary

- AdS boundary: $\rho \to \pi/2$ (global)

  $z \to 0$ (Poincaré)

- AdS metric becomes singular

  $\Rightarrow$ induced metric determined up to conformal rescaling only

- Global: $ds^2_{\rm gl} \sim -d\tau^2 + d\Omega_{D-2}$

  Poincaré: $ds^2_{\rm P} \sim -dt^2 + \sum_{i=1}^{D-2} d(x^i)^2$

  $\Rightarrow$ Different topology: $\mathbb{R} \times S_{D-2}$ and $\mathbb{R}^{D-1}$

- The dual theories live on spacetimes of different topology

Regularization methods

- Decompose metric into AdS part plus deviation

  Bantilan & Pretorius, PRD **85** (2012) 084038

- Factor out appropriate factors of the bulk coordinate

  Chesler & Yaffe, PRL **106** (2011) 021601
  Heller, Janik & Witaszczyk, PRD **85** (2012) 126002

- Factor out singular term of the metric

  Bizoń & Rostworowski, PRL **107** (2011) 031102

- Regularity of the outer boundary may constrain the gauge freedom

  Bantilan & Pretorius, PRD **85** (2012) 084038

Further reading

- Initial data construction

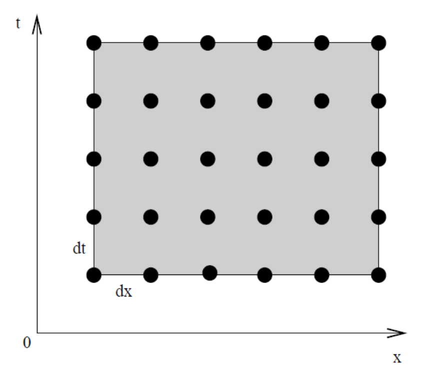
  Cook, Liv. Rev. Rel. **3** (2000) 5

  Pfeiffer, gr-qc/0510016

2.4 Discretization of the equations

Finite differencing

- Consider one spatial, one time dimension $t, \ x$
- Replace computational domain by discrete points
  $x_i = x_0 + i \, dx, \quad t_n = t_0 + n \, dt$
- Function values $f(t_n, x_i) \sim f_{n,i}$



Derivatives and finite derivatives

- Goal: represent $\frac{\partial^m f}{\partial x^m}$ in terms of $f_{n,i}$
- Fix index n; Taylor expansion:

  $f_{i-1} = f_i - f_i' dx + \frac{1}{2} f_i'' dx^2 + \mathcal{O}(dx^3)$

  $f_i = f_i$

  $f_{i+1} = f_i + f_i' dx + \frac{1}{2} f_i'' dx^2 + \mathcal{O}(dx^3)$
- Write f_i' as linear combination: $f_i' = A f_{i-1} + B f_i + C f_{i+1}$
- Insert Taylor expressions and compare coefficients on both sides

  $\Rightarrow 0 = A + B + C, \quad 1 = (-A + B) dx, \quad 0 = \frac{1}{2} A dx^2 + \frac{1}{2} C dx^2$

  $\Rightarrow A = -\frac{1}{2dx},\ B = 0,\ C = \frac{1}{2dx}$

  $\Rightarrow f_i' = \frac{f_{i+1} - f_{i-1}}{2dx} + \mathcal{O}(dx^2)$
- Higher order accuracy $\rightarrow$ more points; works same in time

Mesh refinement

3 Length scales :	BH	$\sim 1\ M$
	Wavelength	$\sim 10...100\ M$
	Wave zone	$\sim 100...1000\ M$

- Critical phenomena

  Choptuik '93

- First used for BBHs

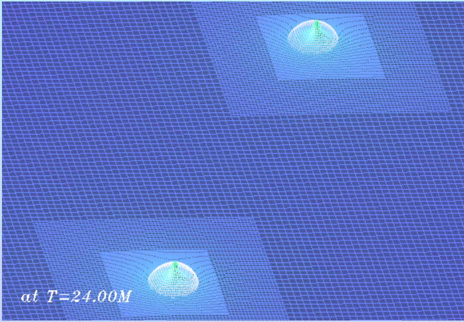
  Brügmann '96

- Available Packages:

  Paramesh MacNeice *et al.* '00

  Carpet Schnetter *et al.* '03

  SAMRAI MacNeice *et al.* '00

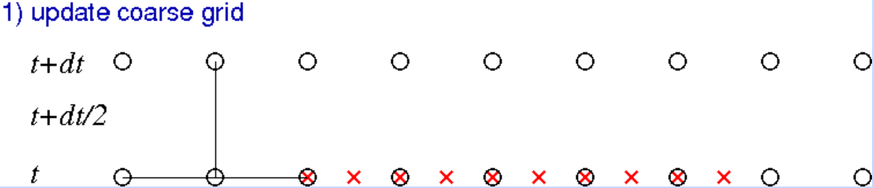


Berger-Oliger mesh refinement

- Goal: Update from t to $t + dt$
- Refinement criteria: numerical error, curvature,...
- Here for $1 + 1$ dimensions

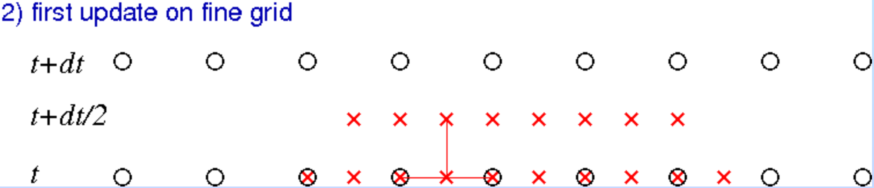
Berger-Oliger mesh refinement

- Goal: Update from t to $t + dt$
- Refinement criteria: numerical error, curvature,...
- Here for $1 + 1$ dimensions



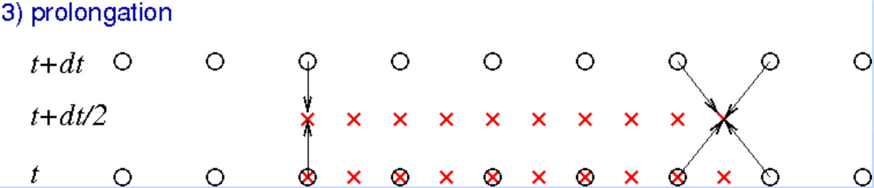
Berger-Oliger mesh refinement

- Goal: Update from t to $t + dt$
- Refinement criteria: numerical error, curvature,...
- Here for $1 + 1$ dimensions



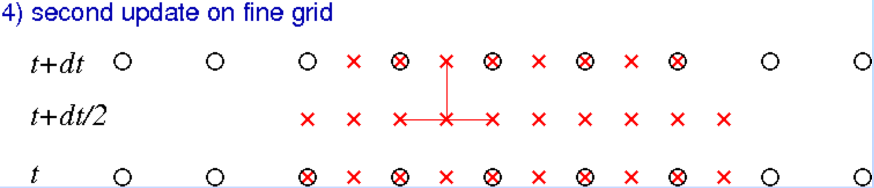
Berger-Oliger mesh refinement

- Goal: Update from t to $t + dt$
- Refinement criteria: numerical error, curvature,...
- Here for $1 + 1$ dimensions



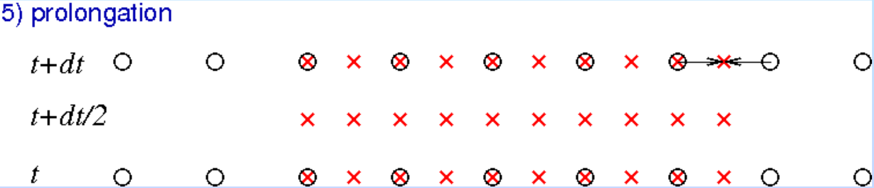
Berger-Oliger mesh refinement

- Goal: Update from t to $t + dt$
- Refinement criteria: numerical error, curvature,...
- Here for $1 + 1$ dimensions



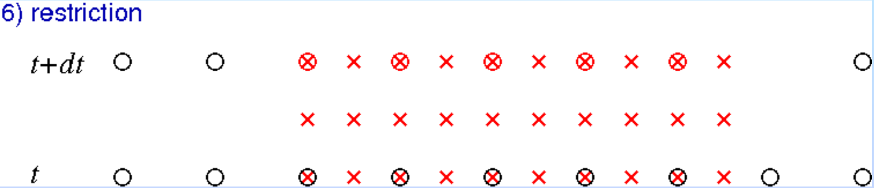
Berger-Oliger mesh refinement

- Goal: Update from t to $t + dt$
- Refinement criteria: numerical error, curvature,...
- Here for $1 + 1$ dimensions



Berger-Oliger mesh refinement

- Goal: Update from t to $t + dt$
- Refinement criteria: numerical error, curvature,...
- Here for $1 + 1$ dimensions



Alternative discretization schemes

- Spectral methods: high accuracy, efficiency, complexity

  Caltech-Cornell-CITA code SpEC

  `http://www.black-holes.org/SpEC.html`

  Applications to moving punctures still in construction

  e.g. Tichy, PRD **80** (2009) 104034

  Also used in symmetric asymptotically AdS spacetimes

  e.g. Chesler & Yaffe, PRL **106** (2011) 021601

- Finite Volume methods
- Finite Element methods

  D. N. Arnold, A. Mukherjee & L. Pouly, gr-qc/9709038
  C. F. Sopuerta, P. Sun & J. Xu, CQG **23** (2006) 251

  C. F. Sopuerta & P. Laguna, PRD **73** (2006) 044028

Further reading

- Numerical methods

  Press et al, "*Numerical Recipes*", Cambridge University Press

2.5 Diagnostics

The subtleties of diagnostics in GR

- Successful numerical simulation $\Rightarrow$ Numbers for grid functions
- Typically: Spacetime metric $g_{\alpha\beta}$ and time derivative or
  ADM variables $\gamma_{ij},\ K_{ij},\ \alpha,\ \beta^i$
- Challenges
  - Coordinate dependence of numbers $\Rightarrow$ Gauge invariants
  - Global quantities at ∞, computational domain finite $\Rightarrow$ Extrapolation
  - Complexity of variables, e.g. GWs $\Rightarrow$ Spherical harmonics
  - Local quantities: meaningful? $\Rightarrow$ Horizons
- AdS/CFT correspondence: Dictionary

