Part III General Relativity

Lecture Notes

Abstract

These notes represent the material covered in the Part III lecture General Relativity. A large part of the mathematical background (mostly up to chapter 9) is based on the more extended lecture notes by Harvey Reall [5] as well as Hawking & Ellis's "The Large Scale Structure of Space-Time" [3] and John Stewart's "Advanced general relativity" [4]. Later sections on the "3+1" formalism of the Einstein equations and the Lagrangian formulation of general relativity have been inspired to quite some extent by Eric Gourgoulhon's "3+1 Formalism and Bases of Numerical Relativity" [2] and Eric Poisson's lecture notes on "Advanced general relativity" [6]. Readers will find all these references valuable sources to explore topics discussed in this lecture in more detail. Primary purpose of the present set of notes is to provide a verbatim description of the material covered in the Part III course on General Relativity. Indeed, they bear a high degree of resemblance to the material as presented on the black board in the lecture theatre.

For further reading on the topic of Einstein's theory of general relativity, there exists a wealth of books more or less directly deidcated to the theory. An incomplete list of books is given as follows.

- J. B. Hartle, "Gravity, An Introduction to Einstein's General Relativity" .
- B. Schutz, "A first course in general relativity" .
- R. M. Wald, "General Relativity" .
- S. M. Carroll: "Spacetime and Geometry: An Introduction to General Relativity" ; cf. also [1] .
- L. Ryder, "Introduction to General Relativity" .
- C. W. Misner, K. S. Thorne & J. A. Wheeler, "Gravitation" .
- S. Weinberg, "Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity" .

Example sheets for this course will be available on the webpage

`http://www.damtp.cam.ac.uk/user/examples/indexP3.html`

Make sure you do not confuse these example sheets with those of the Part II course of the same name on `http://www.damtp.cam.ac.uk/user/examples`.

Note that this course does not cover (in any depth) the topics of Black Holes and Cosmology which are the subject of other Part III Courses.

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Contents

1 The equivalence principle

Special Relativity:
- Physical experiments are the same in any inertial frame
- inertial frames: non-accelerating observers

  related by Lorentz trafos

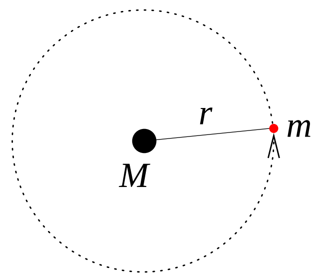
Newtonian gravity: $\boxed{\nabla^2\phi = 4\pi G\rho}$ $\qquad (*)$

$$\Rightarrow \quad \phi(t,\vec{x}) = -G\int \frac{\rho(t,\vec{y})}{|\vec{x}-\vec{y}|}\, d^3y$$

Lorentz trafos mix time and space coordinates $\Rightarrow$ Eq. $(*)$ not invariant

also: finite propagation of signals $\Rightarrow$ Newton's gravity not compatible with SR

Newtonian gravity: good approximation if $v \ll c$



orbiting particle: $\phi = -\dfrac{GM}{r} \quad \Rightarrow \quad \dfrac{v^2}{r} = \dfrac{GM}{r^2}$

then: $v \ll c \quad \Leftrightarrow \quad \dfrac{G}{c^2}\dfrac{M}{r} \ll 1$

Solar system: $\dfrac{G}{c^2}\dfrac{M}{r} < 10^{-5}$

1.1 Statement of the equivalence principle

Newtonian theory

Inertial mass: $\vec{F} = m_I\vec{a}$

Gravitational mass: $\vec{F} = -m_G\vec{\nabla}\phi = m_G\vec{g}\,;\quad \vec{g} := -\vec{\nabla}\phi$

$\Rightarrow$ With suitable scaling: $m_I = m_G$

Experiment: $\dfrac{m_I}{m_G} - 1 = \mathcal{O}(10^{-12})$ "Eötvös" for all kinds of objects

Weak equivalence principle (WEP), version 1: $m_I = m_G$

Newtonian motion: $m_I\,\vec{a} = m_I\,\ddot{\vec{x}} = m_G\,\vec{g} \quad \Rightarrow \quad \ddot{\vec{x}} = \vec{g}$

$\Rightarrow$ **WEP**, version 2: The trajectory of a freely falling test body
depends only on its initial position and velocity
and is independent of its composition.

Comment: "Test body" = body with negligible gravitational self interaction
and size $\ll$ lengthscale on which $\vec{g}$ varies.

<u>accelerated frames</u>

Let $\mathcal{O}$ be inertial frame with coords. $(t, \vec{x})$ in grav. field $\vec{g}$.

Let $\mathcal{O}'$ be a frame accelerated relative to $\mathcal{O}$ with $\vec{a}$.

Coords.: $\big(t,\, \vec{x}' = \vec{x} - \vec{x}_0(t)\big)$ where $\vec{x}_0(t)$ = position of origin of $\mathcal{O}'$ in $\mathcal{O}$ coords.: $\ddot{\vec{x}}_0 = \vec{a}$

$\Rightarrow$ Eq. of motion in $\mathcal{O}'$: $\ddot{\vec{x}}' = \vec{g} - \vec{a}$

$\rightarrow$ different grav. field $\vec{g}' = \vec{g} - \vec{a}$

special cases: 1) $\vec{g} = 0 \quad \Rightarrow \quad \vec{g}' = -\vec{a}$

"uniform acceleration indistinguishable from grav. field"

2) $\vec{g} \neq 0\,, \quad \vec{a} = \vec{g} \quad \Rightarrow \quad \mathcal{O}'$ is a freely falling frame: $\vec{g}' = 0$

<u>Non-uniform grav. fields</u>

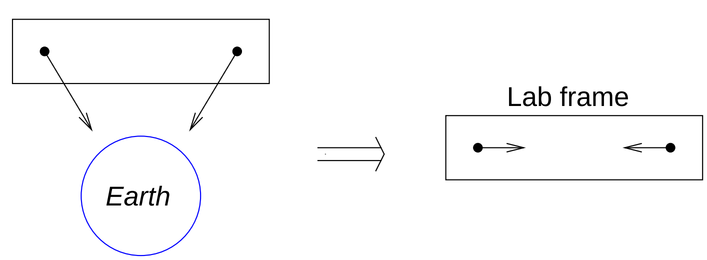
"local inertial frame" = coord. frame (t, x, y, z) defined by freely falling observer in
the same way as in Minkowski space

"local" means "small compared with lengthscale of variations in $\vec{g}$

E.g.: tidal forces:

Lab frame "too large"

$\Rightarrow$ particles accelerated in freely
falling frame due to tidal forces.



The WEP was found in Newtonian physics

Einstein promoted it to be more general:

Einstein EP (EEP): (i) The WEP is valid.

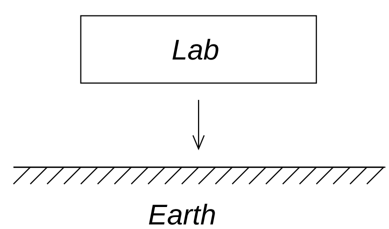
(ii) In a local inertial frame the results of all non-gravitational experiments are indistinguishable from those of the same experiment performed in an inertial frame in Minkowski spacetime.

Schiff's conjecture: The WEP implies the EEP.

argument:
- WEP $\Rightarrow$ (ii) holds for test particles.
- Matter is composed of quarks, electrons etc.
- These are bound by electromagnetic, nuclear forces $\rightarrow$ binding energy makes up part of the bodies mass and appears to also obey (ii)

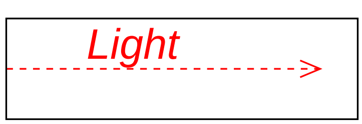
1.2 Bending of light

Consider freely falling lab in uniform grav. field



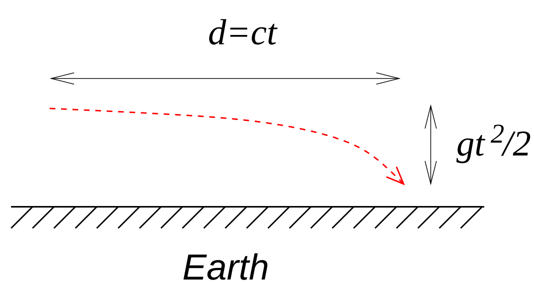
<u>Inside Lab</u>

light moves on straight line



<u>Earth frame</u>

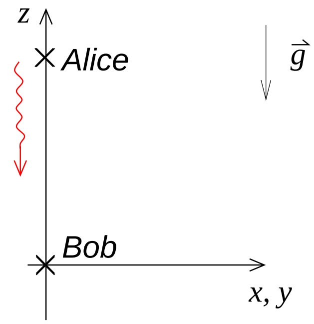
curved path



$$t = \frac{d}{c} \quad \Rightarrow \quad h = \frac{1}{2} g \frac{d^2}{c^2}$$

$$d = 1 \text{ km} \quad \Rightarrow \quad h \approx 5 \cdot 10^{-11} \text{ m}$$

1.3 Gravitational redshift



Consider: $\vec{g} = (0,\, 0,\, -g)$, Alice at $z = h$, Bob at $z = 0$

Alice sends light to Bob.

EP $\Rightarrow$ equivalent to frame accelerated with $(0,\, 0,\, +g)$ in Minkowski spacetime

Assumption: v of Bob, Alice $\ll c$

$\Rightarrow$ ignore $\dfrac{v^2}{c^2}$ and higher-order SR terms

$$\Rightarrow \quad z_A(t) = h + \frac{1}{2} g t^2 \,, \qquad z_B(t) = \frac{1}{2} g t^2 \,, \qquad v_A = v_B = gt \overset{!}{\ll} c \,.$$

- Alice emits first signal at t_1

  $$\Rightarrow z_1(t) = z_A(t_1) - c(t - t_1) = h + \frac{1}{2} g t_1^2 - c(t - t_1)$$

- This reaches Bob at T_1, i.e. $h + \frac{1}{2} g t_1^2 - c(T_1 - t_1) = \frac{1}{2} g T_1^2$ $\qquad$ $(**)$
- Alice emits second signal at $t_2 = t_1 + \Delta\tau_A$.

  This reaches Bob at $T_2 = T_1 + \Delta\tau_B$.

  $$\Rightarrow h + \frac{1}{2} g (t_1 + \Delta\tau_A)^2 - c(T_1 + \Delta\tau_B - t_1 - \Delta\tau_A) = \frac{1}{2} g (T_1 + \Delta\tau_B)^2 \qquad \Big| \text{ subtract } (**)$$

  $$\Rightarrow c(\Delta\tau_A - \Delta\tau_B) + \frac{1}{2} g\, \Delta\tau_A\, (2t_1 + \Delta\tau_A) = \frac{1}{2} g\, \Delta\tau_B\, (2T_1 + \Delta\tau_B)$$

- Assumption: $\Delta\tau_A \ll t_1$, $\Delta\tau_B \ll T_1$, e.g. period in light waves

  $$\Rightarrow c(\Delta\tau_A - \Delta\tau_B) + g\, \Delta\tau_A\, t_1 = g\, \Delta\tau_B\, T_1$$

  $$\Rightarrow \Delta\tau_B\, (g T_1 + c) = \Delta\tau_A\, (g t_1 + c)$$

  $$\Rightarrow \Delta\tau_B = \left(1 + \frac{g T_1}{c}\right)^{-1} \left(1 + \frac{g t_1}{c}\right) \Delta\tau_A \approx \left[1 - \frac{g(T_1 - t_1)}{c}\right] \Delta\tau_A \qquad \Big| \text{ we used } \frac{gt}{c} \ll 1$$

- $(**)$ $\Rightarrow$ $$\frac{h}{c} - (T_1 - t_1) = \frac{1}{2} \underbrace{\frac{g}{c}(T_1 + t_1)}_{\ll 1} \underbrace{(T_1 - t_1)}_{\approx \frac{h}{c}} \approx 0 \qquad \Big| \text{ we used } \frac{gt}{c} \ll 1$$

  $$\Rightarrow T_1 - t_1 = \frac{h}{c} \text{ to leading order.}$$

- $\Rightarrow \boxed{\Delta\tau_B \approx \left(1 - \frac{gh}{c^2}\right)\Delta\tau_A \overset{!}{<} \Delta\tau_A}$

  $\Rightarrow$ Signal appears blue shifted to Bob: $c\,\Delta\tau_B = \lambda_B \approx \left(1 - \frac{gh}{c^2}\right)\lambda_A$

Confirmed in Pound-Rebka experiment (1960): light falling in tower.
Light climbing out of a gravity well is red shifted.

In general: $\boxed{\Delta\tau_B \approx \left(1 + \frac{\phi_B - \phi_A}{c^2}\right)\Delta\tau_A}$ also holds for weak, non-uniform fields

1.4 Curved spacetime

WEP $\Rightarrow$ test bodies move the same way in a grav. field independent of their composition, i.e. their grav. "charge" m. This is not true for other forces!

Einstein: gravity must be a feature of spacetime, i.e. its geometry.

Consider redshift but now in a non-Minkowskian metric

$$c^2\,d\tau^2 = \left[1 + \frac{2\phi(x,y,z)}{c^2}\right] c^2dt^2 - \left[1 - \frac{2\phi(x,y,z)}{c^2}\right](dx^2 + dy^2 + dz^2)\,; \qquad \frac{\phi}{c^2} \ll 1$$

- Alice: $\vec{x}_A$, Bob: $\vec{x}_B$, at fixed positions!
- Alice emits signals at $t_A,\, t_A + \Delta t$

  Bob receives the first at t_B. When does he see the second?
- The spacetime is static: ϕ does not depend on t

  $\Rightarrow$ The two signals travel on identical trajectories, just shifted in time

  $\Rightarrow$ Bob receives the second signal at $t_B + \Delta t$.
- But what proper times do Alice's and Bob's clocks measure?

$$\Delta\tau_A^2 = \left(1 + \frac{2\phi_A}{c^2}\right)\Delta t^2\,, \qquad \Delta\tau_B^2 = \left(1 + \frac{2\phi_B}{c^2}\right)\Delta t^2$$

$$\Rightarrow \Delta\tau_A \approx \left(1 + \frac{\phi_A}{c^2}\right)\Delta t\,, \qquad \Rightarrow \Delta\tau_B \approx \left(1 + \frac{\phi_B}{c^2}\right)\Delta t\,,$$

$$\Rightarrow \Delta\tau_B \approx \left(1 + \frac{\phi_B}{c^2}\right)\left(1 + \frac{\phi_A}{c^2}\right)^{-1}\Delta\tau_A \approx \left(1 + \frac{\phi_B - \phi_A}{c^2}\right)\Delta\tau_A$$

2 Manifolds and tensors

In GR we define spacetime as a manifold: trickier than for Minkowski!

Minkowski: • inertial frames $\rightarrow$ preferred global coordinates

• we can add position vectors $\Rightarrow$ spacetime has structure of vector space

Curved spacetimes: inertial coordinates are local; how about vectors?

