# Non-linear neutron star oscillations viewed as deviations from an equilibrium state

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**Abstract**

A numerical technique is presented which facilitates the evolution of non-linear neutron star oscillations with a high accuracy essentially independent of the oscillation amplitude. We apply this technique to radial neutron star oscillations in a Lagrangian formulation and demonstrate the superior performance of the new scheme compared with "conventional" techniques. The key feature of our approach is to describe the evolution in terms of deviations from an equilibrium configuration. In contrast to standard perturbation analysis we keep all higher order terms in the evolution equations and thus obtain a fully non-linear description. The advantage of our scheme lies in the elimination of background terms from the equations and the associated numerical errors. The improvements thus achieved will be particularly significant in the study of mildly non-linear effects where the amplitude of the dynamic signal is small compared with the equilibrium values but large enough to warrant non-linear effects. We apply the new technique to study the coupling of radial neutron star oscillations due to non-linear effects.

**Keywords:** neutron star oscillations, non-linearity, numerical techniques.

## 1 Introduction

The study of neutron star oscillations has long since been a topic of considerable scientific interest. In recent years the interest in neutron star simulations has further intensified due to the discovery of the gravitational radiation driven instability of the r-modes [1]. A great deal of work has been spent on the development of non-linear, 3-dimensional, relativistic hydrodynamic simulations to investigate such scenarios (see e.g. [3] and references therein). One of the most important questions raised in connection with the r-mode instability concerns the efficiency with which energy is dissipated due to viscosity or non-linear effects. The numerical investigation of non-linear effects in this respect appears to be relying on codes which are not specifically designed to capture mildly non-linear effects in the low-amplitude regime. Results thus obtained have indicated a large saturation amplitude of order unity of the r-modes (see e.g. [4]) but have been called into question by studies which analytically take into account the non-linear mode-coupling and find significant non-linear interaction which prevents a further growth of the r-mode at amplitudes at least 4 orders of magnitude smaller [2]. In the context of simulating non-linear neutron star oscillations at rather small amplitudes it is interesting to note that the signature of the numerical truncation error caused by the intrinsic presence of background terms in the evolution equations has been observed in 3D non-linear evolutions [6]. While these authors find this error to decrease at second order with the grid-resolution and not to seriously affect their results, one has to bear in mind that the impact of the resulting spurious source terms will strongly depend on the amplitude of the dynamic signal. The smaller the deviations from the equilibrium configuration the more significant we expect the numerical contamination to affect the simulations. It appears desirable therefore to have a numerical scheme where the error is determined solely by the amplitude of the dynamic signal as opposed to the equilibrium background. In section 2 and 3 we will show how such an amplitude independent accuracy can be obtained by decomposing the numerical evolution into a static background and time dependent deviations from that background. After testing the resulting code we demonstrate the improvements by comparing numerical evolutions obtained with

the new and a "conventional" scheme. The improvements thus achieved enable us in section 5 to investigate non-linear mode-coupling of radial neutron star oscillations in an amplitude regime comparable to that found relevant for $r$-mode saturation.

## 2 The "conventional" formulation

In order to describe a dynamic, spherically symmetric neutron star we use a Lagrangian approach which straightforwardly facilitates an exact treatment of the moving stellar surface. We thus describe the spacetime in terms of the line element

$$ds^2 = \hat{\lambda}^2 \left(-1 + \frac{w^2}{\hat{\Gamma}}\right) dt^2 + 2\frac{\hat{r}_{,x}\hat{\lambda} w}{\hat{\Gamma}} dt\, dx + \frac{\hat{r}_{,x}^2}{\hat{\Gamma}} dx^2 + \hat{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1}$$