Newton's gravitational constant

- Note: We wrote the Einstein equations for $\Lambda = 0$ as

  $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta}$

- The (areal) horizon radius of a static BH in D dimensions then is

  $r_s^{D-3} = \frac{16\pi G M}{(D-2)\Omega_{D-2}}$,

  where $\Omega_{D-2} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})}$ is the area of the $D-2$ hypersphere

- The Hawking entropy formula is $S = \frac{\mathcal{A}_{AH}}{4G}$

- But Newton's force law picks up geometrical factors:

  $\boldsymbol{F} = \frac{(D-3)8\pi G}{(D-2)\Omega_{D-2}} \frac{Mm}{r^{D-2}} \hat{\boldsymbol{r}}$

  See e.g. Emparan & Reall, Liv. Rev. Rel. **6** (2008)

Global quantities

- Assumptions
  - Asymptotically, the metric is flat and time independent
  - The expressions also refer to Cartesian coordinates
- ADM mass = Total mass-energy of spacetime

  $M_{ADM} = \frac{1}{4\Omega_{D-2}G} \lim_{r\to\infty} \int_{S_r} \sqrt{\gamma}\gamma^{mn}\gamma^{kl}(\partial_n\gamma_{mk} - \partial_k\gamma_{mn})dS_l$
- Linear momentum of spacetime

  $P_i = \frac{1}{8\pi G} \lim_{r\to\infty} \int_{S_r} \sqrt{\gamma}(K^m{}_i - \delta^m{}_i K)dS_m$
- Angular momentum in $D = 4$

  $J_i = \frac{1}{8\pi}\epsilon_{il}{}^m \lim_{r\to\infty} \int_{S_r} \sqrt{\gamma}x^l(K^n{}_m - \delta^n{}_m K)dS_n$
- By construction, these are time independent!

Apparent horizons

- By Cosmic censorship, existence of an apparent horizon implies an event horizon
- Consider outgoing null geodesics with tangent vector k^μ
- Def.: Expansion $\Theta = \nabla_\mu k^\mu$
- Def.: Apparent horizon $=$ outermost surface where $\Theta = 0$
- On a hypersurface Σ_t, the condition for $\Theta = 0$ becomes

  $\hat{D}_m s^m - K + K_{mn} s^m s^n = 0$,

  where $s^i =$ unit normal to the $(D-2)$ dimensional AH surface

  e.g. Thornburg, PRD **54** (1996) 4899

Apparent horizons continued

- Parametrize the horizon by $r = f(\varphi^i)$,
  where r is the radial and φ^i are angular coordinates
- Rewrite the condition $\Theta = 0$ in terms $f(\varphi^i)$
  $\Rightarrow$ Elliptic equation for $f(\varphi^i)$
- This can be solved e.g. with Flow, Newton methods
  Thornburg, PRD **54** (1996) 4899, Gundlach, PRD **57** (1998) 863
  Alcubierre et al, CQG **17** (2000) 2159, Schnetter, CQG **20** (2003) 4719
- Irreducible mass $M_{irr} = \sqrt{\frac{A_{AH}}{16\pi G^2}}$
- BH mass in $D = 4$: $M^2 = M_{irr}^2 + \frac{S^2}{4M_{irr}^2}(+P^2)$,
  where S is the spin of the BH, Christodoulou, PRL **25** (1970) 1596

Gravitational waves in $D = 4$: Newman Penrose

- Construct a Tetrad
  - $n^\alpha =$ Timelike unit normal field
  - Spatial triad $u,\ v,\ w$ through Gram-Schmidt orthogonalization

    E.g. starting with $u^i = [x, y, z]$, $\quad v^i = [xz, yz, -x^2 - y^2]$,

    $$w^i = \epsilon^i{}_{mn} v^m w^n$$
  - $\ell^\alpha = \frac{1}{\sqrt{2}}(n^\alpha + u^\alpha)$, $\quad k^\alpha = \frac{1}{\sqrt{2}}(n^\alpha - u^\alpha)$, $\quad m^\alpha = \frac{1}{\sqrt{2}}(v^\alpha + i w^\alpha)$

    $\Rightarrow -\ell \cdot k = 1 = m \cdot \bar{m}$, $\quad$ all other products vanish
- Newman-Penrose scalar $\Psi_4 = C_{\alpha\beta\gamma\delta} k^\alpha \bar{m}^\beta k^\gamma \bar{m}^\delta$
- In vacuum, $C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}$
- For more details, see e.g.

  Nerozzi, PRD **72** (2005) 024014, Brügmann et al, PRD **77** (2008) 024027

Analysis of Ψ_4

- Multipolar decomposition: $\Psi_4 = \sum_{\ell,m} \psi_{\ell m}(t,r) Y^{-2}_{\ell m}(\theta\phi)$,

  where $\psi_{\ell m} = \int_0^{2\pi} \int_0^{\pi} \Psi_4 \overline{Y^{-2}_{\ell m}} \sin\theta d\theta d\phi$

- Radiated energy: $\frac{dE}{dt} = \lim_{r\to\infty} \left[\frac{r^2}{16\pi} \int_\Omega \left| \int_{-\infty}^{t} \Psi_4 d\tilde{t} \right|^2 d\Omega \right]$

- Momentum: $\frac{dP_i}{dt} = -\lim_{r\to\infty} \left[\frac{r^2}{16\pi} \int_\Omega \ell_i \left| \int_{-\infty}^{t} \Psi_4 d\tilde{t} \right|^2 d\Omega \right]$,

  where $\ell_i = [-\sin\theta\ \cos\phi,\ -\sin\theta\ \sin\phi,\ -\cos\theta]$

- Angular mom.: $\frac{dJ_z}{dt} =$

  $-\lim_{r\to\infty} \left\{ \frac{r^2}{16\pi} \mathrm{Re} \left[\int_\Omega \left(\partial_\phi \int_{-\infty}^{t} \Psi_4 d\tilde{t} \right) \left(\int_{-\infty}^{t} \int_{-\infty}^{\hat{t}} \overline{\Psi}_4 d\tilde{t} d\hat{t} \right) d\Omega \right] \right\}$

see e.g. Ruiz et al, GRG **40** (2008) 2467

Wave extraction in $D \geq 4$

- Newman Penrose formalism: Algebraically special not as powerful
- Landau-Lifshitz pseudo tensor $\Rightarrow$ radiated energy

  Yoshino & Shibata, PRD **80** (2009) 084025, PTPS **189** (2011) 269
- Regge-Wheeler-Zerilli-Moncrief formalism in $D = 4$

  Regge & Wheeler, PR **108** (1957) 1063, Zerilli, PRL **24** (1970) 737, Moncrief, Ann.Phys. **88** (1974) 323

  For applications in NR see e.g.

  Reisswig et al, PRD **83** (2011) 064008, Sperhake et al, PRD **71** (2005) 124042 Rezzolla, gr-qc/0302025
- Generalization to $D > 4 \Rightarrow$ KI formalism

  Kodama & Ishibashi, PTP **110** (2003) 701

The Kodama-Ishibashi formalism

- We follow notation from Witek et al, PRD **82** (2010) 104014
- Assumption: Metric $\approx$ spherically symmetric far from GW sources
- Coordinates adapted to rotational symmetry on S^{D-2}:
  $(t, r, \theta, \bar{\theta}, \phi^1, \ldots, \phi^{D-4})$
- Tangherlini background metric Tangherlini, Nuovo Cim. **27** (1963) 636
  $ds^2_{(0)} = -A(r)dt^2 + A(r)^{-1}dr^2 + r^2[d\bar{\theta}^2 + \sin^2\bar{\theta}(d\theta^2 + \sin^2\theta\, d\Omega_{D-4})]$
  where $A(r) = \left(1 - \frac{r_s^{D-3}}{r^{D-3}}\right)^{-1}$
- Perturbation: $ds^2_{(1)} = h_{ab}dx^a dx^b + h_{a\bar{\theta}}dx^a d\bar{\theta} + h_{\bar{\theta}\bar{\theta}}d\bar{\theta}^2 + h_{\theta\theta}d\Omega_{D-3}$
  where $x^a = (t, r)$

The Kodama-Ishibashi formalism: Axisymmetry

- We now consider the case of $SO(D-2)$ symmetry

  $\Rightarrow h_{a\theta} = h_{\bar{\theta}\theta} = 0$,

  perturbative expansion contains only Scalar harmonics

- Scalar harmonics: $\mathcal{S}(\phi^{\bar{i}})$ which satisfy

  $\Box\mathcal{S} = -k^2\mathcal{S}, \quad k = \ell(\ell + D - 3), \ \ \ell = 0, 1, 2, \ldots$

  where $\Box$ refers to the background metric $\gamma_{\bar{i}\bar{j}}$ induced onto $(\bar{\theta}, \theta, \phi^1, \ldots, \phi^{D-4})$

- Def.: $\mathcal{S}_{\bar{i}} = -\frac{1}{k}\partial_i\mathcal{S}, \quad S_{\bar{i}\bar{j}} = \frac{1}{k^2}\bar{D}_i\partial_j\mathcal{S} + \frac{1}{D-2}\gamma_{\bar{i}\bar{j}}\mathcal{S}$

The Kodama-Ishibashi formalism: Axisymmetry

- The metric perturbations can be written as

  $h_{ab} = f_{ab}\mathcal{S}, \quad h_{a\bar{i}} = r f_a \mathcal{S}_{\bar{i}}, \quad h_{\bar{i}\bar{j}} = 2r^2(H_L \gamma_{\bar{i}\bar{j}}\mathcal{S} + H_T \mathcal{S}_{\bar{i}\bar{j}}),$

  where f_{ab}, f_a, H_L, H_T are functions of (t, r).

  They are obtained from projecting metric components onto spherical harmonics.

- Note: We are suppressing indices ℓ, m here!

  We mean $h^{\ell m}_{ab} = f^{\ell m}_{ab}\mathcal{S}_{\ell m}$, etc.