2.1 Differentiable manifolds

We know how to do calculous in $\mathbb{R}^n$

Goal: develop analog in curved spaces

Def.: n-dim. differentiable manifold $:=$ a set $\mathcal{M}$ with subsets $\mathcal{O}_\alpha$ such that

(1) $\cup_\alpha \mathcal{O}_\alpha = \mathcal{M}$

(2) $\forall_\alpha$ $\exists$ a 1-to-1 and onto map

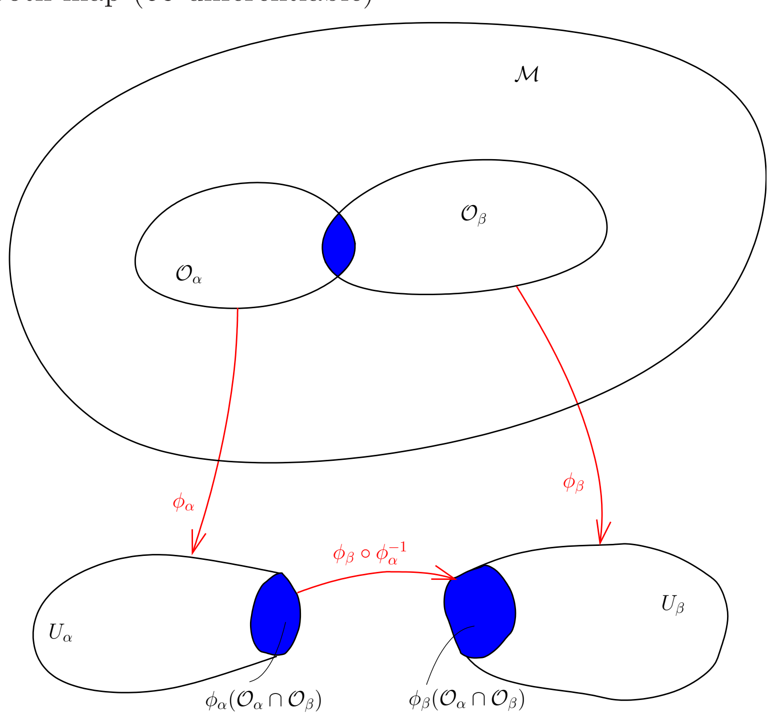
$\phi_\alpha : \mathcal{O}_\alpha \longrightarrow U_\alpha \subset \mathbb{R}^n$ open

(3) If $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \emptyset$, then $\phi_\beta \circ \phi_\alpha^{-1} : \underbrace{[\phi_\alpha(\mathcal{O}_\alpha \cap \mathcal{O}_\beta)]}_{\nwarrow\, \subset U_\alpha \subset \mathbb{R}^n} \longrightarrow \underbrace{[\phi_\beta(\mathcal{O}_\alpha \cap \mathcal{O}_\beta)]}_{\nwarrow\, \subset U_\beta \subset \mathbb{R}^n}$

is a smooth map (∞ differentiable)

The ϕ_α are called "<u>charts</u>"

$\{\phi_\alpha\}$ is an "<u>atlas</u>"



Comments: • For $p \in \mathcal{O}_\alpha$ we often write $\phi_\alpha(p) = \Big(x_\alpha^1(p),\, x_\alpha^2(p),\, x_\alpha^3(p)\Big) = x_\alpha^\mu(p)$

$=$ "coordinates" of p ; the α is often dropped.

• A C^k manifold is defined likewise. We'll assume C^∞

Examples: 1) $\mathbb{R}^n$ is a manifold with an atlas of one chart

$\phi : (x^1, \dots, x^n) \mapsto (x^1, \dots, x^n)$

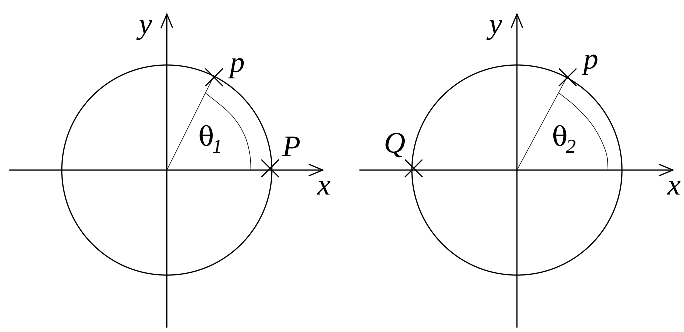
2) $S^1 \equiv$ unit circle $= \big\{(\cos\theta,\, \sin\theta) \in \mathbb{R}^2 \big|\ \theta \in \mathbb{R}\big\}$

$\exists$ no atlas with one chart

$\theta \in [0, 2\pi)$ does not work: not open!

We need 2 charts:

(i) Let $P = (1, 0)$ and $\phi_1 : S^1 - \{P\} \to (0, 2\pi)$, $\phi_1(p) = \theta_1$



(ii) Let $Q = (-1, 0)$ and $\phi_2 : S^1 - \{Q\} \to (-\pi, \pi)$, $\phi_2(p) = \theta_2$

$\{\phi_1,\, \phi_2\}$ form an atlas

Note: On the upper semi circle ($y \geq 0$): $\theta_2 = \phi_2 \circ \phi_1^{-1}(\theta_1) = \theta_1$

On the lower semi circle ($y < 0$): $\theta_2 = \phi_2 \circ \phi_1^{-1}(\theta_1) = \theta_1 - 2\pi$

Comment: $\mathcal{M}$ may admit many atlases

<u>Def.:</u> 2 atlases are compatible $:\Leftrightarrow$ their union is also an atlas

complete atlas $:=$ union of all atlases compatible with a given atlas

$\nwarrow$ contains ∞ atlases

2.2 Smooth functions

Def.: $f : \mathcal{M} \to \mathbb{R}$ is smooth $:\Leftrightarrow$ $\forall$ charts ϕ : $\quad F \equiv f \circ \phi^{-1} : U \subset \mathbb{R}^n \to \mathbb{R}$ is smooth

Sometimes we call f a scalar field

Examples: 1) Consider the S^1 sphere above: $f : S^1 \to \mathbb{R}, \quad (x, y) \mapsto x$

$$f \circ \phi_1^{-1}(\theta_1) = \cos\theta_1 \,, \qquad f \circ \phi_2^{-1}(\theta_2) = \cos\theta_2 \quad \text{both smooth}$$

Let ϕ be some chart $\Rightarrow$ $f \circ \phi^{-1} = \underbrace{\big(f \circ \phi_i^{-1}\big)}_{\text{smooth}} \circ \underbrace{\big(\phi_i \circ \phi^{-1}\big)}_{\substack{\text{smooth}\\ \text{(manifold!)}}}, \quad i = 1,\ 2$

2) Consider manifold $\mathcal{M}$, chart $\phi : \mathcal{O} \subset \mathcal{M} \to U \subset \mathbb{R}^n \,, \quad p \in \mathcal{O} \mapsto \big(x^1(p), \ldots, x^n(p)\big)$

Let ϕ_α be the other charts in the atlas

Let $f : \mathcal{O} \to \mathbb{R} \,, \quad p \mapsto x^1(p)$

$\Rightarrow$ f is smooth: $x^1 \circ \phi_\alpha^{-1}$ is the first component of $\phi \circ \phi_\alpha^{-1}$ which is smooth

3) We can define f through F:

$\{\phi_\alpha\}$ atlas $\Rightarrow$ $F_\alpha : U_\alpha \to \mathbb{R}$ defines $f = F_\alpha \circ \phi_\alpha$

provided F_α is independent of α on overlaps

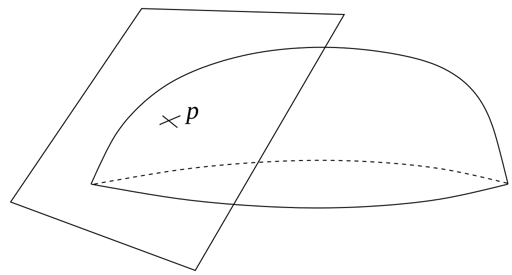
Consider S^1 above: $F_1 : (0, 2\pi) \to \mathbb{R} \,, \quad \theta_1 \mapsto \sin(m\theta_1) \,, \quad m$ integer

$F_2 : (-\pi, \pi) \to \mathbb{R} \,, \quad \theta_2 \mapsto \sin(m\theta_2)$

$\Rightarrow F_1 \circ \phi_1 = F_2 \circ \phi_2$ on overlap: θ_1, θ_2 differ by multiples of 2π

Note: We sometimes do not distinguish between f and F: "$f(x) = F(x)$"

2.3 Curves and vectors



Consider surface $\mathcal{S}$ in $\mathbb{R}^3$, tangent plane at p

$\Rightarrow$ the plane has structure of a 2-dim. vector space;

a tangent vector to a curve in $\mathcal{S}$ at p is in the plane

Goal: formalize this for a manifold

Def.: A smooth curve in a manifold $\mathcal{M}$:= function $\lambda : I \to \mathcal{M}$, where $I \subset \mathbb{R}$ open,

such that $\phi_\alpha \circ \lambda : I \to \mathbb{R}^n$ is smooth for all charts ϕ_α

Directional derivative: let $f : \mathcal{M} \to \mathbb{R}$, $\lambda : I \to \mathcal{M}$ both be smooth

$\Rightarrow f \circ \lambda : I \to \mathbb{R}$ smooth

$$\Rightarrow \frac{d}{dt}\left[(f \circ \lambda)(t)\right] = \frac{d}{dt}\left[f(\lambda(t))\right]$$

Def.: Let $\mathcal{C}^\infty$ be the space of all smooth functions from $\mathcal{M}$ to $\mathbb{R}$.

Let λ be a smooth curve with $\lambda(0) = p \in \mathcal{M}$

$\Rightarrow$ The "tangent vector" to λ is the linear map

$$\boldsymbol{X}_p : \mathcal{C}^\infty(\mathcal{M}, \mathbb{R}) \to \mathbb{R}\,, \quad f \mapsto \boldsymbol{X}_p(f) = \left\{\frac{d}{dt}\left[f(\lambda(t))\right]\right\}_{t=0}$$

Note: (i) Linearity $\Rightarrow$ $\boldsymbol{X}_p(f+g) = \boldsymbol{X}_p(f) + \boldsymbol{X}_p(g)$; $\boldsymbol{X}_p(\alpha f) = \alpha \boldsymbol{X}_p(f)$ for $\alpha = \text{const}$

(ii) $\boldsymbol{X}_p(f\,g) = \boldsymbol{X}_p(f)\,g(p) + f(p)\,\boldsymbol{X}_p(g)$ "Leibniz rule"

(iii) Let $\phi = (x^\mu)$ be a chart defined in a neighbourhood of $p \in \mathcal{M}$, and $F \equiv f \circ \phi^{-1}$

$$\Rightarrow f \circ \lambda = (f \circ \phi^{-1}) \circ (\phi \circ \lambda) = F \circ \phi \circ \lambda\,,$$

$$\boldsymbol{X}_p(f) = \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{d\,x^\mu(\lambda(t))}{dt}\right)_{t=0} = \underbrace{\frac{dx^\mu}{dt}}_{\text{components}}\ \underbrace{\frac{\partial}{\partial x^\mu}}_{\text{basis}} F = \underbrace{\frac{d}{dt}}_{\text{vector}} F\left(x^\mu(\lambda(t))\right) = \frac{d}{dt} f(\lambda(t))$$

Compare with directional derivative in $\mathbb{R}^n$: $\vec{X} \cdot (\vec{\nabla} f)_p$

The set of tangent vectors at $p \in \mathcal{M}$ forms an n-dim. vector space: "Tangent Space" $\mathcal{T}_p(\mathcal{M})$

Proof: (1) "Addition, scalar mult. $\to$ vector"

Let $\lambda,\ \kappa$ be curves through p such that $\lambda(0) = \kappa(0) = p$,

$\boldsymbol{X}_p,\ \boldsymbol{Y}_p$ be their tangent vectors,

$\alpha,\ \beta \in \mathbb{R},\ \phi = (x^\mu)$ be a chart in neighbourhood of p.

Define $\alpha\boldsymbol{X}_p + \beta\boldsymbol{Y}_p : \mathcal{C}^\infty(\mathcal{M}, \mathbb{R}) \to \mathbb{R}\,,\quad f \mapsto \alpha\boldsymbol{X}_p(f) + \beta\boldsymbol{Y}_p(f)$

Consider curve $\nu(t) \equiv \phi^{-1}\big\{\alpha\big[\phi\big(\lambda(t)\big) - \phi(p)\big] + \beta\big[\phi\big(\kappa(t)\big) - \phi(p)\big] + \phi(p)\big\}$

$$\Rightarrow \quad \nu(0) = p$$

Let $\boldsymbol{Z}_p$ be the tangent vector of ν

$$\Rightarrow \quad \boldsymbol{Z}_p(f) = \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)} \left\{\frac{d}{dt}\left[\alpha\big(x^\mu(\lambda(t)) - x^\mu(p)\big) + \beta\big(x^\mu(\kappa(t)) - x^\mu(p)\big)\right]\right\}_{t=0}$$

$$= \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)} \left\{\alpha\left[\frac{dx^\mu(\lambda(t))}{dt}\right]_{t=0} + \beta\left[\frac{dx^\mu(\kappa(t))}{dt}\right]_{t=0}\right\}$$

$$= \alpha\boldsymbol{X}_p(f) + \beta\boldsymbol{Y}_p(f) = (\alpha\boldsymbol{X}_p + \beta\boldsymbol{Y}_p)(f)$$

(2) "n dims.?"

Define for $\mu = 1,\ \ldots,\ n$:

$$\lambda_\mu(t) \equiv \phi^{-1}\left[x^1(p),\ \ldots,\ x^{\mu-1}(p),\ x^\mu(p) + t,\ x^{\mu+1}(p),\ \ldots,\ x^n(p)\right]$$

Let $\left(\dfrac{\partial}{\partial x^\mu}\right)_p$ be the tangent vector to λ_μ

$$\Rightarrow \quad \left(\frac{\partial}{\partial x^\mu}\right)_p (f) = \left.\frac{\partial F}{\partial x^\mu}\right|_{\phi(p)} \qquad (*)$$

Let $\alpha^\mu \in \mathbb{R}$ such that $\alpha^\mu \left(\dfrac{\partial}{\partial x^\mu}\right)_p = 0 \quad \in \quad \mathcal{T}_p(\mathcal{M})$

$$\Rightarrow \alpha^\mu \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)} = 0$$

Let $F(x^\mu) = x^\nu \quad \Rightarrow \quad \dfrac{\partial F}{\partial x^\mu} = \delta^\nu{}_\mu \quad \Rightarrow \quad \alpha^\nu = 0\,.$

Do this for all $\nu = 1,\ \ldots,\ n \quad \Rightarrow$ lin. independence.