where $\hat{\lambda}$, $\hat{\Gamma}$ and $\hat{r}$ are functions of the spatial coordinate $x$ and coordinate time $t$ and we have defined the velocity of the fluid elements $w = \hat{r}_{,t}/\hat{\lambda}$. In this formulation the circumferential radius $\hat{r}$ is a time dependent variable whereas the "co-moving" spatial coordinate $x$ labels the fluid elements. We model the matter as a perfect fluid at zero temperature with a polytropic equation of state $\hat{P} = K\hat{\rho}^\gamma$, where $\hat{P}$ and $\hat{\rho}$ are the pressure and total energy density and the polytropic exponent $\gamma$ and $K$ are constant parameters. The Lagrangian nature of our coordinate system is reflected in the 4-velocity $u^\alpha$ which has a non-vanishing time component $u^0 = 1/\sqrt{-g_{tt}}$ only. The energy momentum tensor for a perfect fluid then follows directly from $T_{\alpha\beta} = (\hat{\rho}+\hat{P})u_\alpha u_\beta + \hat{P} g_{\alpha\beta}$. In order to write the Einstein field equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ and the equations of conservation of energy-momentum $\nabla_\alpha T_\beta{}^\alpha = 0$ it is convenient to introduce the function $\hat{m}$ by

$$\hat{\Gamma} = 1 - 2\frac{\hat{m}}{\hat{r}}. \tag{2}$$

The equations can then be written in the form

$$0 = \hat{\Gamma}(\hat{\Gamma} - w^2)\hat{r}^2\hat{\lambda}_{,x} - \hat{\lambda}\hat{r}_{,x}\left[(\hat{\Gamma} - w^2)\hat{m} + 4\pi\hat{r}^3(w^2\hat{\rho} + \hat{\Gamma}\hat{P})\right], \tag{3}$$

$$0 = (\hat{\Gamma} - w^2)\hat{m}_{,x} - 4\pi\hat{r}^2\hat{r}_{,x}(\hat{\Gamma}\hat{\rho} + w^2\hat{P}), \tag{4}$$

$$0 = \hat{m}_{,t} + 4\pi\hat{r}^2\hat{\lambda} w\hat{P}, \tag{5}$$

$$0 = \hat{r}_{,t} - \hat{\lambda} w, \tag{6}$$

$$0 = (\hat{\Gamma} - w^2)^2\hat{r}^2\hat{\lambda}\hat{P}_{,x} + (\hat{\Gamma} - w^2)w\hat{r}^2\hat{r}_{,x}\hat{P}_{,t} + (\hat{\rho} + \hat{P})\hat{r}_{,x}\left[\hat{\Gamma} w_{,t}\hat{r}^2 + (\hat{\Gamma} - 2w^2)\hat{\lambda}\left(\hat{m} + 4\pi\hat{r}^3\hat{P}\right)\right]. \tag{7}$$

The boundary conditions for this evolution system are given by $\hat{r} = 0$, $w = 0$, $\hat{m} = 0$ at the origin and $\hat{P} = 0$, $\hat{\lambda} = \sqrt{1 - 2\hat{m}/\hat{r}}$ at the surface which matches the line element (1) to an exterior Schwarzschild metric. For the prescription of initial data and the development of our new numerical formulation in section 3 it is essential to discuss the static and the linearized limit of the evolution system (3)-(7). The static limit describes spherically symmetric neutron stars in equilibrium and leads to the well known Tolman-Oppenheimer-Volkoff (TOV) equations

$$0 = r^2\Gamma\lambda_{,x} - \lambda r_{,x}(m + 4\pi r^3 P), \tag{8}$$

$$0 = m_{,x} - 4\pi r^2 r_{,x}\rho, \tag{9}$$

$$0 = r^2\Gamma P_{,x} + r_{,x}(\rho + P)(m + 4\pi r^3 P), \tag{10}$$

where $\Gamma = 1 - 2m/r$. Here we have omitted the "hat" from the time independent equilibrium variables in order to distinguish them from the total quantities. In order to achieve a higher grid resolution near the surface we fix the Lagrangian coordinate $x$ by relating it to the speed of sound via $r_{,x} = \sqrt{\partial P/\partial\rho}$.
The linearized version of the evolution system around the TOV background is best formulated in terms of the rescaled displacement $\zeta = r^2\xi/\lambda$, where $\xi$ is the displacement of the fluid elements in terms of the circumferential radius. The linearized equations then lead to a self adjoint eigenvalue problem for the functions $\zeta(x)$ which has an infinite number of solutions $\zeta_1$, $\zeta_2$,... with ordered eigenvalues $(\omega_1)^2 < (\omega_2)^2 < ....$ Here the $\omega_i$ are the frequencies of the eigenmodes. After appropriate rescaling the eigenfunctions form a complete orthonormal set so that any function $f(x)$ with appropriate boundary conditions can be expanded in a series according to