- Gauge invariant functions: $F = H_L + \frac{1}{D-2} H_T + \frac{1}{r} X_a \hat{D}^a r$

  $F_{ab} = f_{ab} + \hat{D}_b X_a + \hat{D}_a X_b$

  where $X_a = \frac{r}{k}\left(f_a + \frac{r}{k}\hat{D}_a H_T\right)$

  and $\hat{D}_a =$ cov.deriv. of the (t, r) subsector of the backgr. metric

The Kodama-Ishibashi formalism: Master function

- Master function:

$$\partial_t \Phi = (D-2) r^{(D-4)/2} \frac{-F^r{}_t + 2r\partial_t F}{k^2 - D + 2 + \frac{(D-2)(D-1)}{2} \frac{r_s^{D-3}}{r^{D-3}}}$$

- Energy flux; we restore the index ℓ, ($m = 0$ in axisymmetry)

$$\frac{dE_\ell}{dt} = \frac{1}{32\pi} \frac{D-3}{D-2} k^2 (k^2 - D + 2)(\partial_t \Phi^\ell)^2$$

- Total radiated energy:

$$E = \sum_{\ell=2}^{\infty} \int_{-\infty}^{\infty} \frac{dE_\ell}{dt} dt$$

The AdS/CFT dictionary: Fefferman-Graham coords.

- AdS/CFT correspondence

  $\Rightarrow$ Vacuum expectation values $\langle T_{ij} \rangle$ of the field theory given by quasi-local Brown-York stress-energy tensor

  Brown & York, PRD **47** (1993) 1407

- Consider asymptotically AdS metric in Fefferman-Graham coordinates

  $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{L^2}{r^2}(dr^2 + \gamma_{ij} dx^i dx^j)$,

  where

  $\gamma_{ij}(r, x^i) = \gamma_{(0)ij} + r^2 \gamma_{(2)ij} + \cdots + r^D \gamma_{(D)ij} + h_{(D)ij} r^D \log r^2 + \mathcal{O}(r^{D+1})$,

- Note: This asymptotes to Poincaré coordinates as $r \to 0$

The AdS/CFT dictionary: Fefferman-Graham coords.

- Here, the $\gamma_{(a)ij}$, $h_{(D)ij}$ are functions of x^i,
  logarithmic terms only appear for even D,
  powers of r are exclusively even up to order $D-1$
- Vacuum expectation values of CFT momentum tensor for $D=4$ is

  $$\langle T_{ij}\rangle = \frac{4L^3}{16\pi G}\Big\{\gamma_{(4)ij} - \frac{1}{8}\gamma_{(0)ij}[\gamma_{(2)}^2 - \gamma_{(0)}^{km}\gamma_{(0)}^{ln}\gamma_{(2)kl}\gamma_{(2)mn}] - \frac{1}{2}\gamma_{(2)i}{}^m\gamma_{(2)jm} + \frac{1}{4}\gamma_{(2)ij}\gamma_{(2)}\Big\}$$

  where $\gamma_{(n)} \equiv \mathrm{Tr}(\gamma_{(n)ij}) = \gamma_{(0)}^{ij}\gamma_{(n)ij}$

  de Haro et al, Commun.Math.Phys. **217** (2001) 595;   also for other D
- Note: $\gamma_{(2)ij}$ is determined by $\gamma_{(0)ij} \Rightarrow$ CFT freedom given by $\gamma_{(4)ij}$

AdS/CFT: Renormalized stress-tensor

- Again: Brown-York stress-tensor $\rightarrow$ as the VEVs of the field theory
- Divergencies in $\langle T^{ab} \rangle = \frac{\delta S_{\text{eff}}}{\delta \gamma_{ab}}$

  Regularize by adding boundary curvature invariants to S_{eff}

  Balasubramanian & Kraus, Commun.Math.Phys. **208** (1999) 413
- Foliate D dimensional spacetime into timelike hypersurfaces Σ_r

  homoemorphic to the boundary

  $\Rightarrow ds^2 = \alpha^2 dr^2 + \gamma_{ab}(dx^a + \beta^a dr)(dx^b + \beta^b dr)$ (like ADM)
- $\hat{n}^\mu =$ outward pointing normal vector to the boundary

  $\Theta^{\mu\nu} = -\frac{1}{2}(\nabla^\mu \hat{n}^\nu + \nabla^\nu \hat{n}^\mu)$ Extrinsic curvature

AdS/CFT: Renormalized stress-tensor

- Including counter terms, for ADS_5:

  $T^{\mu\nu} = \frac{1}{8\pi G}\left[\Theta^{\mu\nu} - \Theta\gamma^{\mu\nu} - \frac{3}{L}\gamma^{\mu\nu} - \frac{L}{2}\mathcal{G}^{\mu\nu}\right]$

  where $\mathcal{G}_{\mu\nu}$ is the Einstein tensor of the induced metric $\gamma_{\mu\nu}$

- Note: Applying this to the global ADS_5 metric gives $T^{\mu\nu} \neq 0$

  $\Rightarrow$ Casimir energy of quantum field theory on $S^3 \times \mathbb{R}$

- Other D: cf. Balasubramanian & Kraus, Commun.Math.Phys. **208** (1999) 413

- AdS/CFT Dictionary for additional fields, see e.g.

  Skenderis, CQG **19** (2002) 5849

  de Haro et al, Commun.Math.Phys. **217** (2001) 595

Further reading

- Isolated and dynamical horizons

  Ashtekar & Krishnan, Liv. Rev. Rel. **7** (2004) 10

3 Results from BH evolutions

3.1 BHs in GW physics

Gravitational waves

- Weak field limit: $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$
- Trace reversed perturbation $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} h \eta_{\alpha\beta}$

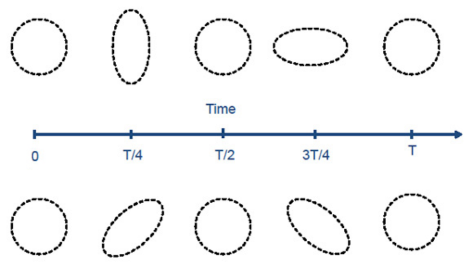
  $\Rightarrow$ Vacuum field eqs.: $\Box \bar{h}_{\alpha\beta} = 0$

- Apropriate gauge $\Rightarrow$

$$\bar{h}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ik_\sigma x^\sigma}$$

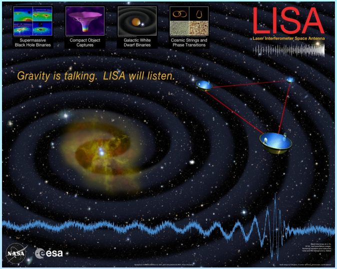
  where k^σ = null vector

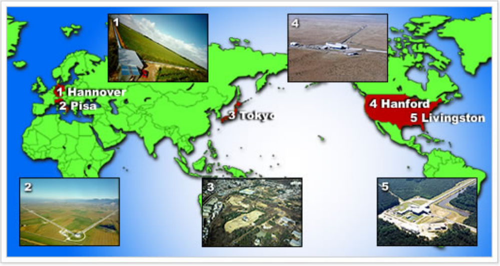
- GWs displace particles

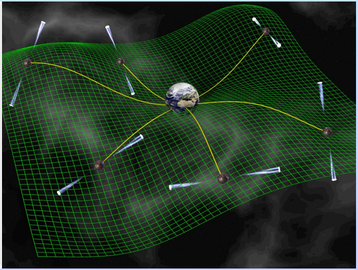


Gravitational wave detectors

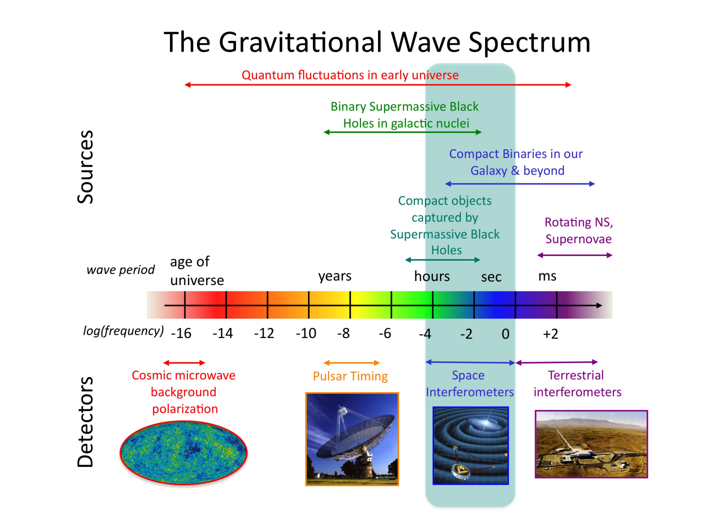
- Accelerated masses ⇒ GWs
- Weak interaction!
- Laser interferometric detectors







The gravitational wave spectrum



Some targets of GW physics

- Confirmation of GR

  Hulse & Taylor 1993 Nobel Prize

- Parameter determination of BHs: M, $\vec{S}$

- Optical counter parts

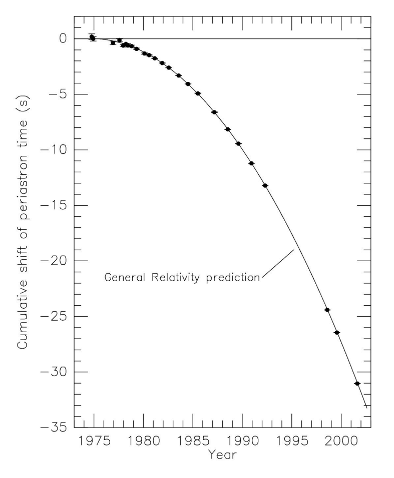
  Standard sirens (candles)

  Mass of graviton

- Test Kerr Nature of BHs

- Cosmological sources

- Neutron stars: EOS



Free parameters of BH binaries

- Total mass M

  Relevant for GW detection: Frequencies scale with M

  Not relevant for source modeling: trivial rescaling

- Mass ratio $q \equiv \frac{M_1}{M_2}$, $\eta \equiv \frac{M_1 M_2}{(M_1+M_2)^2}$

- Spin: $\vec{S}_1$, $\vec{S}_2$ (6 parameters)

- Initial parameters

  Binding energy E_b — Separation

  Orbital ang. momentum L — Eccentricity

  Alternatively: frequency, eccentricity

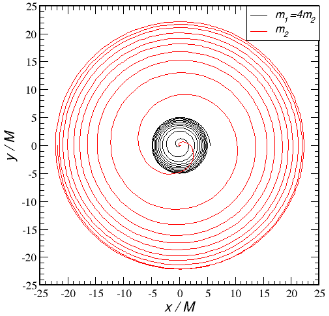
BBH trajectory and waveform

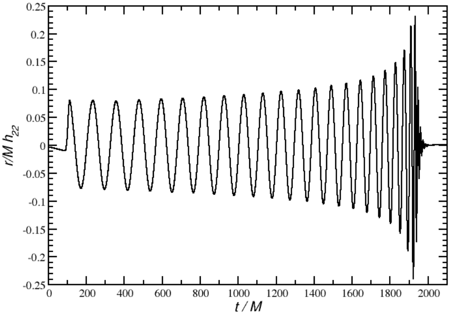
- $q = 4$, non-spinning binary; ~ 11 orbits