(3) "Do we span $\mathcal{T}_p(\mathcal{M})$?"

$$\boldsymbol{X}_p(f) = \left(\frac{dx^\mu(\lambda(t))}{dt}\right)_{t=0} \left(\frac{\partial}{\partial x^\mu}\right)_p (f) \quad \text{for any } f\,!$$

$$\Rightarrow \quad \boldsymbol{X}_p = \left(\frac{dx^\mu(\lambda(t))}{dt}\right)_{t=0} \left(\frac{\partial}{\partial x^\mu}\right)_p \qquad (**)$$

$\Rightarrow$ Any $\boldsymbol{X}_p$ can be written as a linear combination of $\left(\dfrac{\partial}{\partial x^\mu}\right)_p$.

Note:

- $\left(\dfrac{\partial}{\partial x^\mu}\right)_p : \mathcal{C}^\infty(\mathcal{M}, \mathbb{R}) \to \mathbb{R}$ is not the same as the partial derivative $\dfrac{\partial}{\partial x^\mu}$!
- The basis $\left(\dfrac{\partial}{\partial x^\mu}\right)_p$ is chart dependent: "coordinate basis"

Def.: Let $\{\mathbf{e}_\mu\}$, $\mu = 1, \ldots, n$ be a basis of $\mathcal{T}_p(\mathcal{M})$

$\Rightarrow \quad \boldsymbol{X}_p = X_p^\mu \mathbf{e}_\mu\,; \quad X_p^\mu$ are the "components" of $\boldsymbol{X}_p$

Example: **(∗∗)** for coord. basis: $X_p^\mu = \left[\dfrac{dx^\mu\,(\lambda(t))}{dt}\right]_{t=0} =: \dfrac{dx^\mu}{dt}$

Note: When Einstein summation applies: always one index up, one down !

$\left(\dfrac{\partial}{\partial x^\mu}\right)_p$ is a "down" index. Expressions like $X_\mu Y_\mu$ are wrong !

Coordinate transformations

Let $\phi = (x^i)$, $\bar{\phi} = (\bar{x}^i)$ be two charts in a nbhd. of $p \in \mathcal{M}$

$$
\begin{aligned}
\Rightarrow \quad \left(\frac{\partial}{\partial x^\mu}\right)_p (f) \quad &= \quad \frac{\partial}{\partial x^\mu}(f \circ \phi^{-1}) \Big|_{\phi(p)} \\
&= \quad \frac{\partial}{\partial x^\mu}\Big[\big(f \circ \bar{\phi}^{-1}\big) \circ \big(\underbrace{\bar{\phi} \circ \phi^{-1}}_{= \bar{x}^\mu (x^\alpha)} \big) \Big] \Big|_{\phi(p)} \\
&= \quad \frac{\partial}{\partial x^\mu}\Big[\Big(f \circ \bar{\phi}^{-1}\Big)\big(\bar{x}(x)\big)\Big] \\
&= \quad \left[\frac{\partial}{\partial \bar{x}^\alpha}\big(f \circ \bar{\phi}^{-1}\big)(\bar{x})\right]_{\bar{\phi}(p)} \quad \frac{\partial \bar{x}^\alpha}{\partial x^\mu}\Big|_{\phi(p)} \\
&= \quad \left(\frac{\partial}{\partial \bar{x}^\alpha}\right)_p (f) \;\; \frac{\partial \bar{x}^\alpha}{\partial x^\mu}\Big|_{\phi(p)}
\end{aligned}
$$

$$
\Rightarrow \quad \boxed{\left(\frac{\partial}{\partial x^\mu}\right)_p = \left(\frac{\partial \bar{x}^\alpha}{\partial x^\mu}\right)_{\phi(p)} \quad \left(\frac{\partial}{\partial \bar{x}^\alpha}\right)_p}
$$

Components: $\boldsymbol{V} \in \mathcal{T}_p(\mathcal{M})$

$$
\Rightarrow \quad \boldsymbol{V} = V^\mu \left(\frac{\partial}{\partial x^\mu}\right)_p = V^\mu \left(\frac{\partial \bar{x}^\alpha}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial}{\partial \bar{x}^\alpha}\right)_p = \bar{V}^\alpha \left(\frac{\partial}{\partial \bar{x}^\alpha}\right)_p
$$

$$
\Rightarrow \quad \boxed{\bar{V}^\alpha = \left(\frac{\partial \bar{x}^\alpha}{\partial x^\mu}\right) V^\mu}
$$

$$
V^\mu = \text{components of } \boldsymbol{V} \text{ in basis } \left\{ \left(\frac{\partial}{\partial x^\mu}\right) \right\}
$$

$$
\bar{V}^\alpha = \text{components of } \boldsymbol{V} \text{ in basis } \left\{ \left(\frac{\partial}{\partial \bar{x}^\alpha}\right) \right\}
$$

2.4 Covectors

Def.: Let $\mathcal{V}$ be a vector space over $\mathbb{R}$.

"Dual space" $\mathcal{V}^* :=$ vector space of linear maps from $\mathcal{V}$ to $\mathbb{R}$

Lemma: $\mathcal{V}$ n-dimensional $\Rightarrow$ $\mathcal{V}^*$ n-dim.

if $\{\mathbf{e}_\mu\}$, $\mu = 1, \ldots, n$ is a basis of $\mathcal{V}$

$\Rightarrow \{\mathbf{f}^\alpha\}$, $\alpha = 1, \ldots, n$, defined by $\mathbf{f}^\alpha(\mathbf{e}_\mu) = \delta^\alpha{}_\mu$, is the "dual basis" of $\mathcal{V}^*$

Comments:
- $\mathcal{V}$, $\mathcal{V}^*$ are isomorphic ; e.g. $\mathbf{e}_\mu \mapsto \mathbf{f}^\mu$ defines an isomorphism
- The isomorphism is basis dependent
- There is a natural isomorphism between $\mathcal{V}$ and $(\mathcal{V}^*)^*$

Theorem: If $\mathcal{V}$ is finite dim.

$\Rightarrow$ A natural, basis independent isomorphism is given by

$\Phi : \mathcal{V} \to (\mathcal{V}^*)^*$, $\quad \boldsymbol{X} \mapsto \Phi(\boldsymbol{X})$ with $\big(\Phi(\boldsymbol{X})\big)(\boldsymbol{\omega}) := \boldsymbol{\omega}(\boldsymbol{X}) \quad \forall_{\boldsymbol{\omega} \in \mathcal{V}^*}$

Def.: "Cotangent space" $\mathcal{T}_p^*(\mathcal{M})$:= dual space of $\mathcal{T}_p(\mathcal{M})$

Its elements are "covectors" or "1-forms"

If $\{\mathbf{e}_\mu\}$ is a basis of $\mathcal{T}_p(\mathcal{M})$ and $\mathbf{f}^\mu$ the dual basis in $\mathcal{T}_p^*(\mathcal{M})$

$\Rightarrow$ $\boldsymbol{\eta} = \eta_\mu \mathbf{f}^\mu \in \mathcal{T}_p^*(\mathcal{M})$; η_μ are the "components" of $\boldsymbol{\eta}$

Comments:
- $\boldsymbol{\eta}(\mathbf{e}_\mu) = \eta_\nu \mathbf{f}^\nu(\mathbf{e}_\mu) = \eta_\mu$
- $\boldsymbol{X} \in \mathcal{T}_p(\mathcal{M}) \quad \Rightarrow \quad \boldsymbol{\eta}(\boldsymbol{X}) = \boldsymbol{\eta}(X^\mu \mathbf{e}_\mu) = X^\mu \boldsymbol{\eta}(\mathbf{e}_\mu) = X^\mu \eta_\mu$

Def.: Let $f : \mathcal{M} \to \mathbb{R}$ be a smooth function

"gradient of f" at $p :=$ $(\mathbf{d}f)_p \in \mathcal{T}_p^*(\mathcal{M})$ with

$(\mathbf{d}f)_p(\boldsymbol{X}) := \boldsymbol{X}(f) \quad \forall\ \boldsymbol{X} \in \mathcal{T}_p(\mathcal{M})$

Examples (1) Let (x^i) be a coord. chart in nbhd. of $p \in \mathcal{M}$ and $f := x^\mu(p)$ for some μ

$$\Rightarrow \quad \big(\mathbf{d}x^\mu\big)_p \in \mathcal{T}_p^*(\mathcal{M}) \ \text{ with } \ \big(\mathbf{d}x^\mu\big)_p\left(\left(\frac{\partial}{\partial x^\nu}\right)_p\right) = \left.\frac{\partial x^\mu}{\partial x^\nu}\right|_p = \delta^\mu{}_\nu$$

$$\Rightarrow \quad \left\{\big(\mathbf{d}x^\mu\big)_p\right\} \text{ is the dual basis of } \left\{\left(\frac{\partial}{\partial x^\mu}\right)_p\right\}$$

(2) Components of $\big(\mathbf{d}f\big)_p$:

$$\Big[\big(\mathbf{d}f\big)_p\Big]_\mu = \big(\mathbf{d}f\big)_p\left(\left(\frac{\partial}{\partial x^\mu}\right)_p\right) = \left(\frac{\partial}{\partial x^\mu}\right)_p(f) = \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)}$$

Coordinate transformation

If $\phi = (x^i)$, $\bar\phi = (\bar x^i)$ are two charts in nbhd. of $p \in \mathcal{M}$

$$\Rightarrow \ldots \Rightarrow \quad \big(\mathbf{d}x^\mu\big)_p = \left(\frac{\partial x^\mu}{\partial \bar x^\nu}\right)_{\bar\phi(p)} \big(\mathbf{d}\bar x^\nu\big)_p$$

so for $\boldsymbol{\omega} \in \mathcal{T}_p^*(\mathcal{M})$: $\boldsymbol{\omega} = \omega_\mu\,\mathbf{d}x^\mu = \bar\omega_\mu\,\mathbf{d}\bar x^\mu$ with $\bar\omega_\mu = \left(\frac{\partial x^\nu}{\partial \bar x^\mu}\right)_{\bar\phi(p)} \omega_\nu$ "covariant vector"

2.5 Abstract index notation

We have used $\mu, \nu, \ldots$ for components of vectors or 1-forms in a basis

Some expressions are basis dependent, some are not!

E.g.: $\boldsymbol{\eta}(\boldsymbol{X}) = \eta_\mu X^\mu$ independent

$X^\mu = \delta^\mu{}_1$ dependent

Index notation: If a statement is true in any basis, replace $\mu\,\nu, \ldots$ with $a, b, \ldots$

E.g.: $\boldsymbol{\eta}(\boldsymbol{X}) = \eta_a X^a$

Convention: $a, b, \ldots$ do not denote components, but place holders for component indices

X^a is a vector, η_a a 1-form, ... ; "$X^a \neq X^\mu$"

The rules for index positions are the same as for $\mu, \nu, \ldots$. E.g. $\eta_a \omega_a$ is wrong

2.6 Tensors

Tensors in physics: e.g. moment of inertia

In GR many things are tensors

Def.: A tensor of type (r, s) or $\binom{r}{s}$ is a multilinear map

$$\boldsymbol{T} : \underbrace{\mathcal{T}_p^*(\mathcal{M}) \times \ldots \times \mathcal{T}_p^*(\mathcal{M})}_{r \text{ factors}} \times \underbrace{\mathcal{T}_p(\mathcal{M}) \times \ldots \times \mathcal{T}_p(\mathcal{M})}_{s \text{ factors}} \to \mathbb{R}$$

A machine: input: r 1-forms, s vectors; output: a real number

Examples: (1) 1-form $=$ $(0,1)$ tensor : $\mathcal{T}_p(\mathcal{M}) \to \mathbb{R}$

(2) Recall: $\left(\mathcal{T}_p^*(\mathcal{M})\right)^*$ is naturally isomorphic to $\mathcal{T}_p(\mathcal{M})$

$\Rightarrow$ vector $= (1,0)$ tensor : $\mathcal{T}_p^*(\mathcal{M}) \to \mathbb{R}$, $\boldsymbol{\eta} \mapsto \boldsymbol{\eta}(\boldsymbol{X})$ $\forall\, \boldsymbol{\eta} \in \mathcal{T}_p^*(\mathcal{M})$

(3) Define the $(1,1)$ tensor $\boldsymbol{\delta} : \mathcal{T}_p^*(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}$ through

$\boldsymbol{\delta}(\boldsymbol{\eta}, \boldsymbol{X}) := \boldsymbol{\eta}(\boldsymbol{X})$ $\forall\, \boldsymbol{\eta} \in \mathcal{T}_p^*(\mathcal{M})$, $\boldsymbol{X} \in \mathcal{T}_p(\mathcal{M})$

Def.: Let $\{\mathbf{e}_\mu\}$ be a basis of $\mathcal{T}_p(\mathcal{M})$ and $\{\mathbf{f}^\nu\}$ the dual basis of $\mathcal{T}_p^*(\mathcal{M})$

The "components of a (r,s) tensor" $\boldsymbol{T}$ are

$$T^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s} := \boldsymbol{T}(\mathbf{f}^{\mu_1}, \mathbf{f}^{\mu_2}, \ldots, \mathbf{f}^{\mu_r}, \mathbf{e}_{\nu_1}, \mathbf{e}_{\nu_2}, \ldots, \mathbf{e}_{\nu_s})$$

In abstract index notation: $T^{a_1a_2\ldots a_r}{}_{b_1b_2\ldots b_s}$

Comment: Tensors of type (r,s) in $p \in \mathcal{M}$ can be added or multiplied by constants.

They form a vector space of dimension n^{r+s}

Examples: (1) $\boldsymbol{\delta}$ above: $\delta^\mu{}_\nu = \boldsymbol{\delta}(\mathbf{f}^\mu, \mathbf{e}_\nu) = \mathbf{f}^\mu(\mathbf{e}_\nu) = \delta^\mu{}_\nu$

(2) Let $\boldsymbol{\eta}$, $\boldsymbol{\omega} \in \mathcal{T}_p^*(\mathcal{M})$, $\boldsymbol{X} \in \mathcal{T}_p(\mathcal{M})$, $\boldsymbol{T}$ a $(2,1)$ tensor, $\{\mathbf{e}_\mu\}$, $\{\mathbf{f}^\nu\}$ bases

$$\Rightarrow \quad \boldsymbol{T}(\boldsymbol{\eta}, \boldsymbol{\omega}, \boldsymbol{X}) = \boldsymbol{T}(\eta_\mu \mathbf{f}^\mu, \omega_\nu \mathbf{f}^\nu, X^\alpha \mathbf{e}_\alpha)$$
$$= \eta_\mu \omega_\nu X^\alpha \boldsymbol{T}(\mathbf{f}^\mu, \mathbf{f}^\nu, \mathbf{e}_\alpha) = \eta_\mu \omega_\nu X^\alpha T^{\mu\nu}{}_\alpha$$

Index notation: $\eta_a \omega_b X^c T^{ab}{}_c$

Change of basis

Let $\{\mathbf{e}_\mu\}$, $\{\bar{\mathbf{e}}_\nu\}$ be bases of $\mathcal{T}_p(\mathcal{M})$ and $\{\mathbf{f}^\mu\}$, $\{\bar{\mathbf{f}}^\nu\}$ the dual bases of $\mathcal{T}_p^*(\mathcal{M})$

Transformation matrices: $\bar{\mathbf{f}}^\mu = A^\mu{}_\nu \mathbf{f}^\nu$, $\bar{\mathbf{e}}_\mu = B^\nu{}_\mu \mathbf{e}_\nu$

We have: $\delta^\mu{}_\nu = \bar{\mathbf{f}}^\mu(\bar{\mathbf{e}}_\nu) = A^\mu{}_\rho \mathbf{f}^\rho(B^\sigma{}_\nu \mathbf{e}_\sigma) = A^\mu{}_\rho B^\sigma{}_\nu \underbrace{\mathbf{f}^\rho(\mathbf{e}_\sigma)}_{=\delta^\rho{}_\sigma} = A^\mu{}_\rho B^\rho{}_\nu$

$$\Rightarrow B^\mu{}_\nu = \left(A^{-1}\right)^\mu{}_\nu \quad \text{are inverses!}$$

E.g. coord. basis: $A^\mu{}_\nu = \dfrac{\partial \bar{x}^\mu}{\partial x^\nu}$, $B^\nu{}_\mu = \dfrac{\partial x^\nu}{\partial \bar{x}^\mu}$ are obviously inverses

One straightforwardly shows: vector: $\bar{X}^\mu = A^\mu{}_\nu X^\nu$

1-form: $\bar{\eta}_\mu = (A^{-1})^\nu{}_\mu \eta_\nu$

$(2,1)$ tensor: $\bar{T}^{\mu\nu}{}_\rho = A^\mu{}_\alpha A^\nu{}_\beta \,(A^{-1})^\gamma{}_\rho \,T^{\alpha\beta}{}_\gamma$

(r,s) tensor: obvious...