$$f(x) = \sum_i A_i\zeta_i(x) = \sum_i \langle f, \zeta_i\rangle\zeta_i(x), \tag{11}$$

where the scalar product is defined in terms of the TOV-background functions (see [5] for details).

## 3 A "non-linear perturbative" formulation

Before we reformulate the evolution system of section 2 in terms of non-linear deviations from the TOV background we motivate our new approach in the case of a simple toy equation. Let us assume for this purpose that a physical system is described in terms of variables $\hat{f}$, $\hat{g}$ and $\hat{h}$ which all depend on spatial position $x$ and time $t$. We further assume that one of the equations governing the system is $\hat{h}_{,t} = \hat{h}_{,x} + \hat{f}\hat{g}$ and that there exist a non-trivial equilibrium of the system described by $f(x)$, $g(x)$ and $h(x)$ which obey the time independent version of this equation $h_{,x} + fg = 0$. We can now decompose the time dependent functions into static background contributions and time dependent deviations according to $\hat{f}(t,x) = f(x) + \Delta f(t,x)$ and likewise for $\hat{g}$ and $\hat{h}$. Inserting this into the evolution equation we obtain

$$\Delta h_{,t} = \Delta h_{,x} + f\Delta g + g\Delta f + \Delta f \Delta g + (h_{,x} + fg). \tag{12}$$

It is the term in brackets on the right hand side which motivates our reformulation of the problem. From the background equation we know that this term vanishes identically. Numerically, however, this will only be satisfied up to a finite error which represents a spurious source term in the evolution of $\Delta h$. This effect is intrinsic to any numerical scheme based on a "conventional formulation" such as that of section 2. Its severeness will depend on the relative size of the dynamic signal compared with the background values. By virtue of our decomposition we eliminate the problematic terms *prior* to the numerical evolution and instead of Eq. (12) use

$$\Delta h_{,t} = \Delta h_{,x} + f\Delta g + g\Delta f + \Delta f \Delta g. \tag{13}$$

By keeping all higher order terms such as $\Delta f \Delta g$ we ensure that our reformulation is equivalent to the original non-linear problem.

We now reformulate the system of evolution equations (2)-(7) in terms of deviations from the TOV equilibrium background. For this purpose we decompose the variables according to $\hat{r}(t,x) = r(x) + \xi(t,x)$, $\hat{\lambda}(t,x) = \lambda(x) + \Delta\lambda(t,x)$ and likewise for $\hat{m}$, $\hat{\Gamma}$, $\hat{\rho}$ and $\hat{P}$. After eliminating all zero order terms by using the TOV equations (8)-(10) the evolution equations can be written as

$$\xi\hat{\Gamma} + r\Delta\Gamma - \xi + 2\Delta m = 0, \tag{14}$$

$$\begin{aligned}&\xi(2r+\xi)\Gamma^2\lambda_{,x} + \hat{r}^2\Delta\Gamma(2\Gamma+\Delta\Gamma)\lambda_{,x} + \hat{r}^2\hat{\Gamma}^2\Delta\lambda_{,x} - (\xi_{,x}\lambda\Gamma + \hat{r}_{,x}\Delta\lambda\Gamma + \hat{r}_{,x}\hat{\lambda}\Delta\Gamma)(m + 4\pi r^3 P)\\ &+ w^2\left[-\hat{r}^2\hat{\Gamma}\hat{\lambda}_{,x} + \hat{r}_{,x}\hat{\lambda}(\hat{m} - 4\pi\hat{r}^3\hat{\rho})\right] - \hat{r}_{,x}\hat{\lambda}\hat{\Gamma}\left\{\Delta m + 4\pi[(3r^2\xi + 3r\xi^2 + \xi^3)P + \hat{r}^3\Delta P]\right\} = 0,\end{aligned} \tag{15}$$