  US, Brügmann, Müller & Sopuerta '11

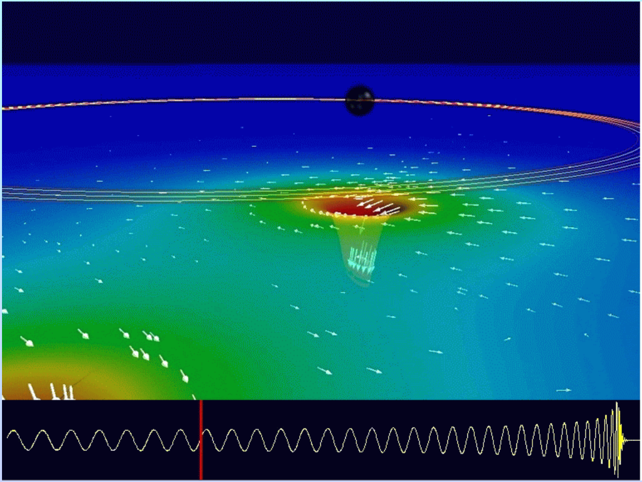
Trajectory

Quadrupole mode





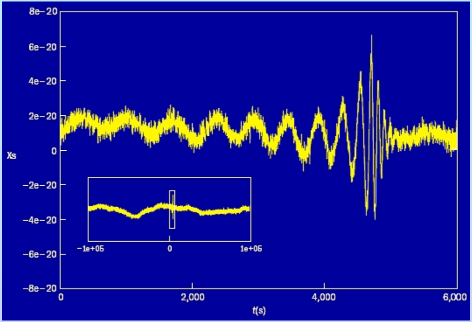
Morphology of a BBH inspiral



Thanks to Caltech, Cornell, CITA

Matched filtering

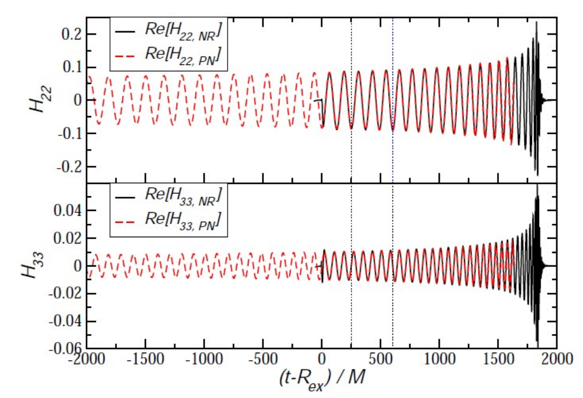
- BH binaries have 7 parameters: 1 mass ratio, 2×3 for spins
- Sample parameter space, generate waveform for each point



- NR + PN
- Effective one body
- Ninja, NRAR Projects
- GEO 600 noise
  chirp signal

Template construction

- Stitch together PN and NR waveforms
- EOB or phenomenological templates for $\geq$ 7-dim. par. space



Template construction

- Phenomenological waveform models
  - Model phase, amplitude with simple functions $\rightarrow$ Model parameters
  - Create map between physical and model parameters
  - Time or frequency domain

  Ajith et al, CQG **24** (2007) S689, PRD **77** (2008) 104017, CQG **25** (2008) 114033, PRL **106** (2011) 241101; Santamaria et al, PRD **82** (2010) 064016, Sturani et al, arXiv:1012.5172 [gr-qc]

- Effective-one-body (EOB) models
  - Particle in effective metric, PN, ringdown model
    Buonanno & Damour PRD **59** (1999) 084006, PRD **62** (2000) 064015
  - Resum PN, calibrate pseudo PN parameters using NR
    Buonanno et al, PRD **77** (2008) 026004, Pan et al, PRD **81** (2010) 084041, PRD **84** (2012) 124052; Damour et al, PRD **77** (2008) 084017, PRD **78** (2008) 044039, PRD **83** (2011) 024006

The Ninja project

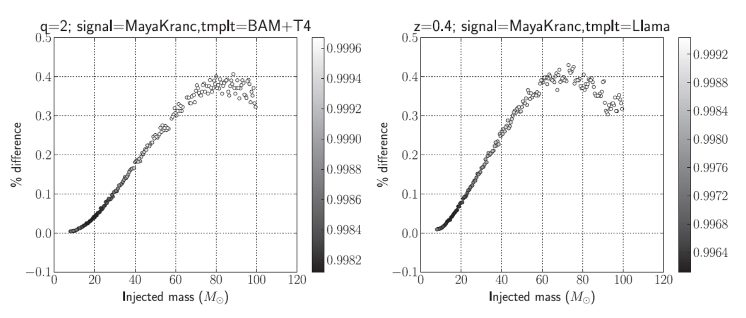
https://www.ninja-project.org/

Aylott et al, CQG **26** (2009) 165008, CQG **26** (2009) 114008

Ajith et al, CQG **29** (2012) 124001

- Use PN/NR hybrid waveforms in GW data analysis
- Ninja2: 56 hybrid waveforms from 8 NR groups
- Details on hybridization procedures
- Overlap and mass bias study:
  - Take one waveform as signal, fixing M_{tot}
  - Search with other waveform (same config.) varying $t_0,\ \phi_0,\ M_{tot}$

The Ninja project



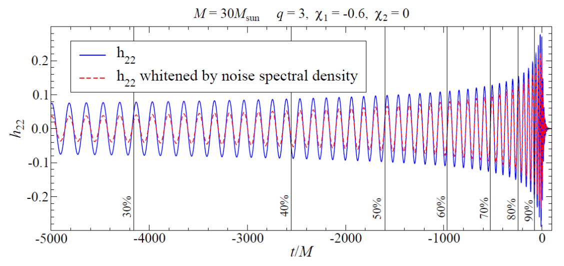
- Left: $q = 2$, non-spinning waveforms, MayaKranc, BAM + T4
- Right: $q = 1$, $\chi_1 = \chi_2 = 0.4$ waveform, MayaKranc, Llama + T4
- Mass bias < 0.5 %

The NRAR project

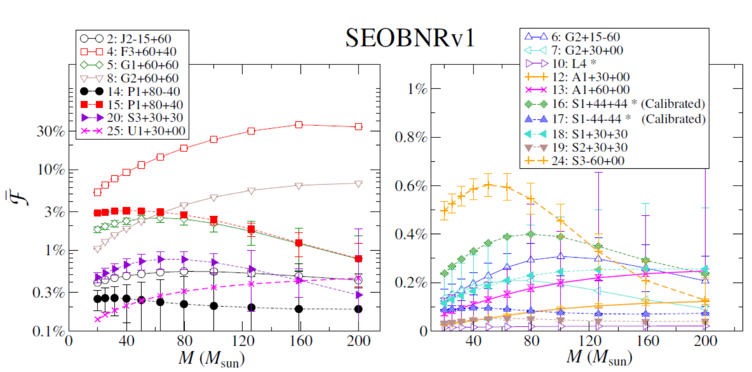
https://www.ninja-project.org/doku.php?id=nrar:home

Hinder, Buonanno et al, in preparation

- Pool efforts from 9 NR groups
- 11M core hours on XSEDE Kraken
- 22 waveforms, including precessing runs
- Common, automatized analysis, uncertainty measures



The NRAR project

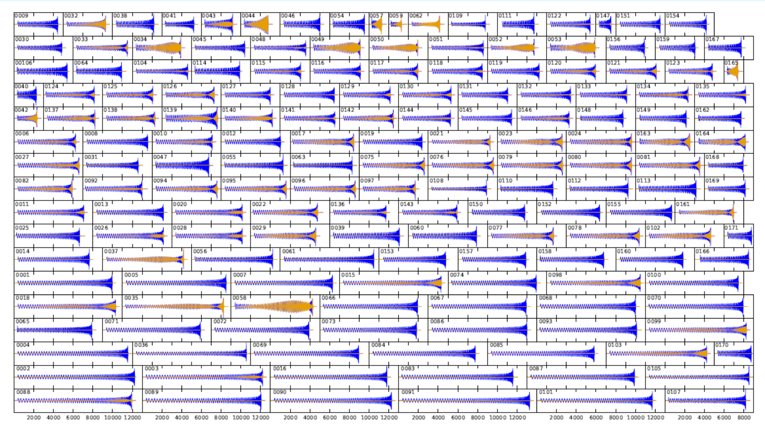


- Unfaithfulness $\bar{\mathcal{F}} = 1-$ best overlap varying $t_0,\ \phi_0$
- $\bar{\mathcal{F}}$ between SEOBNRv1 and NR waveforms

Tools of mass production

- SpEC catalog: 171 waveforms: $q \leq 8$, 90 precessing, ≤ 34 orbits

  Mroué et al, arXiv:1304.6077[gr-qc]



Strategies in parameter space

- SpEC: 16 orbits in 40 hours
- Still, 7-dimensional parameter space $\rightarrow$ $N \sim 10^7$ waveforms?
- Probably too many...
- Reduce # of parameters describing dominant spin effects

  Ajith et al, PRL **106** (2011) 241101, PRD **84** (2011) 084037,

  Pürrer et al, arXiv:1306.2320 [gr-qc]
- Spin-robit resonances $\Rightarrow$ preferred regions in parameter space?

  Gerosa et al, arXiv:1302.4442 [gr-qc]
- Trade-off: Quantity or quality of waveforms?

  Both affects parameter estimation!