Def.: "Contraction of (r,s) tensor" := Summation over 1 upper and 1 lower index

$\rightarrow$ $(r-1, s-1)$ tensor

Example: Let $\boldsymbol{T}$ be a $(3,2)$ tensor

$\Rightarrow$ $(2,1)$ tensor $\boldsymbol{S}(\boldsymbol{\omega},\boldsymbol{\eta},\boldsymbol{X}) := \boldsymbol{T}(\mathbf{f}^\mu,\boldsymbol{\omega},\boldsymbol{\eta},\mathbf{e}_\mu,\boldsymbol{X})$

This is basis independent:

$$\boldsymbol{T}(\bar{\mathbf{f}}^\mu,\boldsymbol{\omega},\boldsymbol{\eta},\bar{\mathbf{e}}_\mu,\boldsymbol{X}) = \boldsymbol{T}\left(A^\mu{}_\nu \mathbf{f}^\nu,\boldsymbol{\omega},\boldsymbol{\eta},\left(A^{-1}\right)^\rho{}_\mu \mathbf{e}_\rho,\boldsymbol{X}\right) = \boldsymbol{T}(\mathbf{f}^\nu,\boldsymbol{\omega},\boldsymbol{\eta},\mathbf{e}_\nu,\boldsymbol{X})$$

Components: $S^{\mu\nu}{}_\rho = T^{\alpha\mu\nu}{}_{\alpha\rho}$

Abstract index notation: $S^{ab}{}_c = T^{dab}{}_{dc}$

Note: In general $T^{dab}{}_{dc} \neq T^{abd}{}_{dc}$ $\Rightarrow$ index position important!

Def.: Let $\boldsymbol{S}$ be a (p,q) tensor, $\boldsymbol{T}$ a (r,s) tensor

"outer product" $\boldsymbol{S}\otimes\boldsymbol{T}$ is a $(p+r,q+s)$ tensor with

$$\big(\boldsymbol{S}\otimes\boldsymbol{T}\big)(\boldsymbol{\omega}_1,\,\dots,\,\boldsymbol{\omega}_p,\,\boldsymbol{\eta}_1,\,\dots,\,\boldsymbol{\eta}_r,\,\boldsymbol{X}_1,\,\dots,\,\boldsymbol{X}_q,\,\boldsymbol{Y}_1,\,\dots,\,\boldsymbol{Y}_s)$$

$$:=\boldsymbol{S}(\boldsymbol{\omega}_1,\,\dots,\,\boldsymbol{\omega}_p,\,\boldsymbol{X}_1,\,\dots,\,\boldsymbol{X}_q)\,\boldsymbol{T}(\boldsymbol{\eta}_1,\,\dots,\,\boldsymbol{\eta}_r,\,\boldsymbol{Y}_1,\,\dots,\,\boldsymbol{Y}_s)$$

One straightforwardly shows:

(1) $\big(S\otimes T\big)^{a_1\dots a_p b_1\dots b_r}{}_{c_1\dots c_q d_1\dots d_s}=S^{a_1\dots a_p}{}_{c_1\dots c_q}\,T^{b_1\dots b_r}{}_{d_1\dots d_s}$

(2) In a coord. basis, a $(2,1)$ tensor can be written as

$$\boldsymbol{T}=T^{\mu\nu}{}_{\rho}\left(\frac{\partial}{\partial x^\mu}\right)_p\otimes\left(\frac{\partial}{\partial x^\nu}\right)_p\otimes\big(\mathbf{d}x^\rho\big)_p$$

likewise (r,s) tensor

Comment: We always first put in 1-forms into a tensor, then vectors.

This is not necessary. We can define

$$\boldsymbol{T}:\mathcal{T}_p^*\times\mathcal{T}_p\times\mathcal{T}_p^*\to\mathbb{R}\,,\quad(\boldsymbol{\eta},\boldsymbol{X},\boldsymbol{\omega})\mapsto\boldsymbol{T}(\boldsymbol{\eta},\boldsymbol{X},\boldsymbol{\omega})$$

and this is isomorphic to

$$\boldsymbol{T}:\mathcal{T}_p^*\times\mathcal{T}_p^*\times\mathcal{T}_p\to\mathbb{R}\,,\quad(\boldsymbol{\eta},\boldsymbol{\omega},\boldsymbol{X})\mapsto\boldsymbol{T}(\boldsymbol{\eta},\boldsymbol{\omega},\boldsymbol{X})\,.$$

So we do not distinguish between them.

But be careful with index positions: In general $T^{ab}{}_c\eta_a\omega_b=T^{ba}{}_c\eta_b\omega_a\neq T^{ba}{}_c\eta_a\omega_b$

Def.: Let $\boldsymbol{T}$ be a $(0,2)$ tensor.

"Symmetrization": $S_{ab} := \frac{1}{2}(T_{ab} + T_{ba}) =: T_{(ab)}$

"Anti-symmetrization": $A_{ab} := \frac{1}{2}(T_{ab} - T_{ba}) =: T_{[ab]}$

Can be applied to a subset of indices: $T^{(ab)c}{}_d = \frac{1}{2}(T^{abc}{}_d + T^{bac}{}_d)$

Over $n > 2$ indices:
- sum over all permutations
- apply sign of permutation for anti-symm.
- divide by $n!$

E.g.: $T^a{}_{[bcd]} = \frac{1}{3!}\left(T^a{}_{bcd} + T^a{}_{dbc} + T^a{}_{cdb} - T^a{}_{dcb} - T^a{}_{cbd} - T^a{}_{bdc}\right)$

For non-adjacent indices: $T_{(a|bc|d)} := \frac{1}{2}\left(T_{abcd} + T_{dbca}\right)$

2.7 Tensor Fields

So far: tensors at point $p \in \mathcal{M}$; Now: fields

Def.: vector field := a map $\boldsymbol{X} : \mathcal{M} \to \mathcal{T}_p(\mathcal{M})$, $p \to \boldsymbol{X}_p$

Let $f : \mathcal{M} \to \mathbb{R}$ be smooth

$\Rightarrow$ $\boldsymbol{X}(f)$ is a function $\boldsymbol{X}(f) : \mathcal{M} \to \mathbb{R}$, $p \mapsto \boldsymbol{X}_p(f)$

$\boldsymbol{X}$ is smooth $:\Leftrightarrow$ $\boldsymbol{X}(f)$ smooth for all smooth f

Example: Let $\phi = (x^\mu)$ be a chart and $\boldsymbol{\partial}_\mu := \left(\frac{\partial}{\partial x^\mu}\right)$ be the vector field defined by $p \mapsto \left(\frac{\partial}{\partial x^\mu}\right)_p$

$$\Rightarrow \boldsymbol{\partial}_\mu(f) : \mathcal{M} \to \mathbb{R} \,, \quad p \mapsto \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)} \quad \text{where} \quad F = f \circ \phi^{-1}$$

Note: Everything is smooth: $\phi(p)$, $F(x^\mu)$, $\frac{\partial F}{\partial x^\mu}$ $\Rightarrow$ $\boldsymbol{\partial}_\mu(f) : \mathcal{M} \to \mathbb{R}$ is smooth.

ϕ may only cover a subset of $\mathcal{M}$ and the map only part of $\mathcal{M} \to \phi_\alpha$ on patches $\mathcal{O}_\alpha$

Comment: $\left(\dfrac{\partial}{\partial x^\mu}\right)_p$ is basis of $\mathcal{T}_p$

$\Rightarrow$ Expand vector field: $\boldsymbol{X} = X^\mu \left(\dfrac{\partial}{\partial x^\mu}\right) = X^\mu \boldsymbol{\partial}_\mu$

$\boldsymbol{\partial}_\mu$ smooth $\Rightarrow$ $\big(\boldsymbol{X}$ smooth $\Leftrightarrow$ X^μ smooth functions$\big)$

Def.: Covector field $\boldsymbol{\omega} :=$ map $: \mathcal{M} \to \mathcal{T}_p^*(\mathcal{M})$, $p \mapsto \boldsymbol{\omega}_p$

Note: a vector field $\boldsymbol{X}$ and covector field $\boldsymbol{\omega}$ define a function

$\boldsymbol{\omega}(\boldsymbol{X}) : \mathcal{M} \to \mathbb{R}$, $p \mapsto \boldsymbol{\omega}_p(\boldsymbol{X}_p)$

$\boldsymbol{\omega}$ smooth $:\Leftrightarrow$ $\boldsymbol{\omega}(\boldsymbol{X})$ is smooth for all smooth $\boldsymbol{X}$

Example: $\mathbf{d}f : \mathcal{M} \to \mathcal{T}_p^*(\mathcal{M})$, $p \mapsto (\mathbf{d}f)_p$

f, $\boldsymbol{X}$ smooth $\Rightarrow$ $\mathbf{d}f(\boldsymbol{X}) = \boldsymbol{X}(f)$ is a smooth function

$\Rightarrow$ $\mathbf{d}f$ is smooth, "gradient"

Set $f = x^\mu$ $\Rightarrow$ $\mathbf{d}x^\mu$ is a smooth covector field

Def.: (r,s) Tensor field $:=$ map $\boldsymbol{T} : \mathcal{M} \to$ (r,s) tensor at $p \in \mathcal{M}$

Smooth vector, covector fields $\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_r, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_s$ define a function

$$\boldsymbol{T}(\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_r, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_s) : \mathcal{M} \to \mathbb{R}\,, \quad p \mapsto \boldsymbol{T}_p\Big((\boldsymbol{\eta}_1)_p, \ldots, (\boldsymbol{\eta}_r)_p, (\boldsymbol{X}_1)_p, \ldots, (\boldsymbol{X}_s)_p\Big)$$

$\boldsymbol{T}$ smooth $:\Leftrightarrow$ this function is smooth $\forall$ smooth $\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_r, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_s$

Note: One can show: $\boldsymbol{T}$ smooth $\Leftrightarrow$ its components in coord. basis are smooth

From now on: assume all our tensors are smooth

2.8 The commutator

Let $\boldsymbol{X}$, $\boldsymbol{Y}$ be vector fields, f, g functions

$\Rightarrow$ $\boldsymbol{Y}(f)$ is a function $\Rightarrow$ $\boldsymbol{X}\big(\boldsymbol{Y}(f)\big)$ is a function

But: $\boldsymbol{X}\big(\boldsymbol{Y}(f\,g)\big) = \boldsymbol{X}\big(f\,\boldsymbol{Y}(g) + g\,\boldsymbol{Y}(f)\big)$

$$= f\,\boldsymbol{X}\big(\boldsymbol{Y}(g)\big) + \boldsymbol{X}(f)\,\boldsymbol{Y}(g) + \boldsymbol{X}(g)\,\boldsymbol{Y}(f) + g\,\boldsymbol{X}\big(\boldsymbol{Y}(f)\big)$$

$$\neq f\,\boldsymbol{X}\big(\boldsymbol{Y}(g)\big) + g\,\boldsymbol{X}\big(\boldsymbol{Y}(f)\big) \qquad \text{“no Leibniz”!}$$

$\Rightarrow$ The map $f \mapsto \boldsymbol{X}\big(\boldsymbol{Y}(f)\big)$ does not define a vector field. But: ...

Def.: Commutator of 2 vectorfields $\boldsymbol{X}$, $\boldsymbol{Y}$:

$$[\boldsymbol{X},\boldsymbol{Y}](f) := \boldsymbol{X}\big(\boldsymbol{Y}(f)\big) - \boldsymbol{Y}\big(\boldsymbol{X}(f)\big) \;\; \text{satisfies Leibniz!}$$

$[\boldsymbol{X},\boldsymbol{Y}]$ is indeed a vectorfield

Proof: coord. chart (x^μ)

$$\begin{aligned}
\Rightarrow \;\; [\boldsymbol{X},\boldsymbol{Y}](f) &= \boldsymbol{X}\left(Y^\nu \frac{\partial F}{\partial x^\nu}\right) - \boldsymbol{Y}\left(X^\mu \frac{\partial F}{\partial x^\mu}\right) \\
&= X^\mu \frac{\partial}{\partial x^\mu}\left(Y^\nu \frac{\partial F}{\partial x^\nu}\right) - Y^\nu \frac{\partial}{\partial x^\nu}\left(X^\mu \frac{\partial F}{\partial x^\mu}\right) \\
&= X^\mu \frac{\partial Y^\nu}{\partial x^\mu}\frac{\partial F}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu}\frac{\partial F}{\partial x^\mu} \\
&= \underbrace{\left(X^\nu \frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu}\right)}_{[X,Y]^\mu :=} \frac{\partial F}{\partial x^\mu}
\end{aligned}$$

$$f \text{ arbitrary} \;\; \Rightarrow \;\; [\boldsymbol{X},\boldsymbol{Y}] = [X,Y]^\mu \left(\frac{\partial}{\partial x^\mu}\right) \qquad \square$$

Example: Let $\boldsymbol{X} = \dfrac{\partial}{\partial x^1}$, $\quad \boldsymbol{Y} = x^1 \dfrac{\partial}{\partial x^2} + \dfrac{\partial}{\partial x^3}$

$$\Rightarrow \;\; [X,Y]^\mu = \frac{\partial Y^\mu}{\partial x^1} = \delta^\mu{}_2$$

$$\Rightarrow \;\; [\boldsymbol{X},\boldsymbol{Y}] = \frac{\partial}{\partial x^2}$$

One can show: $[\boldsymbol{X}, \boldsymbol{Y}] = -[\boldsymbol{Y}, \boldsymbol{X}]$

$$[\boldsymbol{X}, \boldsymbol{Y} + \boldsymbol{Z}] = [\boldsymbol{X}, \boldsymbol{Y}] + [\boldsymbol{X} + \boldsymbol{Z}]$$

$$[\boldsymbol{X}, f\,\boldsymbol{Y}] = f\,[\boldsymbol{X}, \boldsymbol{Y}] + \boldsymbol{X}(f)\,\boldsymbol{Y}$$

$$[\boldsymbol{X}, [\boldsymbol{Y}, \boldsymbol{Z}]] + [\boldsymbol{Y}, [\boldsymbol{Z}, \boldsymbol{X}]] + [\boldsymbol{Z}, [\boldsymbol{X}, \boldsymbol{Y}]] = 0 \quad \text{“Jacobi identity”}$$

Note: $\left[\dfrac{\partial}{\partial x^\mu}, \dfrac{\partial}{\partial x^\nu}\right] = 0$ (coord. basis $\Rightarrow$ commutators vanish)

Conversely, one can show:

If $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$, $m \leq \dim(\mathcal{M})$ are vector fields which are

lin. indep. $\forall p \in \mathcal{M}$ and whose commutators all vanish

$\Rightarrow$ In a nbhd. of p one can find coords. (x^μ)

such that $\boldsymbol{X}_i = \dfrac{\partial}{\partial x^i}$, $i = 1, \ldots, m$

2.9 Integral curves

<u>**Def.:**</u> Let $\boldsymbol{X}$ be a VF and $p \in \mathcal{M}$.