$$\Delta\Gamma m_{,x} + \hat{\Gamma}\Delta m_{,x} - w^2(\hat{m}_{,x} + 4\pi\hat{r}^2\hat{r}_{,x}\hat{P}) - 4\pi\left[\xi(2r+\xi)r_{,x}\Gamma\rho + \hat{r}^2(r_{,x}\Delta\Gamma\rho + r_{,x}\hat{\Gamma}\Delta\rho + \xi_{,x}\hat{\rho}\hat{\Gamma})\right] = 0, \tag{16}$$

$$\Delta m_{,t} + 4\pi\hat{r}^2\hat{\lambda}w\hat{P} = 0, \tag{17}$$

$$\xi_{,t} - \hat{\lambda}w = 0, \tag{18}$$

$$\begin{aligned}&\hat{\lambda}(w^4 - 2\hat{\Gamma}w^2)\hat{r}^2\hat{P}_{,x} + w\hat{r}^2\hat{r}_{,x}(\hat{\Gamma} - w^2)\Delta P_{,t} + (\hat{\rho} + \hat{P})\hat{r}_{,x}\left[\hat{r}^2\hat{\Gamma}w_{,t} - 2\hat{\lambda}w^2(\hat{m} + 4\pi\hat{r}^3\hat{P})\right]\\ &+ (\Delta\lambda\Gamma + \hat{\lambda}\Delta\Gamma)\left[\hat{r}^2\hat{\Gamma}\hat{P}_{,x} + (\hat{\rho} + \hat{P})\hat{r}_{,x}(\hat{m} + 4\pi\hat{r}^3\hat{P})\right] + \lambda\Gamma\left\{\left[(\Delta\rho + \Delta P)r_{,x} + (\hat{\rho} + \hat{P})\xi_{,x}\right](m + 4\pi r^3 P)\right.\\ &\left.+\hat{r}^2\hat{\Gamma}\Delta P_{,x} + \xi(2r+\xi)\Gamma P_{,x} + \hat{r}^2\Delta\Gamma P_{,x} + (\hat{\rho} + \hat{P})\hat{r}_{,x}\left[\Delta m + 4\pi(3r^2\xi + 3r\xi^2 + \xi^3)P + 4\pi\hat{r}^3\Delta P\right]\right\} = 0.\end{aligned} \tag{19}$$

We solve this system with an implicit, second order in space and time numerical scheme similar to the Crank-Nicholson scheme.

We have tested the resulting numerical code in three independent ways which cover a large range of amplitudes of the deviations. First we evolve an isolated eigenmode of a background neutron star model with polytropic parameters $\gamma = 2$, $K = 150\,\mathrm{km}^2$, mass $M = 1.48\,M_\odot$ and radius $R = 11.3\,\mathrm{km}$. For a small amplitude of the eigenmode (we choose 10 cm) the exact solution will be well approximated by the harmonic oscillations predicted by the linearized equations. We find our code to reproduce this analytic solution with a relative point-wise error of about $10^{-3}$ for 200 grid points. Secondly we confirmed second order convergence of the code both for eigenmode evolutions with various amplitudes and the collapse of unstable neutron stars. Finally we have simulated the collapse of a spherically symmetric, initially homogeneous dust cloud, i.e. a fluid with zero pressure ("Oppenheimer-Snyder dust collapse"). We find our code to reproduce the analytic solution with a relative accuracy better than $10^{-3}$ for 800 grid points throughout the whole collapse.