Limits in the parameter space

- Mass ratio $q = 100$

  Lousto & Zlochower, PRL **106** (2011) 041101

  Head-on case: Sperhake et al, PRD **84** (2011) 084038

- Spin magnitude $\chi = 0.97$

  Superposed Kerr-Schild data (non-conformally flat)

  Lovelace et al, CQG **29** (2012) 045003

- Separations $D = 100\ M$; few orbits

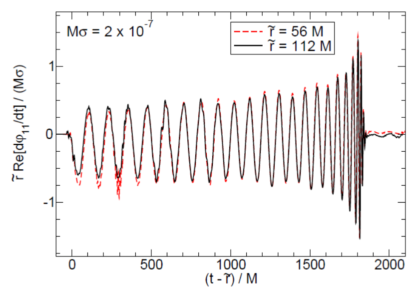
  Lousto & Zlochower, arXiv:1304.3937 [gr-qc]

Going beyond GR: Scalar-tensor theory of gravity

- Brans-Dicke theory: 1 parameter ω_{BD}; well constrained
- Bergmann-Wagoner theories: Generalize $\omega = \omega(\phi)$
- No-hair theorem: BHs solutions same as in GR
  e.g. Hawking, Comm.Math.Phys. **25** (1972) 167
  Sotiriou & Faraoni, PRL **108** (2012) 081103
- Circumvent no-hair theorem: Scalar bubble

  Healey et al, arXiv:1112.3928 [gr-qc]
- Circumvent no-hair theorem: Scalar gradient

  Berti et al, arXiv:1304.2836 [gr-qc]



3.2 BHs in Astrophysics

Evidence for astrophysical black holes

- X-ray binaries

  e. g. Cygnus X-1 (1964)

  MS star + compact star

  $\Rightarrow$ Stellar Mass BHs
  $\sim 5 \ldots 50\ M_\odot$

- Stellar dynamics

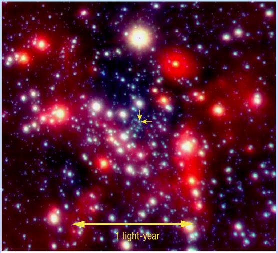
  near galactic centers,

  iron emission line profiles

  $\Rightarrow$ Supermassive BHs

  $\sim 10^6 \ldots 10^9\ M_\odot$

  AGN engines

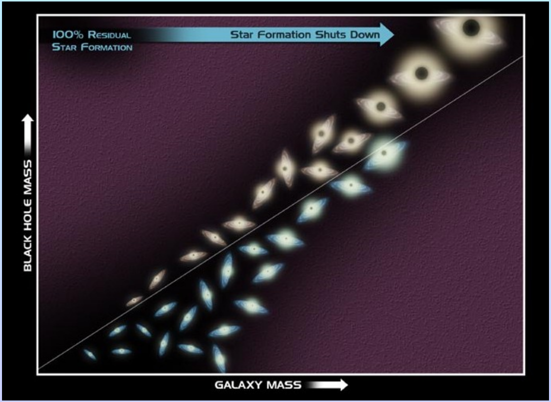


The Centre of the Milky Way
(VLT YEPUN + NACO)
ESO PR Photo 23a/02 (9 October 2002)
©European Southern Observatory

Correlation of BH and host galaxy properties

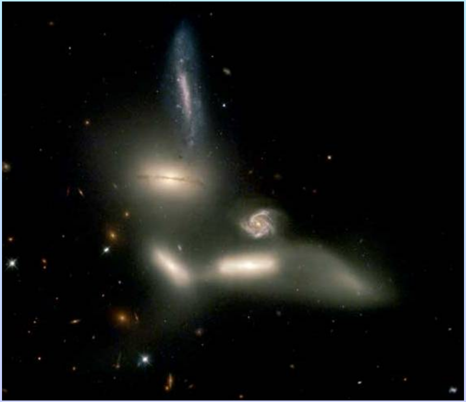
- Galaxies ubiquitously harbor BHs
- BH properties correlated with bulge properties

  e. g. J.Magorrian *et al.*, AJ 115, 2285 (1998)



SMBH formation

- Most widely accepted scenario for galaxy formation: hierarchical growth; “bottom-up”
- Galaxies undergo frequent mergers $\Rightarrow$ BH merger



Gravitational recoil

- Anisotropic GW emission ⇒ recoil of remnant BH

  Bonnor & Rotenburg '61, Peres '62, Bekenstein '73

- Escape velocities:

Globular clusters	30 $\mathrm{km/s}$
dSph	20 − 100 $\mathrm{km/s}$
dE	100 − 300 $\mathrm{km/s}$
Giant galaxies	$\sim$ 1000 $\mathrm{km/s}$

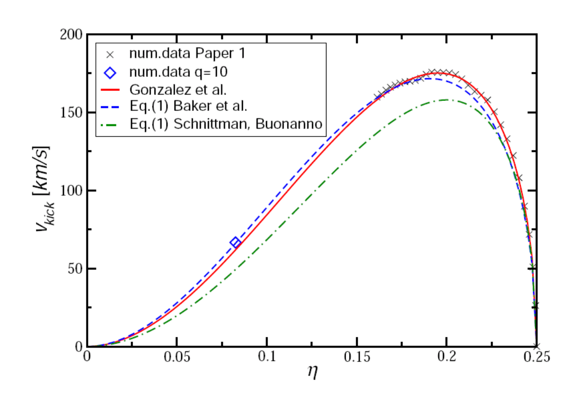
Ejection / displacement of BH ⇒

- Growth history of SMBHs
- BH populations, IMBHs
- Structure of galaxies

Kicks from non-spinning BHs

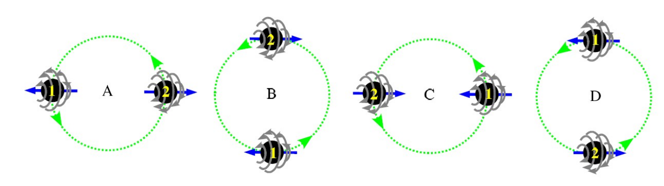
- Max. kick: $\sim 180\ km/s$, harmless!

  González et al., PRL 98, 091101 (2009)



Spinning BHs: Superkicks

- Superkick configuration:



- Kicks up to $v_{\max} \approx 4\,000\ \mathrm{km/s}$

  Campanelli *et al.*, PRL **98** (2007) 231102

  González *et al.* PRL **98** (2007) 231101

- Suppression via spin alignment and Resonance effects in inspiral
  Schnittman, PRD **70** (2004) 124020

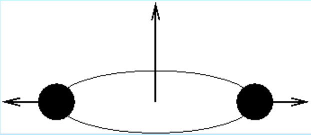
  Bogdanovicź et al, ApJ **661** (2007) L147

  Kesden et al, PRD **81** (2010) 084054, ApJ **715** (2010) 1006

Even larger kicks: superkick and hang-up

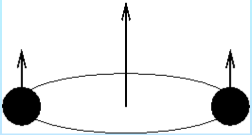
Lousto & Zlochower, arXiv:1108.2009 [gr-qc]

Superkicks

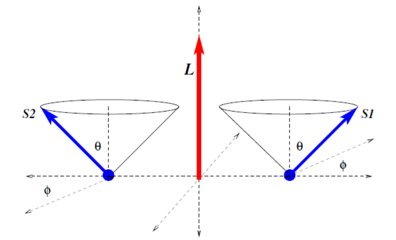


- Moderate GW generation
- Large kicks

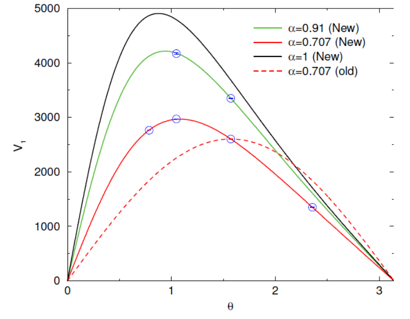
Hangup



- Strong GW generation
- No kicks



Superkicks and orbital hang-up



- Maximum kick about 25 % larger: $v_{\rm max} \approx 5\,000$ km/s
- Distribution asymmetric in θ; v_{max} for partial alignment
- Supression through resonances still works

Berti et al, PRD **85** (2012) 124049

Spin precession and flip

- X-shaped radio sources

  Merrit & Ekers, Science **297** (2002) 1310

- Jet along spin axis

- Spin re-alignment

  $\Rightarrow$ new + old jet

- Spin precession 98°

  Spin flip 71°

  Campanelli et al, PRD **75** (2006) 064030

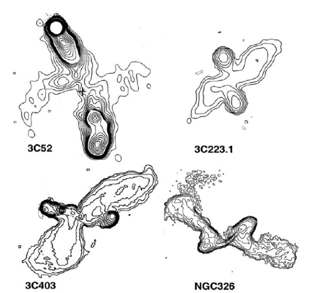
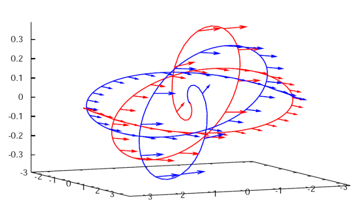


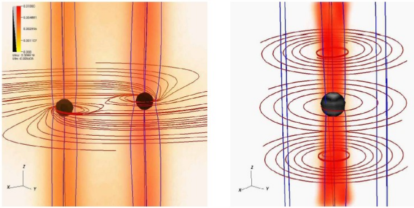
Fig. 2. A composite of four clear examples of the X-shape morphology. VLA radio observations of 3C52 (28), 3C223.1 (29), 3C403 (29), and NGC 326 (12).



Jets generated by binary BHs

Palenzuela et al, PRL **103** (2009) 081101, Science **329** (2010) 927

- Non-spinning BH binary
- Einstein-Maxwell equtions with "force free" plasma
- Electromagnetic field extracts energy from **L** $\Rightarrow$ jets
- Optical signature: double jets



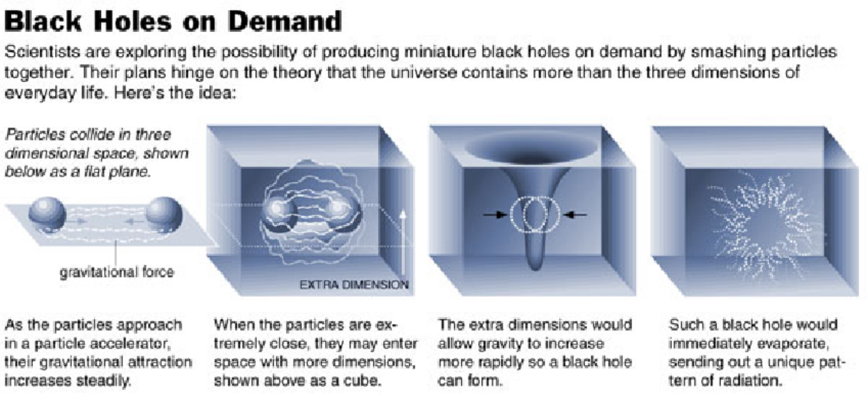
(a) $-8.2\,M_8$ hrs (b) $4.6\,M_8$ hrs

3.3. High-energy collisions of BHs

The Hierarchy Problem of Physics

- Gravity $\approx 10^{-39} \times$ other forces
- Higgs field $\approx \mu_{obs} \approx 250\ GeV = \sqrt{\mu^2 - \Lambda^2}$
  where $\Lambda \approx 10^{16}\ GeV$ is the grand unification energy
- Requires enormous finetuning!!!
- Finetuning exist: $\frac{987654321}{123456789} = 8.0000000729$
- Or E_{Planck} much lower? Gravity strong at small r?
  $\Rightarrow$ BH formation in high-energy collisions at LHC
- Gravity not measured below $0.16\ mm$! Diluted due to...
  - Large extra dimensions   Arkani-Hamed, Dimopoulos & Dvali '98
  - Extra dimension with warp factor   Randall & Sundrum '99

Stages of BH formation



- Matter does not matter at energies well above the Planck scale

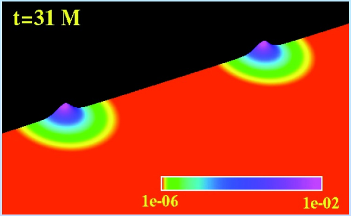
$\Rightarrow$ Model particle collisions by black-hole collisions

Banks & Fischler, gr-qc/9906038; Giddings & Thomas, PRD **65** (2002) 056010

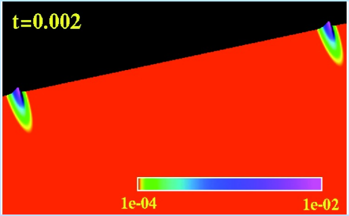
Does matter “matter”?