“integral curve of $\boldsymbol{X}$ through p”

:= curve through p whose tangent at every point q is $\boldsymbol{X}_q$

Let λ be an integral curve of $\boldsymbol{X}$, $\lambda(0) = p$, (x^μ) be a coord. chart

$$\Rightarrow \frac{dx^\mu\big(\lambda(t)\big)}{dt} = X^\mu\Big(x^\alpha\big(\lambda(t)\big)\Big), \quad x^\mu\big(\lambda(0)\big) = x^\mu_p \qquad (*)$$

ODE theory guarantees existence, uniqueness of solution

$\Rightarrow$ $\exists$ unique integral curve of $\boldsymbol{X}$ through $p \in \mathcal{M}$

Example: Chart $\phi = (x^\mu)$, let $\boldsymbol{X} = \dfrac{\partial}{\partial x^1} + x^1 \dfrac{\partial}{\partial x^2}$, $x^\mu(p) = (0, \ldots, 0)$

$$(*) \quad \Rightarrow \quad \frac{dx^1}{dt} = 1\,, \quad \frac{dx^2}{dt} = x^1$$

$$\Rightarrow \ldots \Rightarrow \quad x^1 = t\,, \qquad x^2 = \frac{t^2}{2}\,, \qquad x^i = 0 \;\text{ for }\; i = 3, \ldots, n$$

3 The metric tensor

3.1 Metrics

We want to measure things $\rightarrow$ need metric!

E.g.: $\mathbb{R}^3$, scalar product: maps 2 vectors to $\mathbb{R}$

$\Rightarrow$ metric should be $(0,2)$ tensor

Def.: A metric at $p \in \mathcal{M}$:= $(0,2)$ tensor that is:

(i) symmetric: $\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{g}(\boldsymbol{Y}, \boldsymbol{X}) \quad \forall\, \boldsymbol{X}, \boldsymbol{Y} \in \mathcal{T}_p(\mathcal{M}) \quad \Leftrightarrow \quad g_{ab} = g_{ba}$

(ii) non-degenerate: $\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y}) = 0 \;\; \forall\, \boldsymbol{Y} \in \mathcal{T}_p(\mathcal{M}) \quad \Leftrightarrow \quad \boldsymbol{X} = 0$

Notation: $\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y}) = \langle \boldsymbol{X}, \boldsymbol{Y} \rangle = \boldsymbol{X} \cdot \boldsymbol{Y}$

Comment: a metric defines an isomorphism between vectors and 1-forms:

$$\boldsymbol{X} \mapsto \boldsymbol{g}(\boldsymbol{X}, .) =: \underline{\boldsymbol{X}}\,, \;\text{ i.e. }\; \underline{\boldsymbol{X}} : \mathcal{T}_p(\mathcal{M}) \rightarrow \mathbb{R}\,, \quad \boldsymbol{Y} \mapsto \underline{\boldsymbol{X}}(\boldsymbol{Y}) := \boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y})$$

with the metric inverse (see below), we can raise and lower indices of tensors

Signature

$\boldsymbol{g}$ symmetric $\Rightarrow$ components of $\boldsymbol{g}$ at $p \in \mathcal{M}$ are a symmetric matrix

$\Rightarrow$ $\exists$ basis where $g_{\mu\nu}$ is diagonal

$\boldsymbol{g}$ non-degenerate $\Rightarrow$ all diagonal elements are $\neq 0$

$\Rightarrow$ we can rescale the basis such that the diagonal elements $= \pm 1$

"orthonormal basis" $\leftarrow$ basis non-unique!

"Sylvester's law" $\Rightarrow$ the number of $+1$ and -1 entries is independent of basis

Def.: "signature" := sum $+1$, -1 over all diagonal elements

Riemannian metrics: signature $= + + \ldots +$ or $+n = \#$ of dims.

Lorentzian metrics: $- + + \ldots +$ or $n - 2$. Some people use $+ - - \ldots -$

Note: Equivalence principle

$\Rightarrow$ in a local inertial frame, the laws of SR hold

$\Rightarrow$ $\exists$ chart: metric $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ "Lorentz invariant"

Only possible locally! At $q \neq p$, $g_{\mu\nu} \neq \eta_{\mu\nu}$ in general

Def.: "A Riemannian (Lorentzian) manifold"

:= $(\mathcal{M}, \boldsymbol{g})$ where $\mathcal{M}$ is a diff. manifold and $\boldsymbol{g}$ a Riemannian (Lorentzian) metric

"spacetime" := Lorentzian manifold

Notation: in coord. basis: $\boldsymbol{g} = g_{\mu\nu}\, \mathbf{d}x^\mu \otimes \mathbf{d}x^\nu$

often used: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

Comment: Let $\lambda : (a,b) \subset \mathbb{R} \to \mathcal{M}$ be a smooth curve on a Riemannian manifold, $\boldsymbol{X}$ be the tangent vector of λ

Then the length of λ is: $\displaystyle\int_a^b \sqrt{\boldsymbol{g}(\boldsymbol{X},\boldsymbol{X})_{\lambda(t)}}\,dt$

re-parametrize: $t = t(u)$ with $\dfrac{dt}{du} > 0\,,\quad u \in (c,d)\,,\quad t(c) = a\,,\quad t(d) = b$

$\Rightarrow$ the curve $\kappa(u) := \lambda\big(t(u)\big)$ has tangent vector $\boldsymbol{Y} = \dfrac{dt}{du}\boldsymbol{X}$

$\Rightarrow$ the length of κ is the same as that of λ.

<u>Examples</u>

(1) Euclidean metric in $\mathbb{R}^n$ with coords. $x^1, \ldots, x^n$:

$$\boldsymbol{g} = \mathbf{d}x^1 \otimes \mathbf{d}x^1 + \ldots + \mathbf{d}x^n \otimes \mathbf{d}x^n\,.$$

A coord. chart of $(\mathbb{R}^n, \boldsymbol{g})$ where $g_{\mu\nu} = \mathrm{diag}(1, \ldots, 1)$ is called "Cartesian"

(2) Minkowski metric in $\mathbb{R}^4$ with coords. x^0, x^1, x^2, x^3:

$$\boldsymbol{\eta} = -(\mathbf{d}x^0)^2 + (\mathbf{d}x^1)^2 + (\mathbf{d}x^2)^2 + (\mathbf{d}x^3)^2\,, \qquad (\mathbf{d}x^0)^2 \equiv \mathbf{d}x^0 \otimes \mathbf{d}x^0\,, \ldots$$

A coord. chart which covers $\mathbb{R}^4$ such that $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is called "inertial frame". $(\mathbb{R}^4, \boldsymbol{\eta}) =:$ Minkowski spacetime

(3) Let (θ, ϕ) be spherical coords. on S^2 $\Rightarrow$ $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

This is positive definite on $\theta \in (0, \pi)$ but not on all S^2

$\Rightarrow$ We need second chart, e.g. θ', ϕ' with

$$x = -\sin\theta' \cos\phi'\,, \quad y = \cos\theta'\,, \quad z = \sin\theta' \sin\phi'$$

<u>Def.:</u> $\boldsymbol{g}$ non-degenerate $\Rightarrow$ $\boldsymbol{g}$ invertible

"inverse metric" $= \boldsymbol{g}^{-1} :=$ symmetric $(2,0)$ tensor g^{ab} with $g^{ab}g_{bc} = \delta^a{}_c$

Example: On S^2 on the chart (θ, ϕ) we have $g^{\mu\nu} = \mathrm{diag}(1,\, 1/\sin^2\theta)$

Comment: $\boldsymbol{g}^{-1}$ maps 1-forms to vectors: $\big(\boldsymbol{g}^{-1}(\boldsymbol{\eta},\ .\ )\big)(\boldsymbol{\omega}) := \boldsymbol{g}^{-1}(\boldsymbol{\eta},\boldsymbol{\omega})$

The metric mappings between vectors and 1-forms are inverses of each other:

$\boldsymbol{g}^{-1}\big(\boldsymbol{g}(\boldsymbol{X},\ .\ ),\ .\ \big) = \boldsymbol{X}\,, \quad \boldsymbol{g}\big(\boldsymbol{g}^{-1}(\boldsymbol{\eta},\ .\ ),\ .\ \big) = \boldsymbol{\eta}$

$\rightarrow$ natural isomorphism

Example: Let $\boldsymbol{T}$ be a $(3,2)$ tensor: $T^{a}{}_{b}{}^{cde} = g_{bf}g^{dh}g^{ej}T^{afc}{}_{hj}$

– we use the same letter T irrespective of the up or down position of indices

– the order of indices is preserved!

3.2 Lorentzian signature

Note: indices typically chosen to run from $0\ldots3$

At any $p\in\mathcal{M}$ of a Lorentzian manifold:

we can choose orthonormal basis (ONB): $\boldsymbol{g}(\mathbf{e}_\mu,\mathbf{e}_\nu) = \eta_{\mu\nu} = \mathrm{diag}(-1,\,1,\,1,\,1)$

This basis is not unique:

$$\bar{\mathbf{e}}_\mu = \big(A^{-1}\big)^{\nu}{}_{\mu}\mathbf{e}_\nu$$

$\Rightarrow \bar{g}_{\mu\nu} = \boldsymbol{g}(\bar{\mathbf{e}}_\mu,\bar{\mathbf{e}}_\nu) = \big(A^{-1}\big)^{\rho}{}_{\mu}\big(A^{-1}\big)^{\sigma}{}_{\nu}\,\boldsymbol{g}(\mathbf{e}_\rho,\mathbf{e}_\sigma) = \big(A^{-1}\big)^{\rho}{}_{\mu}\big(A^{-1}\big)^{\sigma}{}_{\nu}\,\eta_{\rho\sigma} \stackrel{!}{=} \eta_{\mu\nu}$

$\Rightarrow A^{\mu}{}_{\rho}A^{\nu}{}_{\sigma}\eta_{\mu\nu} = \eta_{\rho\sigma}$ “Lorentz trafos of SR!

$\Rightarrow$ ONBs are related by Lorentz trafos

$\Rightarrow$ locally at p we recover SR

<u>**Def.:**</u> Let $(\mathcal{M},\boldsymbol{g})$ be a Lorentzian manifold, $\boldsymbol{X}\in\mathcal{T}_p(\mathcal{M})$, $\boldsymbol{X}\neq0$

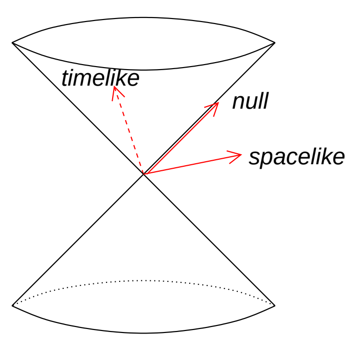
$\boldsymbol{X}$ is timelike $:\Leftrightarrow\ \boldsymbol{g}(\boldsymbol{X},\boldsymbol{X})<0$

null $:\Leftrightarrow\ \boldsymbol{g}(\boldsymbol{X},\boldsymbol{X})=0$

spacelike $:\Leftrightarrow\ \boldsymbol{g}(\boldsymbol{X},\boldsymbol{X})>0$

In an ONB, $g_{\mu\nu}=\eta_{\mu\nu}$ locally

$\Rightarrow$ locally we have the light cone structure of SR



One can show: If $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{T}_p(\mathcal{M})$, $\boldsymbol{X}, \boldsymbol{Y} \neq 0$ with $\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y}) = 0$. Then

$\boldsymbol{X}$ timelike $\Rightarrow$ $\boldsymbol{Y}$ spacelike

$\boldsymbol{X}$ null $\Rightarrow$ $\boldsymbol{Y}$ spacelike or null

$\boldsymbol{X}$ spacelike $\Rightarrow$ $\boldsymbol{Y}$ spacelike, timelike or null

Principle of proof: Apply spatial rotation such that $\boldsymbol{X}$ has simple space components.

E.g. timelike $\boldsymbol{X}$ $\rightarrow X^\mu = (X^0,\, X^1,\, 0,\, 0)$

Def.: On a Riemannian manifold:

"norm" of $\boldsymbol{X} \in \mathcal{T}_p(\mathcal{M})$: $|\boldsymbol{X}| := \sqrt{\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{X})}$

"angle" between $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{T}_p(\mathcal{M})$: $\theta := \arccos\left(\dfrac{\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y})}{|\boldsymbol{X}|\,|\boldsymbol{Y}|}\right)$

Same for spacelike vectors in Lorentzian manifold.

Def.: A curve is timelike (null, spacelike)

$:\Leftrightarrow$ its tangent vector is timelike (null, spacelike) everywhere

Comments:

- curves often change their character between timelike, null, spacelike
- the length of a spacelike curve λ is

$$s = \int_{t_0}^{t_1} \sqrt{\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{X})|_{\lambda(t)}}\, dt$$

- for timelike curves we define the proper time along the curve as

$$\tau := \int_{t_0}^{t_1} \sqrt{-\,\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{X})|_{\lambda(t)}}\, dt$$

In a coord. chart: $X^\mu = \dfrac{dx^\mu}{dt}$, so we often write:

$$d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu\,, \quad \tau = \int d\tau$$

Def.: "4-velocity" of a timelike curve λ

:= tangent vector of the curve parametrized by the proper time

$$u^\mu = \frac{dx^\mu}{d\tau}\bigg|_{\lambda(\tau)}$$

Note: along this curve:

$$\tau = \int_{\tau_0}^{\tau_1} \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau = \int_{\tau_0}^{\tau_1} \sqrt{-g_{\mu\nu}u^\mu u^\nu}d\tau \qquad \bigg| \ \frac{d}{d\tau}$$

$$\Rightarrow 1 = \sqrt{-g_{\mu\nu}u^\mu u^\nu}$$

$$\Rightarrow g_{\mu\nu}u^\mu u^\nu = -1$$

3.3 Curves of extremal proper time

Let $p, q \in \mathcal{M}$ be connected by a timelike curve λ

A small deformation of λ is still timelike

Which curve connecting p, q extremizes the proper time along it?