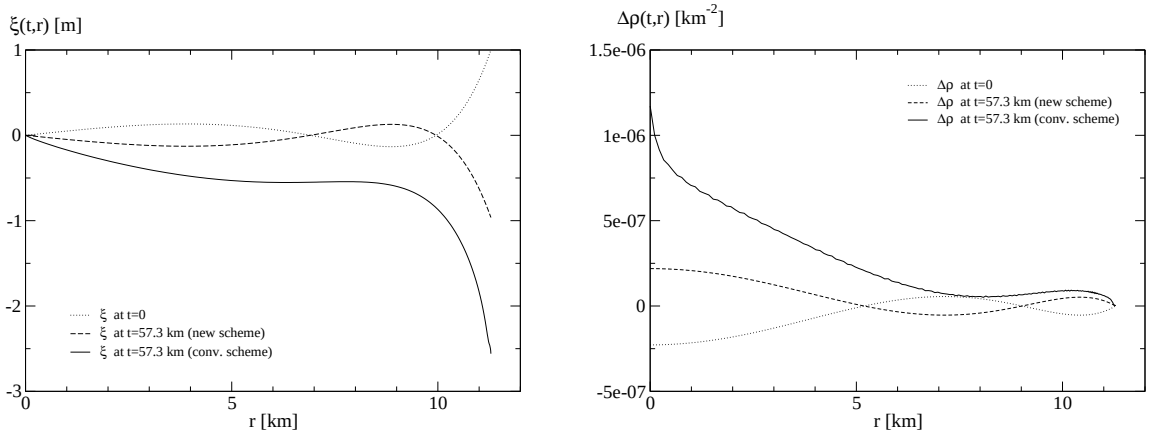


Figure 1: Snap shots of the evolution of $\xi$ (right panel) and $\Delta\rho$ (left panel) obtained with a non-linear perturbative and a conventional scheme.

# 4 Comparing the numerical performance of the two schemes

In order to compare the numerical schemes outlined in sections 2 and 3 we consider the evolution of the third eigenmode of the above model with an amplitude of 1 m using 200 grid points. For this amplitude the non-linear effects are rather small (albeit measurable as we will see below) so that we expect the evolution to be close to the harmonic oscillations predicted by the linearized equations. In Fig. 1 we show snapshots of the evolution of the displacement $\xi$ (left panel) and the energy density deviation $\Delta\rho$ (right panel). The dotted lines represent the initial data, the dashed lines the results obtained with our non-linear perturbative scheme of section 3 and the solid lines those obtained with the conventional scheme of section 2. The results demonstrate that the new scheme produces the expected oscillations while the "conventional" scheme leads to severely distorted profiles. The numerical noise visible in the solid curve of $\Delta\rho$ arises from the spurious formation of shocks near the stellar surface. We emphasize that these results have been obtained with a single numerical code. For our new scheme we use a TOV-background while the conventional formulation is emulated by using a trivial flat space vacuum background ($\lambda = 1$, $m = 0$, $\rho = 0$) for which the two systems of equations (14)-(19) and (2)-(7) become identical. The only significant difference between the two runs is therefore the presence or absence of the background error terms analogous to the term in brackets on the right hand side of Eq. (12). We conclude that the presence of these terms gives rise to spurious numerical effects such as mode coupling and shock formation. We find the significance of this effect to decrease with higher grid resolution and larger amplitude of the initial data. Our new scheme will therefore be particularly suitable for studying mildly non-linear effects at amplitudes notably smaller than the background values.