- Hoop conjecture $\Rightarrow$ kinetic energy triggers BH formation
- Einstein plus minimally coupled, massive, complex scalar filed

“Boson stars”   Pretorius & Choptuik, PRL **104** (2010) 111101



$\gamma = 1$



$\gamma = 4$

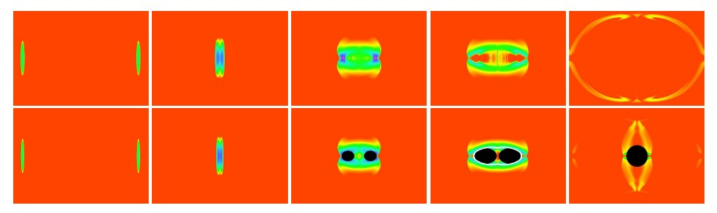
- BH formation threshold: $\gamma_{\rm thr} = 2.9 \pm 10\ \% \sim 1/3\ \gamma_{hoop}$
- Model particle collisions by BH collisions

Does matter "matter"?

- Perfect fluid "stars" model
- $\gamma = 8 \ldots 12$; BH formation below Hoop prediction

  East & Pretorius, PRL **110** (2013) 101101

- Gravitational focussing $\Rightarrow$ Formation of individual horizons



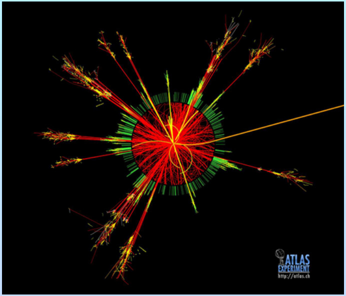
- Type-I critical behaviour
- Extrapolation by 60 orders would imply no BH formation at LHC

  Rezzolla & Tanaki, CQG **30** (2013) 012001

Experimental signature at the LHC

Black hole formation at the LHC could be detected by the properties of the jets resulting from Hawking radiation. `BlackMax`, `Charybdis`

- Multiplicity of partons: Number of jets and leptons
- Large transverse energy
- Black-hole mass and spin are important for this!



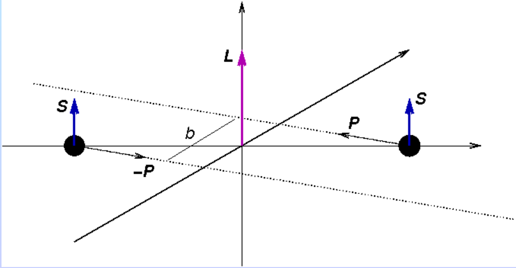
ToDo:

- Exact cross section for BH formation
- Determine loss of energy in gravitational waves
- Determine spin of merged black hole

$D = 4$: Initial setup: 1) Aligned spins

- Orbital hang-up   Campanelli et al, PRD **74** (2006) 041501
- 2 BHs: Total rest mass: $M_0 = M_{\mathrm{A},\,0} + M_{\mathrm{B},\,0}$

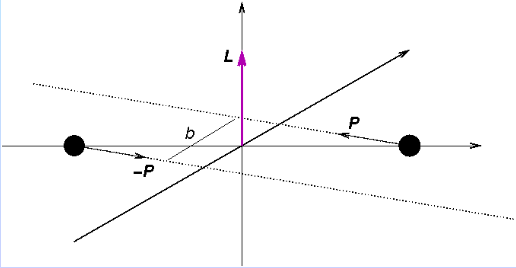
  Boost: $\gamma = 1/\sqrt{1-v^2}$, $M = \gamma M_0$
- Impact parameter: $b \equiv \frac{L}{P}$



$D = 4$: Initial setup: 2) No spins

- Orbital hang-up   Campanelli et al, PRD **74** (2006) 041501
- 2 BHs: Total rest mass: $M_0 = M_{\mathrm{A},\,0} + M_{\mathrm{B},\,0}$

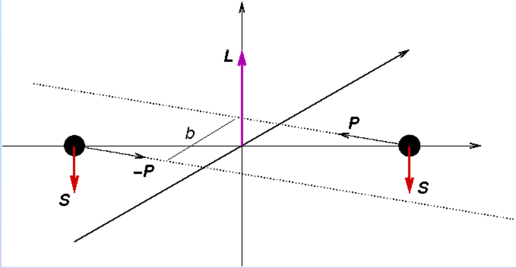
  Boost: $\gamma = 1/\sqrt{1-v^2}$, $M = \gamma M_0$
- Impact parameter: $b \equiv \frac{L}{P}$



$D = 4$: Initial setup: 3) Anti-aligned spins

- Orbital hang-up   Campanelli et al, PRD **74** (2006) 041501
- 2 BHs: Total rest mass: $M_0 = M_{\mathrm{A},\,0} + M_{\mathrm{B},\,0}$

  Boost: $\gamma = 1/\sqrt{1-v^2}$, $M = \gamma M_0$
- Impact parameter: $b \equiv \frac{L}{P}$

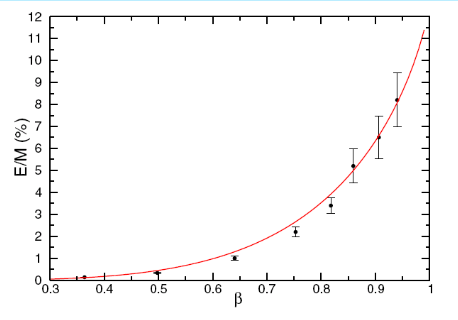


$D = 4$: Head-on: $b = 0$, $\vec{S} = 0$

- Total radiated energy: 14 ± 3 % for $v \to 1$

  US et al, PRL **101** (2008) 161101

  About half of Penrose '74



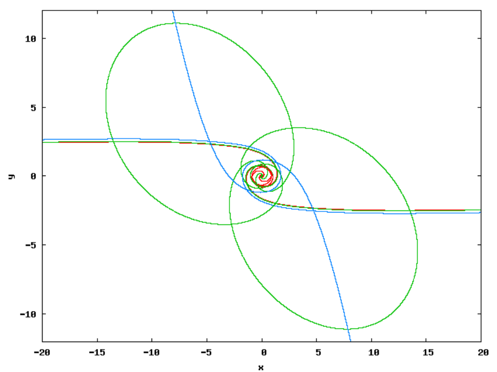
- Agreement with approximative methods

  Flat spectrum, GW multipoles   Berti et al, PRD **83** (2011) 084018

$D = 4$: Grazing: $b \neq 0, \quad \vec{S} = 0, \quad \gamma = 1.52$

- Radiated energy up to at least 35 % M
- Immediate vs. Delayed vs. No merger

US et al, PRL **103** (2009) 131102



$D = 4$: Scattering threshold b_{scat} for $\vec{S} = 0$

- $b < b_{\rm scat} \quad \Rightarrow \quad$ Merger

  $b > b_{\rm scat} \quad \Rightarrow \quad$ Scattering

- Numerical study: $\quad b_{\rm scat} = \frac{2.5 \pm 0.05}{v} M$

  Shibata et al, PRD **78** (2008) 101501(R)

- Independent study US et al, PRL **103** (2009) 131102, arXiv:1211.6114

  $\gamma = 1.23 \ldots 2.93$:

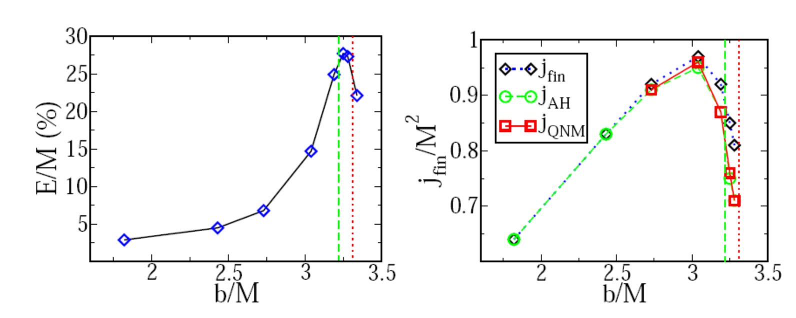
  $\chi = -0.6,\ 0,\ +0.6$ (anti-aligned, nonspinning, aligned)

- Limit from Penrose construction: $b_{\rm crit} = 1.685\ M$

  Yoshino & Rychkov, PRD **74** (2006) 124022

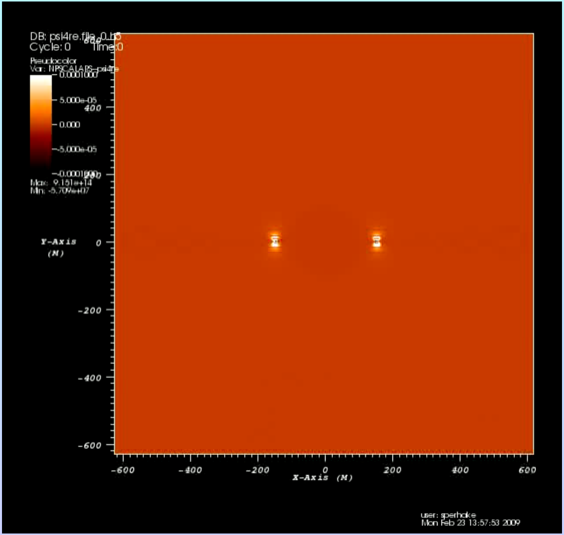
$D = 4$: Radiated quantities $b \neq 0, \quad \vec{S} = 0$