Let u be a parameter such that $\lambda(u=0) = p$, $\lambda(u=1) = q$. Let $\dot{} := \frac{d}{du}$.

$$\Rightarrow \tau[\lambda] = \int_0^1 G\big(x(u), \dot{x}(u)\big)du \quad \text{with} \quad G = \sqrt{-g_{\mu\nu}\big(x(u)\big)\,\dot{x}^\mu(u)\,\dot{x}^\nu(u)}$$

$$\text{and} \quad x(u) := x\big(\lambda(u)\big)$$

This is an Euler-Lagrange problem

$\Rightarrow$ the extremal curve satisfies $\displaystyle\frac{d}{du}\left(\frac{\partial G}{\partial \dot{x}^\mu}\right) - \frac{\partial G}{\partial x^\mu} = 0$

We have: $$\frac{\partial G}{\partial \dot{x}^\mu} = -\frac{1}{2G}\,2g_{\mu\nu}\dot{x}^\nu = -\frac{1}{G}g_{\mu\nu}\dot{x}^\nu$$

$$\frac{\partial G}{\partial x^\mu} = -\frac{1}{2G}\,\partial_\mu g_{\nu\rho}\,\dot{x}^\nu\dot{x}^\rho\,, \qquad \text{where} \quad \partial_\mu := \frac{\partial}{\partial x^\mu}$$

Now change to the proper time as a parameter: $\displaystyle\tau = \int \sqrt{-g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du}}du$

$$\Rightarrow \quad \left(\frac{d\tau}{du}\right)^2 = -g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = G^2$$

$$\Rightarrow \quad \frac{d\tau}{du} = G$$

$$\Rightarrow \quad \frac{d}{du} = G\frac{d}{d\tau}$$

$$\Rightarrow \ldots \Rightarrow \text{ Euler-Lagrange Eq.: } \frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) - \frac{1}{2}\partial_\mu g_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

$$\Rightarrow \quad g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} + \underbrace{\partial_\rho g_{\mu\nu}}_{\stackrel{!}{=}\partial_\rho g_{(\mu\nu)}} \underbrace{\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau}}_{\text{symm. in } \rho,\nu} - \frac{1}{2}\partial_\mu g_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \quad \Big| \cdot g^{\alpha\mu}$$

$$\Rightarrow \quad \boxed{\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0} \qquad (*)$$

with $\boxed{\Gamma^\alpha_{\nu\rho} = \frac{1}{2}g^{\alpha\mu}\left(\partial_\rho g_{\mu\nu} + \partial_\nu g_{\rho\mu} - \partial_\mu g_{\nu\rho}\right)}$ "Christoffel symbols"

Comments:
- $\Gamma^\alpha_{\nu\rho} = \Gamma^\alpha_{\rho\nu}$
- $\Gamma^\alpha_{\nu\rho}$ are not tensor components
- The individual terms of $(*)$ are not vector components, but the sum is
- $(*)$ is called the "geodesic equation"
- In Minkowski: $\Gamma^\alpha_{\nu\rho} = 0 \quad \Rightarrow \quad \frac{d^2x^\alpha}{d\tau^2} = 0$

  $\Rightarrow$ The eqs. of motion of a free particle extremize proper time

Postulate: Massive particles in GR follow curves of extremal proper time, i.e. follow $(*)$

Comments:
- massless particles follow a similar equation
- In Minkowski: curves of extremal proper time maximize proper time between 2 points.

  In GR: This holds locally; the max. may not be a global one.

<u>One can show the following:</u>

(1) $(*)$ are the Euler-Lagrange eqs. of $L = -g_{\mu\nu}\big(x(\tau)\big)\,\dfrac{dx^\mu}{d\tau}\dfrac{dx^\nu}{d\tau}$

$\rightarrow$ easy way to calculate $\Gamma^\alpha_{\nu\rho}$

(2) L has no explicit τ dependence: $\dfrac{\partial L}{\partial \tau} = 0$

with EL eqs.: $\Rightarrow \ldots \Rightarrow \quad L - \dfrac{\partial L}{\partial \dot{x}^\mu}\,\dot{x}^\mu = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = g_{\mu\nu}\dfrac{dx^\mu}{d\tau}\dfrac{dx^\nu}{d\tau}$

is conserved along curves of extremal proper time: $\dfrac{d}{d\tau}\,\boldsymbol{\cdot} = 0$

It better be! 4-velocity $u^\mu = \dfrac{dx^\mu}{d\tau}\,, \quad g_{\mu\nu}u^\mu u^\nu = -1$

Example: Schwarzschild metric in Schwarzschild coords.:

$$ds^2 = -f\,dt^2 + f^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2\,, \quad f = 1 - \frac{2M}{r}\,, \quad M = \text{const}$$

$\Rightarrow \quad L = f\dot{t}^2 - f^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\,\dot{\phi}^2$

EL for $t(\tau)$: $\dfrac{d}{d\tau}\big(2f\dot{t}\big) = 0 \quad \Rightarrow \quad \dfrac{d^2t}{d\tau^2} + f^{-1}\dfrac{df}{dr}\,\dot{t}\,\dot{r} = 0$

$\Rightarrow \quad \Gamma^t_{tr} = \Gamma^t_{rt} = \dfrac{df/dr}{2f}\,, \quad \Gamma^t_{\mu\nu} = 0$ otherwise

cf. Example sheet 1

4 Covariant derivative

4.1 Introduction

Physical laws involve derivatives.

For functions we have: $\dfrac{\partial f}{\partial x^\mu}$ are the components of the gradient $\mathbf{d}f$

Vectors and tensors: This does not work. We cannot take the difference between vectors at different points: $\boldsymbol{U} \in \mathcal{T}_p(\mathcal{M})$, $\boldsymbol{V} \in \mathcal{T}_q(\mathcal{M})$

$\rightarrow$ Covariant derivative ∇ on manifold $\mathcal{M}$

Def.: "Covariant derivative ∇" := map from two smooth vectorfields $\boldsymbol{X}$, $\boldsymbol{Z}$ to a smooth vectorfield $\nabla_{\boldsymbol{X}}\boldsymbol{Z}$ with

(1) $\nabla_{f\boldsymbol{X}+g\boldsymbol{Y}}\boldsymbol{Z} = f\,\nabla_{\boldsymbol{X}}\boldsymbol{Z} + g\,\nabla_{\boldsymbol{Y}}\boldsymbol{Z}$, f, g functions

(2) $\nabla_{\boldsymbol{X}}(\boldsymbol{Y}+\boldsymbol{Z}) = \nabla_{\boldsymbol{X}}\boldsymbol{Y} + \nabla_{\boldsymbol{X}}\boldsymbol{Z}$

(3) $\nabla_{\boldsymbol{X}}(f\boldsymbol{Y}) = f\,\nabla_{\boldsymbol{X}}\boldsymbol{Y} + (\nabla_{\boldsymbol{X}}f)\,\boldsymbol{Y}$ "Leibniz"; $\nabla_{\boldsymbol{X}}f := \boldsymbol{X}(f)$

Comments: we can view $\nabla\boldsymbol{Y} : \mathcal{T}_p(\mathcal{M}) \to \mathcal{T}_p(\mathcal{M})$, $\boldsymbol{X} \mapsto \nabla_{\boldsymbol{X}}\boldsymbol{Y}$

or $\nabla\boldsymbol{Y} : \mathcal{T}^*_p(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}$, $(\boldsymbol{\eta}, \boldsymbol{X}) \mapsto \boldsymbol{\eta}(\nabla_{\boldsymbol{X}}\boldsymbol{Y})$; $\binom{1}{1}$ tensor

Def.: The $\binom{1}{1}$ tensor $\nabla\boldsymbol{Y}$ is the covariant derivative of $\boldsymbol{Y}$:

Notation: $(\nabla Y)^a{}_b = \nabla_b Y^a = Y^a{}_{;b}$

Comment:
- for a function f: $\nabla f : \boldsymbol{X} \mapsto \nabla_{\boldsymbol{X}}f = \boldsymbol{X}(f)$ is a $\binom{0}{1}$ tensor
- we cannot view $\nabla : (\boldsymbol{X}, \boldsymbol{Y}) \mapsto \nabla_{\boldsymbol{X}}\boldsymbol{Y}$ as a $\binom{1}{2}$ tensor field: not linear in $\boldsymbol{Y}$.

Def.: Let $\{\mathbf{e}^\mu\}$ be a basis. We define the "connection components" $\Gamma^\mu_{\nu\rho}$: $\nabla_\rho\mathbf{e}_\nu := \nabla_{\mathbf{e}_\rho}\mathbf{e}_\nu = \Gamma^\mu_{\nu\rho}\mathbf{e}_\mu$

Example: The Christoffel symbols are one connection: the "Levi-Civita" connection in a coord. basis; cf. below.

Comment: For a vectorfield $\boldsymbol{V}$ and a coord. basis,

we can define $T^{\mu}{}_{\nu} := \partial_\nu V^\mu = \dfrac{\partial V^\mu}{\partial x^\nu}$.

This is not chart independent and, hence, not a tensor.

We are missing the variation of the basis vectors!

For an arbitrary basis $\{\mathbf{e}_\mu\}$ write:

$\boldsymbol{X} = X^\mu \mathbf{e}_\mu \,, \quad \boldsymbol{Y} = Y^\mu \mathbf{e}_\mu$

$$\begin{aligned}
\Rightarrow \quad \nabla_{\boldsymbol{X}} \boldsymbol{Y} &= \nabla_{\boldsymbol{X}}(Y^\mu \mathbf{e}_\mu) = \boldsymbol{X}(Y^\mu)\mathbf{e}_\mu + Y^\mu \nabla_{\boldsymbol{X}} \mathbf{e}_\mu \\
&= X^\nu \mathbf{e}_\nu(Y^\mu)\, \mathbf{e}_\mu + Y^\mu \nabla_{X^\nu \mathbf{e}_\nu} \mathbf{e}_\mu \\
&= X^\nu \mathbf{e}_\nu(Y^\mu)\, \mathbf{e}_\mu + Y^\mu X^\nu \underbrace{\nabla_\nu \mathbf{e}_\mu}_{=\Gamma^\rho_{\mu\nu}\mathbf{e}_\rho} \\
&= X^\nu \Big(\mathbf{e}_\nu(Y^\mu) + \Gamma^\mu_{\rho\nu} Y^\rho \Big) \mathbf{e}_\mu
\end{aligned}$$

$$\Rightarrow \quad (\nabla_{\boldsymbol{X}} \boldsymbol{Y})^\mu = X^\nu \mathbf{e}_\nu(Y^\mu) + \Gamma^\mu_{\rho\nu} Y^\rho X^\nu \qquad \Big| \ \boldsymbol{X} \text{ arbitrary}$$

$$\Rightarrow \quad (\nabla \boldsymbol{Y})^\mu{}_\nu = \nabla_\nu Y^\mu = Y^\mu{}_{;\nu} = \mathbf{e}_\nu(Y^\mu) + \Gamma^\mu_{\rho\nu} Y^\rho$$

Coord. basis $\Rightarrow$ $\boxed{\nabla_\nu Y^\mu = \partial_\nu Y^\mu + \Gamma^\mu_{\rho\nu} Y^\rho}$

Change of basis

$\tilde{\mathbf{e}}_\mu = (A^{-1})^\nu{}_\mu \mathbf{e}_\nu$

$$\Rightarrow \ldots \Rightarrow \quad \tilde{\Gamma}^\mu_{\nu\rho} = A^\mu{}_\tau (A^{-1})^\lambda{}_\nu (A^{-1})^\sigma{}_\rho \Gamma^\tau_{\lambda\sigma} + \underbrace{A^\mu{}_\tau (A^{-1})^\sigma{}_\rho\, \mathbf{e}_\sigma\big((A^{-1})^\tau{}_\nu\big)}_{\text{independent of } \Gamma \ !} \qquad \Big| \ \text{Ex. sheet 2}$$

$\Rightarrow$ Difference of 2 connections $\Gamma^\mu_{\nu\rho} - \tilde{\Gamma}^\mu_{\nu\rho}$ transforms as tensor

Covariant derivative of tensors

Obtained from Leibniz rule; (r,s) tensor $\boldsymbol{T} \quad \mapsto \quad \nabla\boldsymbol{T}$ is $(r,s+1)$ tensor

E.g. 1-form: $\big(\nabla_{\boldsymbol{X}}\boldsymbol{\eta}\big)(\boldsymbol{Y}) := \nabla_{\boldsymbol{X}}\big(\boldsymbol{\eta}(\boldsymbol{Y})\big) - \boldsymbol{\eta}(\nabla_{\boldsymbol{X}}\boldsymbol{Y})$

$\nabla\boldsymbol{\eta}$ is a $(0,2)$ tensor since:

$$\begin{aligned}
\big(\nabla_{\boldsymbol{X}}\boldsymbol{\eta}\big)(\boldsymbol{Y}) &= \nabla_{\boldsymbol{X}}(\eta_\mu Y^\mu) - \eta_\mu(\nabla_{\boldsymbol{X}}\boldsymbol{Y})^\mu \\
&= \boldsymbol{X}(\eta_\mu)Y^\mu + \underbrace{\eta_\mu\boldsymbol{X}(Y^\mu) - \eta_\mu\big(X^\nu\mathbf{e}_\nu(Y^\mu)}_{=0} + \Gamma^\mu_{\rho\nu}Y^\rho X^\nu\big) \\
&= X^\nu\mathbf{e}_\nu(\eta_\mu)Y^\mu - \Gamma^\mu_{\rho\nu}\eta_\mu Y^\rho X^\nu \\
&= \big(\mathbf{e}_\nu(\eta_\rho) - \Gamma^\mu_{\rho\nu}\eta_\mu\big)X^\nu Y^\rho \qquad \text{is linear in } \boldsymbol{X},\ \boldsymbol{Y}
\end{aligned}$$

$$\begin{array}{rl}
\text{Components:} & \boxed{\begin{aligned} \eta_{\mu\,;\nu} = \nabla_\mu\eta_\mu &= \mathbf{e}_\nu(\eta_\mu) - \Gamma^\rho_{\mu\nu}\eta_\rho \\ &= \partial_\nu\eta_\mu - \Gamma^\rho_{\mu\nu}\eta_\rho \end{aligned}} \\
\text{Coord. basis:} &
\end{array}$$

Covariant derivative of (r,s) tensor:

$$\boxed{\begin{aligned}
\nabla_\rho T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_2} = \ &\partial_\rho T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} + \Gamma^{\mu_1}_{\sigma\rho}\,T^{\sigma\mu_2\ldots\mu_r}{}_{\nu_1\ldots\nu_2} + \ldots + \Gamma^{\mu_r}_{\sigma\rho}\,T^{\mu_1\ldots\mu_{r-1}\sigma}{}_{\nu_1\ldots\nu_s} \\
&- \Gamma^\sigma_{\nu_1\rho}\,T^{\mu_1\ldots\mu_r}{}_{\sigma\nu_2\ldots\nu_s} - \ldots - \Gamma^\sigma_{\nu_s\rho}\,T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_{s-1}\sigma}
\end{aligned}}$$

Higher derivatives

$f_{,\mu\nu} = \partial_\nu\partial_\mu f \quad$ or $\quad X^a{}_{;bc} = \nabla_c\nabla_b X^a$

Note order of indices! Derivatives sometimes commute, sometimes not.