# 5 Non-linear coupling of eigenmodes

In section 1 we have already noted the importance of a detailed understanding of non-linear coupling of eigenmodes for addressing questions such as the saturation amplitudes of neutron star oscillations. Analytic studies of mode-coupling effects normally view the eigenmode coefficients $A_i$ (cf. Eq. 11) as harmonic oscillators and the non-linear interaction between different modes is represented in the form of a series of driving terms with increasing order in the amplitudes which is truncated at second or third order (see e.g. [7]). While the investigation of neutron star oscillations in 3-dimensions such as r-modes is beyond the scope of our work, we will demonstrate in the case of radial oscillations that our scheme presents a numerical alternative to study non-linear coupling of eigenmodes without any restrictions due to the emission of higher order terms. For this purpose we consider the neutron star model of section 3 and provide initial data in the form of one isolated eigenmode. The index $j$ of this mode and the initial amplitude given by the initial displacement of the stellar surface $\xi_s$ in m from the equilibrium radius are the two free parameters. During the fully non-linear evolution we make use of Eq. (11) which enables us to expand the time dependent displacement function according to $\zeta(t,x) = \sum_i A_i(t)\zeta_i(x)$. The time dependent eigenmode coefficients are then given by $A_i(t) = \langle \zeta(t,x), \zeta_i(x) \rangle$ and can be calculated at each time step. In the amplitude range considered here we typically find these coefficients to oscillate with a frequency close to the value predicted by linear theory. We therefore measure the degree to which a mode is present in an evolution by taking the maximum of $|A_i(t)|$ which we denote by $A_i$. The

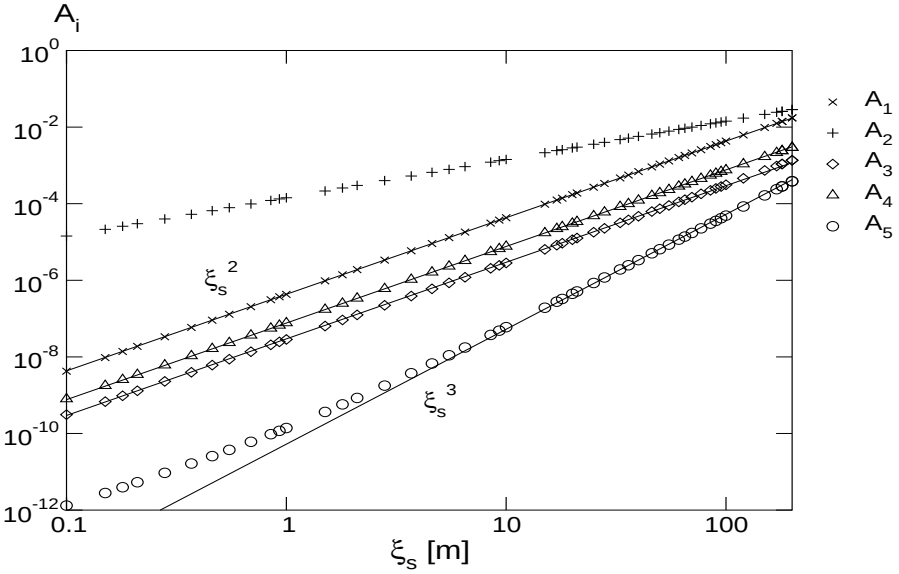


Figure 2: The eigenmode coefficients $A_i$ are shown for initial data in the form the second eigenmode as a function of the initial amplitude $\xi_s$.

integration time for these runs is $T \approx 4\,\mathrm{ms}$ which corresponds to about 10 oscillation periods of the fundamental mode. In Fig. (2) we show the eigenmode coefficients $A_i$ of the first 5 eigenmodes thus obtained for $j = 2$ as a function of the initial surface displacement. The only mode initially present in these evolutions is the second mode and we find the corresponding eigenmode coefficient $A_2$ ("+" in the figure) to depend linearly on the initial surface displacement $\xi_s$. Other eigenmodes while not present in the initial data, are excited in the course of the non-linear evolution, so that we find non-zero values for the other $A_i$. We clearly see that the degree of excitation increases with the initial amplitude. A more detailed analysis reveals that the dependence of the $A_i$ on $\xi_s$ can be well approximated by power laws with integer index as shown in the figure. While $A_1$, $A_3$ and $A_4$ are well approximated by a quadratic power law, we find that $A_5$ grows like the cube of $\xi_s$. We have frequently observed that the excitation of higher order modes is modeled by larger integer power law indices, although we find exceptions to this rule. We emphasize the importance of the amplitude independent accuracy provided by our scheme for being able to measure these effects. The deviations of $A_5$ from the power law behavior observed at small amplitudes are most likely caused by the accuracy limits encountered even when using our scheme with the large resolution of 3200 grid points.

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