- $b-$sequence with $\gamma = 1.52$
- Threshold of immediate merger Pretorius & Khurana, CQG **24** (2007) S83
- $E_{\rm rad} \sim 35$ % for $\gamma = 2.93$; about 10 % of Dyson luminosity

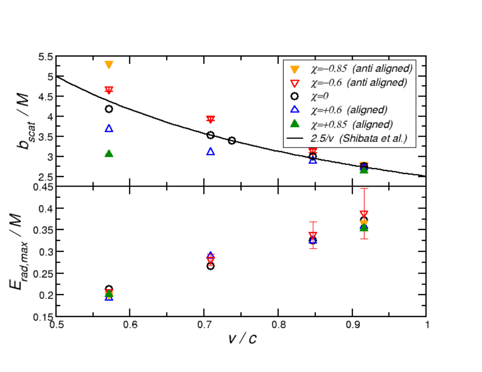


US *et al.*, PRL **103** (2009) 131102

$D = 4$: Gravitational radiation: Delayed merger



$D = 4$: Scattering threshold and radiated energy $\vec{S} \neq 0$

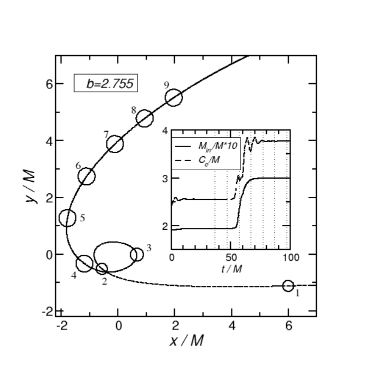


US et al, arXiv:1211.6114

- At speeds $v \gtrsim 0.9$ spin effects washed out
- E_{rad} always below $\lesssim 50\ \%\ M$

$D = 4$: Absorption

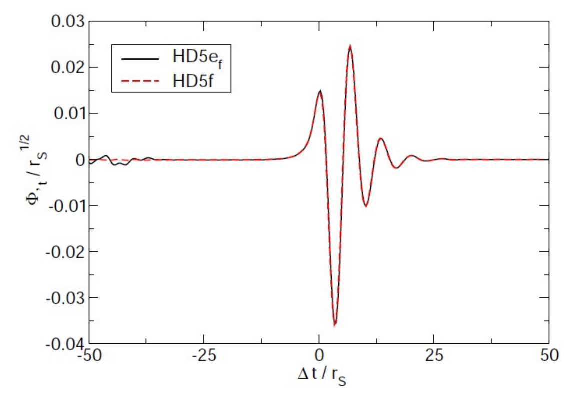
- For large γ: $E_{kin} \approx M$
- If E_{kin} is not radiated, where does it go?
- Answer: ~ 50 % into E_{rad}, ~ 50 % is absorbed



US et al, arXiv:1211.6114

$D = 5$: GWs from head-on

Wave extraction based on Kodama & Ishibashi, PTP **110** (2003) 701

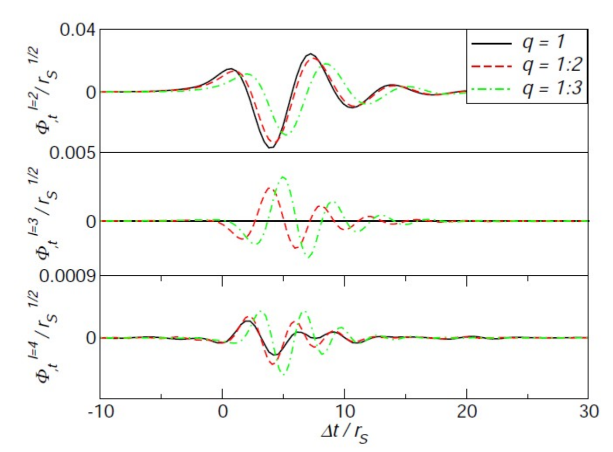


$E_{\rm rad} = 0.089\ \%M$ cf. $0.055\ \%M$ in $D = 4$

Witek *et al.*, PRD **82** (2010) 104014

$D = 5$: Unequal-mass head-on

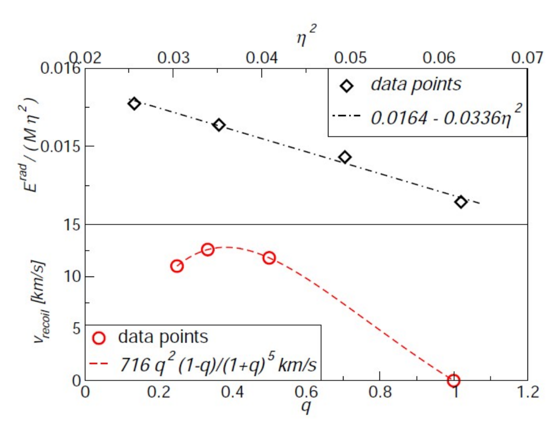
Kodama-Ishibashi multipoles



Witek *et al.*, PRD **83** (2011) 044017

$D = 5$: Unequal-mass head-on

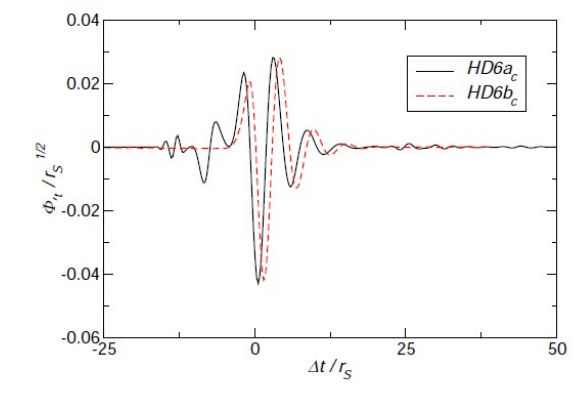
Radiated energy and momentum



Agreement within < 5 % with extrapolated point particle calculations

$D = 6$: First black-hole collisions

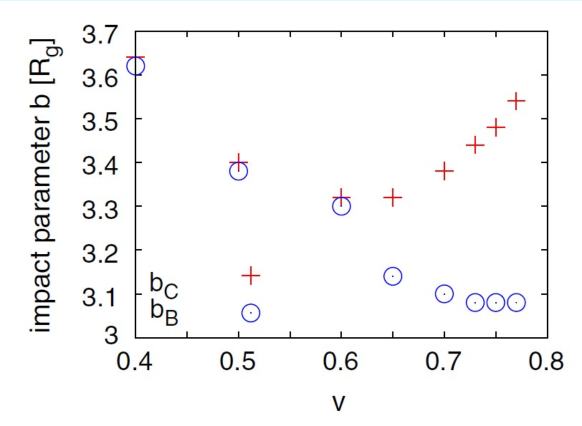
Witek *et al. '10*



- Adjust shift parameters
- Use LaSh system Witek et al, PRD **83** (2011) 104041

$D = 5$: Scattering threshold

Okawa et al, PRD **83** (2011) 121501

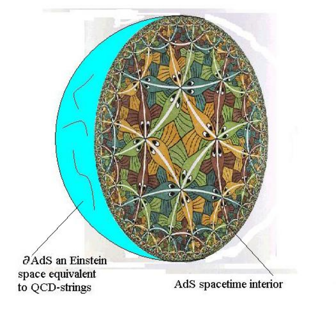


Numerical stability still an issue...

3.4. BH Holography

Large N and holography

- Holography
  - BH entropy $\propto A_{Hor}$
  - For a Local Field Theory
    entropy $\propto V$
  - Gravity in D dims
    $\Leftrightarrow$ local FT in $D-1$ dims



- Large N limit
  - Perturbative expansion of gauge theory in g^2N
    $\sim$ loop expansion in string theory
  - N: # of "colors"
    g^2N: t'Hooft coupling

The AdS/CFT conjecture

Maldacena, Adv.Theor.Math.Phys. **2** (1998) 231

- "strong form": Type IIb string theory on $AdS_5 \times S^5$

  $\Leftrightarrow \mathcal{N} = 4$ super Yang-Mills in $D = 4$

  Hard to prove; non-perturbative Type IIb String Theory?

- "weak form": low-energy limit of string-theory side

  $\Rightarrow$ Type IIb Supergravity on $AdS_5 \times S^5$

- Some assumptions, factor out S^5

  $\Rightarrow$ General Relativity on AdS_5

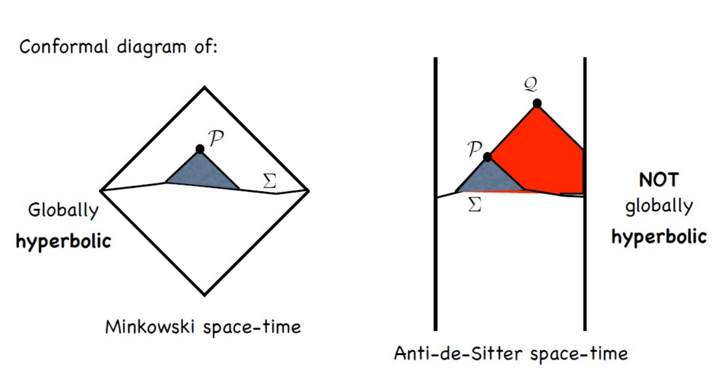
- Corresponds to limit of large N, $g^2 N$ in the field theory

- E. g. Stationary AdS BH $\Leftrightarrow$ Thermal Equil. with T_{Haw} in dual FT

  Witten, Adv.Theor.Math.Phys. **2** (1998) 253

The boundary in AdS

- Dictionary between metric properties and vacuum expectation values of CFT operators.
  E. g. $T_{\alpha\beta}$ operator of CFT $\leftrightarrow$ transverse metric on *AdS* boundary.
- The boundary plays an active role in *AdS*!   Metric singular!