E.g. $\quad \partial_\nu\partial_\mu f = \partial_\mu\partial_\nu f$

$$\begin{aligned}
\text{but } \nabla_\nu\nabla_\mu f &= \nabla_\nu\partial_\mu f = \partial_\nu\partial_\mu f - \Gamma^\rho_{\mu\nu}\,\partial_\rho f \\
&= \nabla_\mu\nabla_\nu f - \Gamma^\rho_{\mu\nu}\,\partial_\rho f + \Gamma^\rho_{\nu\mu}\,\partial_\rho f \\
&= \nabla_\mu\nabla_\nu f - 2\Gamma^\rho_{[\mu\nu]}\,\partial_\rho f
\end{aligned}$$

Def.: "Torsion tensor" $\ T_{\mu\nu}{}^\lambda := \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}$

Γ is torsion free $\ :\Leftrightarrow \quad \Gamma^\lambda_{[\mu\nu]} = 0$

Lemma: Γ torsion free, $\ \boldsymbol{X}, \boldsymbol{Y}\ $ vector fields $\ \Rightarrow \quad \nabla_{\boldsymbol{X}}\boldsymbol{Y} - \nabla_{\boldsymbol{Y}}\boldsymbol{X} = [\boldsymbol{X}, \boldsymbol{Y}]$

Proof: Coord. basis

$$\Rightarrow \; X^\nu \nabla_\nu Y^\mu - Y^\nu \nabla_\nu X^\mu = X^\nu \partial_\nu Y^\mu + X^\nu \Gamma^\mu_{\rho\nu} Y^\rho - Y^\nu \partial_\nu X^\mu - Y^\nu \Gamma^\mu_{\rho\nu} X^\rho$$
$$= [X, Y]^\mu + 2\Gamma^\mu_{[\rho\nu]} X^\nu Y^\rho = [X, Y]^\mu$$

Note: Even with torsion-free connection, 2^{nd} cov. derivs. of tensor fields generally do not commute.

4.2 The Levi-Civita connection

A metric singles out a preferred connection.

Fundamental theorem of Riemannian geometry:

On a manifold $\mathcal{M}$ with metric g, there exists a unique, torsion-free connection with $\nabla \boldsymbol{g} = 0$: The "Levi-Civita connection"

Proof: 1) Uniqueness

Let ∇ be a Levi-Civita connection, $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$ vector fields

$$\Rightarrow \; \boldsymbol{X}\big(\boldsymbol{g}(\boldsymbol{Y},\boldsymbol{Z})\big) = \nabla_{\boldsymbol{X}}\big(\boldsymbol{g}(\boldsymbol{Y},\boldsymbol{Z})\big) = \boldsymbol{g}\big(\nabla_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}\big) + \boldsymbol{g}\big(\boldsymbol{Y},\nabla_{\boldsymbol{X}}\boldsymbol{Z}\big) + 0$$
$$\boldsymbol{Z}\big(\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y})\big) = \nabla_{\boldsymbol{Z}}\big(\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y})\big) = \boldsymbol{g}\big(\nabla_{\boldsymbol{Z}}\boldsymbol{X},\boldsymbol{Y}\big) + \boldsymbol{g}\big(\boldsymbol{X},\nabla_{\boldsymbol{Z}}\boldsymbol{Y}\big) + 0$$
$$\boldsymbol{Y}\big(\boldsymbol{g}(\boldsymbol{Z},\boldsymbol{X})\big) = \nabla_{\boldsymbol{Y}}\big(\boldsymbol{g}(\boldsymbol{Z},\boldsymbol{X})\big) = \boldsymbol{g}\big(\nabla_{\boldsymbol{Y}}\boldsymbol{Z},\boldsymbol{X}\big) + \boldsymbol{g}\big(\boldsymbol{Z},\nabla_{\boldsymbol{Y}}\boldsymbol{X}\big) + 0$$
$$\Rightarrow \boldsymbol{X}\big(\boldsymbol{g}(\boldsymbol{Y},\boldsymbol{Z})\big) + \boldsymbol{Y}\big(\boldsymbol{g}(\boldsymbol{Z},\boldsymbol{X})\big) - \boldsymbol{Z}\big(\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y})\big)$$
$$= \boldsymbol{g}\big(\nabla_{\boldsymbol{X}}\boldsymbol{Y} + \nabla_{\boldsymbol{Y}}\boldsymbol{X},\boldsymbol{Z}\big) - \boldsymbol{g}\big(\nabla_{\boldsymbol{Z}}\boldsymbol{X} - \nabla_{\boldsymbol{X}}\boldsymbol{Z},\boldsymbol{Y}\big) + \boldsymbol{g}\big(\nabla_{\boldsymbol{Y}}\boldsymbol{Z} - \nabla_{\boldsymbol{Z}}\boldsymbol{Y},\boldsymbol{X}\big)$$

Torsion free: $\nabla_{\boldsymbol{X}}\boldsymbol{Y} - \nabla_{\boldsymbol{Y}}\boldsymbol{X} = [\boldsymbol{X},\boldsymbol{Y}]$; permute $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$

$$\Rightarrow \boldsymbol{X}\big(\boldsymbol{g}(\boldsymbol{Y},\boldsymbol{Z})\big) + \boldsymbol{Y}\big(\boldsymbol{g}(\boldsymbol{Z},\boldsymbol{X})\big) - \boldsymbol{Z}\big(\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y})\big)$$
$$= 2\boldsymbol{g}\big(\nabla_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}\big) - \boldsymbol{g}\big([\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z}\big) - \boldsymbol{g}\big([\boldsymbol{Z},\boldsymbol{X}],\boldsymbol{Y}\big) + \boldsymbol{g}\big([\boldsymbol{Y},\boldsymbol{Z}],\boldsymbol{X}\big)$$
$$\Rightarrow \boldsymbol{g}\big(\nabla_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}\big) = \frac{1}{2}\Big\{\boldsymbol{X}\big(\boldsymbol{g}(\boldsymbol{Y},\boldsymbol{Z})\big) + \boldsymbol{Y}\big(\boldsymbol{g}(\boldsymbol{Z},\boldsymbol{X})\big) - \boldsymbol{Z}\big(\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y})\big)$$
$$+\boldsymbol{g}\big([\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z}\big) + \boldsymbol{g}\big([\boldsymbol{Z},\boldsymbol{X}],\boldsymbol{Y}\big) - \boldsymbol{g}\big([\boldsymbol{Y},\boldsymbol{Z}],\boldsymbol{X}\big)\Big\} \qquad (*)$$

$\boldsymbol{g}$ non-degenerate $\Rightarrow$ unique expression for $\nabla_{\boldsymbol{X}}\boldsymbol{Y}$

2) <u>Existence:</u> Is $\nabla_{\boldsymbol{X}}$ thus defined a connection?

Check (1) of the definition of the covariant derivative.

Let f be a function; use $(*)$ with $\boldsymbol{X} \to f\,\boldsymbol{X}$

$$\begin{aligned}
\Rightarrow \boldsymbol{g}\big(\nabla_{f\boldsymbol{X}}\boldsymbol{Y}, \boldsymbol{Z}\big) &= \frac{1}{2}\big\{f\,\boldsymbol{X}\big(\boldsymbol{g}(\boldsymbol{Y},\boldsymbol{Z})\big) + \boldsymbol{Y}\big(f\,\boldsymbol{g}(\boldsymbol{Z},\boldsymbol{X})\big) - \boldsymbol{Z}\big(f\,\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y})\big) \\
&\qquad + \boldsymbol{g}\big([f\,\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z}\big) + \boldsymbol{g}\big([\boldsymbol{Z},f\,\boldsymbol{X}],\boldsymbol{Y}\big) - f\,\boldsymbol{g}\big([\boldsymbol{Y},\boldsymbol{Z}],\boldsymbol{X}\big)\big\} \\
&= \frac{1}{2}\Big\{f\,\boldsymbol{X}\big(\boldsymbol{g}(\boldsymbol{Y},\boldsymbol{Z})\big) + f\,\boldsymbol{Y}\big(\boldsymbol{g}(\boldsymbol{Z},\boldsymbol{X})\big) \underline{+\boldsymbol{Y}(f)\,\boldsymbol{g}(\boldsymbol{Z},\boldsymbol{X})} - f\,\boldsymbol{Z}\big(\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y})\big) \\
&\qquad \underline{-\boldsymbol{Z}(f)\,\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y})} + f\,\boldsymbol{g}\big([\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z}\big) \underline{-\boldsymbol{Y}(f)\,\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Z})} \\
&\qquad + f\,\boldsymbol{g}\big([\boldsymbol{Z},\boldsymbol{X}],\boldsymbol{Y}\big) \underline{+\boldsymbol{Z}(f)\,\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y})} - f\,\boldsymbol{g}\big([\boldsymbol{Y},\boldsymbol{Z}],\boldsymbol{X}\big)\Big\} \\
&= \frac{f}{2}\big\{\boldsymbol{X}\big(\boldsymbol{g}(\boldsymbol{Y},\boldsymbol{Z})\big) + \boldsymbol{Y}\big(\boldsymbol{g}(\boldsymbol{Z},\boldsymbol{X})\big) - \boldsymbol{Z}\big(\boldsymbol{g}(\boldsymbol{X},\boldsymbol{Y})\big) \\
&\qquad + \boldsymbol{g}\big([\boldsymbol{X},\boldsymbol{Y}],\boldsymbol{Z}\big) + \boldsymbol{g}\big([\boldsymbol{Z},\boldsymbol{X}],\boldsymbol{Y}\big) - \boldsymbol{g}\big([\boldsymbol{Y},\boldsymbol{Z}],\boldsymbol{X}\big)\big\} \\
&= f\,\boldsymbol{g}\big(\nabla_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}\big) = \boldsymbol{g}\big(f\,\nabla_{\boldsymbol{X}}\boldsymbol{Y},\boldsymbol{Z}\big)
\end{aligned}$$

$$\Rightarrow \quad \boldsymbol{g}\big(\nabla_{f\boldsymbol{X}}\boldsymbol{Y} - f\,\nabla_{\boldsymbol{X}}\boldsymbol{Y}, \boldsymbol{Z}\big) = 0 \qquad \forall\,\boldsymbol{Z}$$

$\boldsymbol{g}$ non-degenerate $\Rightarrow \quad \nabla_{f\boldsymbol{X}}\boldsymbol{Y} = f\,\nabla_{\boldsymbol{X}}\boldsymbol{Y} \quad \Box$

(2), (3) of the definition of the cov. deriv. can be shown similarly.

<u>Components of Levi-Civita connection in coord. basis</u>

Use $(*)$ with $[\mathbf{e}_\mu, \mathbf{e}_\nu] = 0$

$$\Rightarrow \boldsymbol{g}\big(\underbrace{\nabla_\rho \mathbf{e}_\nu}_{=\Gamma^\mu_{\nu\rho}\mathbf{e}_\mu}, \mathbf{e}_\sigma\big) = \frac{1}{2}\big[\mathbf{e}_\rho(g_{\nu\sigma}) + \mathbf{e}_\nu(g_{\sigma\rho}) - \mathbf{e}_\sigma(g_{\rho\nu})\big]$$

$$\Rightarrow \boldsymbol{g}\big(\Gamma^\mu_{\nu\rho}\mathbf{e}_\mu, \mathbf{e}_\sigma\big) = \Gamma^\mu_{\nu\rho} g_{\mu\sigma} = \frac{1}{2}\big(\partial_\rho g_{\nu\sigma} + \partial_\nu g_{\sigma\rho} - \partial_\sigma g_{\rho\nu}\big) \qquad \Big| \; \cdot g^{\lambda\sigma}$$

$$\Rightarrow \delta^\lambda{}_\mu \Gamma^\mu_{\nu\rho} = \boxed{\Gamma^\lambda_{\nu\rho} = \frac{1}{2} g^{\lambda\sigma}\big(\partial_\rho g_{\nu\sigma} + \partial_\nu g_{\sigma\rho} - \partial_\sigma g_{\rho\nu}\big)} \qquad \leftarrow \text{ Christoffel symbols!}$$

Comment: In GR we take the Levi-Civita connection.

Different connection $\rightarrow \Delta\Gamma$ which is a tensor

$\rightarrow$ can be viewed as matter source

4.3 Geodesics

Curves extremizing proper time: $\boxed{\dfrac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\big(x(\tau)\big)\dfrac{dx^\nu}{d\tau}\dfrac{dx^\rho}{d\tau} = 0}$ $(*)$

here: $\tau =$ proper time along curve

$X^\mu = \dfrac{dx^\mu}{d\tau} =$ tangent vector along curve

Let's extend $\boldsymbol{X}$ to be a vectorfield in a neighbourhood of curve.

$$\Rightarrow \frac{d^2x^\mu}{d\tau^2} = \frac{dX^\mu\big(x(\tau)\big)}{d\tau} = \frac{dx^\nu}{d\tau}\frac{\partial X^\mu}{\partial x^\nu} = X^\nu \partial_\nu X^\mu$$

LHS independent of the extension $\Rightarrow$ RHS too.

$(*) \Rightarrow X^\nu\big(\partial_\nu X^\mu + \Gamma^\mu_{\nu\rho}X^\rho\big) = X^\nu \nabla_\nu X^\mu = 0 \quad \text{or} \quad \nabla_{\boldsymbol{X}}\boldsymbol{X} = 0\,.$

We derived this for the Levi-Civita connection but define for any connection:

<u>Def.:</u> "affinely parametrized geodesic"

$:=$ integral curve of vector field $\boldsymbol{X}$ with $\nabla_{\boldsymbol{X}}\boldsymbol{X} = 0$

Comment: Let u be another parameter of the curve

such that $\tau = \tau(u)\,, \quad \dfrac{d\tau}{du} > 0\,.$

$\Rightarrow$ tangent vector now: $\boldsymbol{Y} = h\boldsymbol{X}$ with $h := \dfrac{d\tau}{du}$

$$\Rightarrow \nabla_{\boldsymbol{Y}}\boldsymbol{Y} = \nabla_{h\boldsymbol{X}}(h\boldsymbol{X}) = h\nabla_{\boldsymbol{X}}(h\boldsymbol{X}) = h^2 \underbrace{\nabla_{\boldsymbol{X}}\boldsymbol{X}}_{=0} + \boldsymbol{X}(h)\,h\boldsymbol{X} = \frac{dh}{d\tau}\boldsymbol{Y}$$

$\Rightarrow \nabla_{\boldsymbol{Y}}\boldsymbol{Y} = \dfrac{dh}{d\tau}\boldsymbol{Y}$ describes the same geodesic.