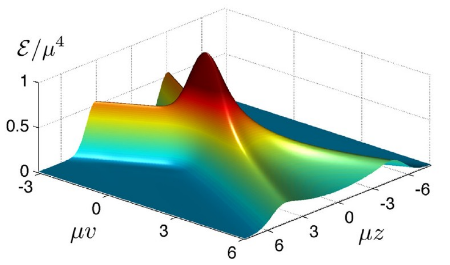
Collision of planar shockwaves in $\mathcal{N} = 4$ SYM

- Dual to colliding gravitational shock waves in AADS
- Characteristic study with translational invariance

  Chesler & Yaffe PRL **102** (2009) 211601, PRD **82** (2010) 026006, PRL **106** (2011) 021601

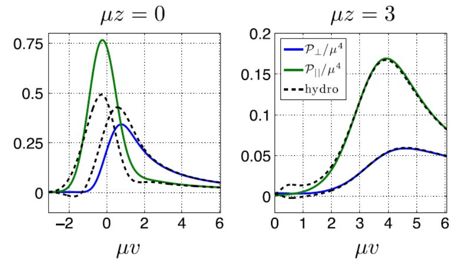
- Initial data: 2 superposed shockwaves

  $$ds^2 = r^2[-dx_+ dx_- + d\mathbf{x}_\perp] + \frac{1}{r^2}[dr^2 + h(x_\pm)dx_\pm^2]$$



Collision of planar shockwaves in $\mathcal{N} = 4$ SYM

- Initially system far from equilibrium
- Isotropization after $\Delta v \sim 4/\mu \sim 0.35\ fm/c$
- Confirms hydro sims. of QGP $\sim 1\ fm/c$ Heinz, nucl-th/0407067



- Non-linear vs. linear Einstein Eqs. agree within $\sim 20\ \%$
  Heller et al, PRL **108** (2012) 191601
- Thermalization in ADM formulation Heller et al, PRD **85** (2012) 126002

Cauchy ("4+1") evolutions in asymptotically AdS

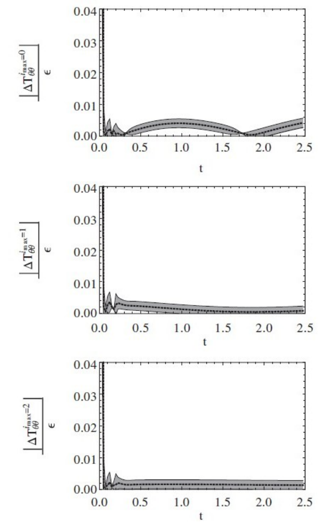
- Characteristic coordinates successful numerical tool in AdS/CFT
- But: restricted to symmetries, caustics problem...
- Cauchy evolution needed for general scenarios? Cf. BBH inspiral!!
- Cauchy scheme based on generalized harmonic formulation

  Bantilan & Pretorius, PRD **85** (2012) 084038

  - $SO(3)$ symmetry
  - Compactify "bulk radius"
  - Asymptotic symmetry of AdS_5: $SO(4,2)$
  - Decompose metric into AdS_5 piece and deviation
  - Gauge must preserve asymptotic fall-off

Cauchy (“4+1”) evolutions in asymptotically AdS

- Scalar field collapse
- BH formation and ringdown
- Low order QNMs $\sim$ perturbative studies, but mode coupling
- CFT stress-energy tensor consistent with thermalized $\mathcal{N} = 4$ SYM fluid
- Difference of CFT $T_{\theta\theta}$ and hydro ($+1^{st}$, 2^{nd} corrs.)



3.5 Fundamental properties of BHs

Stability of AdS

- $m = 0$ scalar field in as. flat spacetimes   Choptuik, PRL **70** (1993) 9

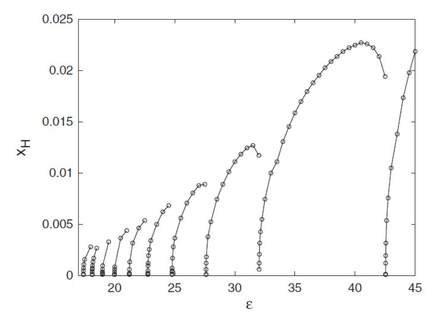
  $p > p^* \Rightarrow$ BH, $p < p^* \Rightarrow$ flat

- $m = 0$ scalar field in as. AdS Bizoń & Rostworowski, PRL **107** (2011) 031102

- Similar behaviour for "Geons"
  Dias, Horowitz & Santos '11

- $D > 4$ dimensions
  Jałmużna et al, PRD **84** (2011) 085021

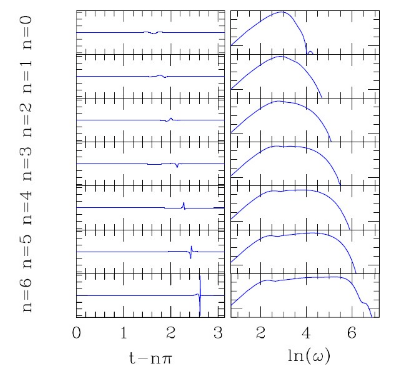
- $D = 3$: Mass gap: smooth solutions
  Bizoń & Jałmużna, arXiv:1306.0317



Stability of AdS

- Pulses narrow under successive reflections

  Buchel et al, PRD **86** (2012) 123011



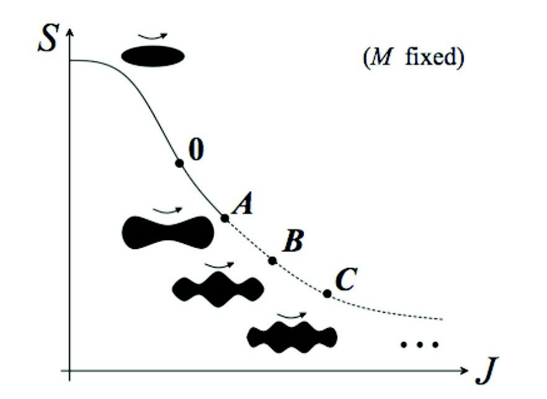
- ∃ Non-linearly stable solutions in AdS

  Dias et al, CQG **29** (2012) 235019, Buchel et al, arXiv:1304.4166,

  Maliborski & Rostworowski arXiv:1303.3186

Bar mode instability of Myers-Perry BH

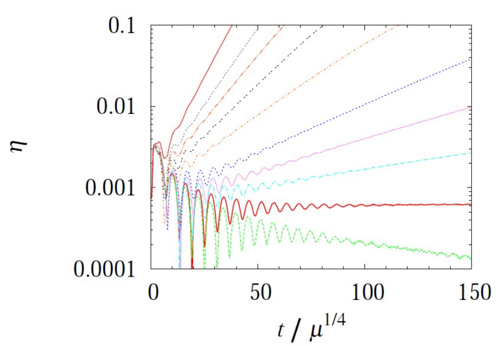
- MP BHs (with single ang.mom.) should be unstable.
- Linearized analysis Dias et al, PRD **80** (2009) 111701(R)



Non-linear analysis of MP instability

Shibata & Yoshino, PRD **81** (2010) 104035

- Myers-Perry metric; transformed to Puncture like coordinate
- Add small bar-mode perturbation
- Deformation $\eta := \frac{2\sqrt{(l_0 - l_{\pi/2})^2 + (l_{\pi/4} - l_{3\pi/4})^2}}{l_0 + l_{\pi/2}}$



Superradiant instability

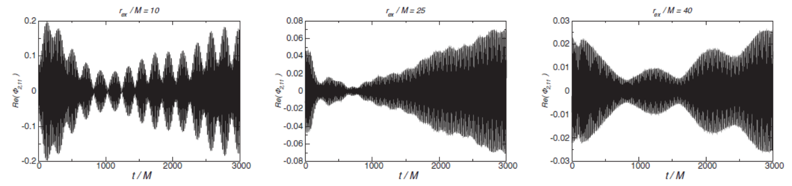
- Scattering of waves with $\mathrm{Re}[\omega]$ off BH with ang. horizon velocity Ω_H

  $\Rightarrow$ amplification $\Leftrightarrow$ $\mathrm{Re}[\omega] < m\Omega_H$

- Measure photon mass? Pani et al, PRL **109** (2012) 131102

- Numerical simulations Dolan, arXiv:1212.1477 Witek et al

- Instability of spinning BHs, Beating effects



Witek et al, PRD **87** (2013) 043513

Cosmic Censorship in $D = 5$

Pretorius & Lehner, PRL **105** (2010) 101102

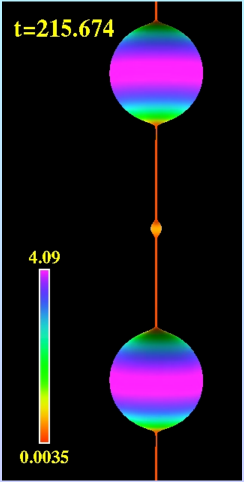
- Axisymmetric code
- Evolution of black string...
- Gregory-Laflamme instability

  cascades down

  in finite time

  until string has zero width

  $\Rightarrow$ naked singularity

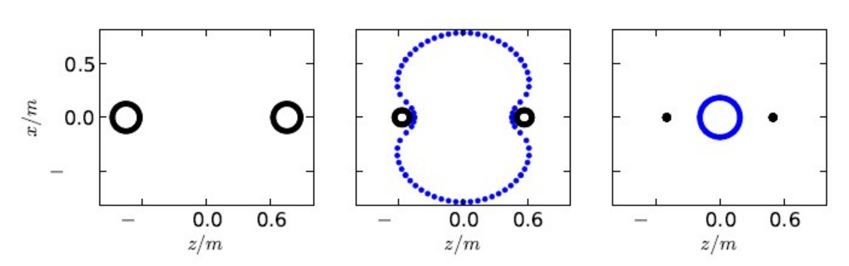


Cosmic Censorship in $D = 4$ de Sitter

Zilhão et al, PRD **85** (2012) 124062

- Two parameters: MH, d
- Initial data: McVittie type binaries McVittie, MNRAS **93** (1933) 325
- "Small BHs": $d < d_{crit} \Rightarrow$ merger

  $d > d_{crit} \Rightarrow$ no common AH
- "Large" holes at small d: Cosmic Censorship holds



Further reading

- Reviews about numerical relativity

  Centrella et al, Rev. Mod. Phys. **82** (2010) 3069

  Pretorius, arXiv:0710.1338

  Sperhake et al, arXiv:1107.2819

  Pfeiffer, CQG **29** (2012) 124004

  Hannam, CQG **26** (2009) 114001

  Sperhake, IJMPD **22** (2013) 1330005