Unless $\dfrac{dh}{d\tau} = 0\,,$ it is not affinely parameterized.

u is also an affine parameter $\Leftrightarrow$ h constant $\Leftrightarrow$ $u = a\tau + b\,, \quad a,\, b = \text{const}$

$\Rightarrow$ 2 parameter family of affine parameters

Note:
- For any connection, we can write the geodesic eq. $(*)$ for some affine parameter.
- Curves of extremal proper time are timelike geodesics.

  We can also define spacelike geodesics through $(*)$.

  Then τ is not proper time but arc length often denoted by s.

Theorem: Let $\mathcal{M}$ be a manifold with connection, $p \in \mathcal{M}$, $\boldsymbol{X}_p \in \mathcal{T}_p(\mathcal{M})$

$\Rightarrow$ $\exists$ unique affinely parametrized geodesic with tangent vector $\boldsymbol{X}_p$ in p

Proof: Let x^μ be a coord. chart in nbhd. of p, X_p^μ components of $\boldsymbol{X}_p$

geodesic eq.: $$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

with initial conditions $x^\mu(0) = x^\mu(p)\,, \quad \left.\frac{dx^\mu}{d\tau}\right|_{\tau=0} = X_p^\mu$

This is a system of ODEs for x^μ. Theory of ODEs

$\Rightarrow$ unique solution exists.

Note: Levi-Civita connection, $\nabla_{\boldsymbol{X}}\boldsymbol{X} = 0$ along affinely parametrized geodesic implies:

$\nabla_{\boldsymbol{X}}\big(\boldsymbol{g}(\boldsymbol{X},\boldsymbol{X})\big) = \big(\nabla_{\boldsymbol{X}}\boldsymbol{g}\big)(\boldsymbol{X},\boldsymbol{X}) + 2\boldsymbol{g}\big(\nabla_{\boldsymbol{X}}\boldsymbol{X},\boldsymbol{X}\big) = 0 + 0$

$\Rightarrow \boldsymbol{g}(\boldsymbol{X},\boldsymbol{X})$ const. along curve

$\Rightarrow$ tangent vector cannot change time, space or null character

$\Rightarrow$ geodesic is either time, spacelike or null

Postulate: massive (massless) particles in GR move on timelike (null) geodesics

Note: Null geodesics have no analogue of proper time or arc length,

but still affine parameters.

4.4 Normal coordinates

Def.: Let $\mathcal{M}$ be a manifold, Γ a connection, $p \in \mathcal{M}$.

"exponential map" $:= e : \mathcal{T}_p \to \mathcal{M}\,, \quad \boldsymbol{X}_p \mapsto q$ with

$q :=$ point a unit affine parameter distance along

geodesic through p with tangent $\boldsymbol{X}_p$

Comments: 1) e can be shown to be one-to-one and onto locally,

(geodesics can cross globally)

2) The vector $\boldsymbol{X}_p$ fixes the parametrization of the geodesic:

One can show that $t\,\boldsymbol{X}_p\,,\quad 0 \le t \le 1$ is mapped to point at

affine par. distance t along the geodesic of $\boldsymbol{X}_p$. **(∗∗)**

Def.: Let $\{\mathbf{e}_\mu\}$ be a basis of $\mathcal{T}_p(\mathcal{M})$. "Normal coords. in nbhd. of $p \in \mathcal{M}$":

chart that assigns to $q = e(\boldsymbol{X}) \in \mathcal{M}$ the coordinates X^μ

Note: The coords. X^μ are not fixed by the vector $\boldsymbol{X}$.

We still have the freedom to choose a basis for $\mathcal{T}_p(\mathcal{M})$.

Lemma: In normal coordinates, $\Gamma^\mu_{(\nu\rho)} = 0$ at p.

If Γ is torsion free, then $\Gamma^\mu_{\nu\rho} = 0$.

Proof: From **(∗∗)** $\Rightarrow$ affinely parametrized geodesic is given by

$x^\mu(t) = t\,X^\mu_p$ in normal coords.

$$\Rightarrow \text{ geodesic eq.: } 0 + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = \Gamma^\mu_{\nu\rho}X^\nu_p X^\rho_p = 0 \ \text{ at } p \ \ \forall \boldsymbol{X} \in \mathcal{T}_p(\mathcal{M})$$

$\Rightarrow \Gamma^\mu_{(\nu\rho)} = 0$

torsion free $\Rightarrow$ $\Gamma^\mu_{[\nu\rho]} = 0$ $\Rightarrow$ $\Gamma^\mu_{\nu\rho} = 0$

Note: in general $\Gamma^\mu_{\nu\rho} \neq 0$ away from p!

Lemma: with metric, we can use the Levi-Civita connection

$\Rightarrow$ in normal coords. at p: $\partial_\rho g_{\mu\nu} = g_{\mu\nu,\rho} = 0$

Proof: $\Gamma^\rho_{\mu\nu} = 0 \quad \Rightarrow \quad 2 g_{\sigma\rho}\Gamma^\rho_{\mu\nu} = \partial_\nu g_{\sigma\mu} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu} = 0$

symmetrize on σ, μ, add $\Rightarrow \partial_\nu g_{\sigma\mu} = 0$

Note: Again valid only at p!

In general we cannot make $\partial_\nu g_{\sigma\mu}$ vanish away from p.

Lemma: Let $\mathcal{M}$ be a manifold with metric $g_{\mu\nu}$ and torsion free connection.

$\Rightarrow$ we can choose normal coords. such that at p:

$\partial_\rho g_{\mu\nu} = 0\,, \quad g_{\mu\nu} = \eta_{\mu\nu} \quad$ (or $\delta_{\mu\nu}$ in Riemannian case)

Proof: Choose an orthonormal basis $\{\mathbf{e}_\mu\}$ for $\mathcal{T}_p(\mathcal{M})$.

Let $\boldsymbol{X}$ be a vector field.

$\Rightarrow$ at p: $\boldsymbol{X} = X^1\mathbf{e}_1 + \ldots + X^n\mathbf{e}_n$ defines normal coords. $\hat{x}^\mu = X^\mu$

Consider vector $\dfrac{\partial}{\partial \hat{x}^1} \quad \Rightarrow \quad$ its integral curve is $\hat{x}^\mu(t) = (t,\, 0,\, \ldots,\, 0)$

because $\dfrac{d}{dt} = \dfrac{d\hat{x}^\mu}{dt}\dfrac{\partial}{\partial \hat{x}^\mu} = \delta^\mu{}_1 \dfrac{\partial}{\partial \hat{x}^\mu} = \dfrac{\partial}{\partial \hat{x}^1}$

The components of the tangent vector to the curve $\hat{x}^\mu(t) = (t,\, 0,\, \ldots,\, 0)$

are also: $\dfrac{d\hat{x}^\mu}{dt} = (1,\, 0,\, \ldots,\, 0)$

$\Rightarrow$ The tangent vector is $\mathbf{e}_1 \quad \Rightarrow \quad \mathbf{e}_1 = \dfrac{\partial}{\partial \hat{x}^1}$

Likewise: $\mathbf{e}_\mu = \dfrac{\partial}{\partial \hat{x}^\mu}$

$\rightarrow \quad \left\{ \dfrac{\partial}{\partial \hat{x}^\mu} \right\}$ defines a coordinate orthonormal basis.

Summary: Locally, we can choose coordinates such that the metric is $\eta_{\mu\nu} = \mathrm{diag}(-1,\, 1,\, 1,\, 1)$ and its first derivatives vanish.

Def.: "local inertial frame at $p \in \mathcal{M}$" := normal coord. chart with these properties

5 Physical laws in curved spacetime

5.1 Covariance

"general covariance": Physical laws should be independent of the choice of charts and basis.

"special covariance" in special relativity: laws independent of inertial frame

Recipe for converting SR laws → GR laws

1) $\eta_{\mu\nu} \to g_{\mu\nu}$ Minkowski → curved metric

2) $\partial \to \nabla$ partial → covariant derivs.

3) $\mu,\, \nu, \ldots \to a,\, b,\, \ldots$ coord. indices → abstract indices

Examples:

1) Let x^μ be inertial frame coords., $\eta_{\mu\nu}$ the Minkowski metric.

$\Rightarrow$ scalar wave eq.: $\eta^{\mu\nu}\partial_\mu\partial_\nu\phi = 0$ in SR

$$\to \quad g^{ab}\nabla_a\nabla_b\phi = \nabla^a\nabla_a\phi = \phi_{;a}{}^a = 0$$

2) Electromagnetic field in SR:

$F_{\mu\nu} = F_{[\mu\nu]}$ with $F_{0i} = -E_i\,,\quad F_{ij} = \epsilon_{ijk}B_k\,,\qquad (i,\, j,\, k = 1\ldots 3)$

vacuum Maxwell eqs.: $\eta^{\mu\nu}\partial_\mu F_{\nu\rho} = 0\,,\quad \partial_{[\mu}F_{\nu\rho]} = 0$

$\to$ in GR: $g^{ab}\nabla_a F_{bc} = 0\,,\quad \nabla_{[a}F_{bc]} = 0$

Lorentz force in SR: $\dfrac{d^2x^\mu}{d\tau^2} = \dfrac{q}{m}\eta^{\mu\nu}F_{\nu\rho}\underbrace{\dfrac{dx^\rho}{d\tau}}_{= u^\rho = \text{4-velocity}}\,;\quad \tau = \text{proper time}$

$\to$ in GR: $u^b\nabla_b u^a = \dfrac{q}{m}g^{ab}F_{bc}u^c$

Comment: This procedure satisfies the EP. In a local inertial frame:

$$\Gamma^{\mu}_{\nu\rho}\Big|_p = 0\,, \quad g_{\mu\nu}\Big|_p = \eta_{\mu\nu}$$

$\Rightarrow \quad \nabla \to \partial\,,$ so in a LIF we have SR.

But: The step SR $\to$ GR is not unique.

E.g. we can add curvature terms to the GR eqs.

Such terms are zero in SR (see below).

Ultimate test: experiment.

5.2 Energy momentum tensor

Energy, momentum, mass source gravity. Ho do we describe them in GR?

Particles

1) in SR: Associate rest mass with particle

$\Rightarrow$ 4-momentum $P^{\mu} = mu^{\mu} = (E,\, P^i)$ in this frame

4-velocity of observer in particle's rest frame: $v^{\mu} = (1,\, 0,\, 0,\, 0)$

particle energy measured by this observer: $E = -\eta_{\mu\nu} v^{\mu} P^{\nu}$

particle's rest mass: $\eta_{\mu\nu} P^{\mu} P^{\nu} = -E^2 + \vec{p}^2 = -m^2\,;$ note: $c = 1$

2) in GR: EP $\Rightarrow \quad P^a = mu^a \quad \Rightarrow \quad g_{ab} P^a P^b = -m^2$

Particle energy measured by observer: $E = -g_{ab}(p)\, v^a(p)\, P^b(p)$

works only if both are at p

An observer at $p \in \mathcal{M}$ cannot measure the energy of a particle at q

<u>**electromagnetic field**</u>

1) pre-relativistic notation, Cartesian coordinates:

energy density: $\epsilon = \frac{1}{8\pi}(E_i E_i + B_i B_i)$

momentum density, energy flux: $S_i = \frac{1}{4\pi}\epsilon_{ijk} E_j B_k$ "Poynting vector"

Maxwell eqs. $\Rightarrow \quad \frac{\partial \epsilon}{\partial t} + \partial_i S_i = 0$

stress tensor: $t_{ij} = \frac{1}{4\pi}\left[\frac{1}{2}(E_k E_k + B_k B_k)\delta_{ij} - E_i E_j - B_i B_j\right]$

conservation law: $\frac{\partial S_i}{\partial t} + \partial_j t_{ij} = 0$

Force exerted on surface element dA with normal n_i: $t_{ij} n_j dA$

2) Special relativity:

energy momentum tensor (= stress tensor = stress-energy tensor) in IF:

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}F^{\rho\sigma}F_{\rho\sigma}\eta_{\mu\nu}\right) = T_{\nu\mu}$$

$T_{00} = \epsilon\,, \quad T_{0i} = -S_i\,, \quad T_{ij} = t_{ij}\,;$ from 1)

Conservation: $\partial^\mu T_{\mu\nu} = \eta^{\mu\sigma}\partial_\sigma T_{\mu\nu} = 0$

3) GR: we define by covariance:

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}F^{cd}F_{cd}g_{ab}\right)$$

Maxwell eqs.: $\nabla^a T_{ab} = 0\,;$ cf. example sheet 2

<u>**Postulate:**</u> In GR, continuous matter is described by a conserved, symmetric $(0,2)$ tensor which contains the information about the matter's energy, momentum and stress.

The energy momentum tensor is conserved: $\nabla^a T_{ab} = 0$.

Comment: Let $\mathcal{O}$ be an observer with 4-velocity u^a.

Consider a LIF at p where $\mathcal{O}$ is at rest.

Choose orthonormal basis $\{\mathbf{e}_\mu\}$ at p aligned with the coord. axes of this LIF.

$\Rightarrow \quad e_0^a = u^a\,,\quad$ spatial basis vectors $e_i^a\,,\quad i = 1,\,2,\,3$

EP $\Rightarrow$ $\epsilon = T_{00} = T_{ab}e_0^a e_0^b = T_{ab}u^a u^b$ = energy density at p measured by $\mathcal{O}$

$S_i = -T_{0i}$ = momentum density

$j^a = -T^a{}_b u^b = (\epsilon,\, S_i)$ in this basis = energy momentum current

$t_{ij} = T_{ij}$ = stress tensor as measured by $\mathcal{O}$

Comment: Consider an IF in Minkowski spacetime.

local conservation $\partial^\mu T_{\mu\nu} \xrightarrow{\text{integration}}$ global conservation

E.g.: $\dfrac{\partial\epsilon}{\partial t} + \partial_i S_i = 0 \quad\Rightarrow\quad \dfrac{d}{dt}\displaystyle\int_V \epsilon\, dV = -\int_{\partial V} \vec{S}\cdot\vec{n}\, dA$

in GR: This is not possible! The grav. field contains energy,

but there is no invariant definition for it.

Newtonian analogue $-\dfrac{1}{8\pi}(\vec{\nabla}\phi)^2$ does not work because

metric derivatives vanish in normal coordinates.

$\Rightarrow$ energy only defined for global spacetime or special cases, e.g. horizons

Example: A perfect fluid is matter described by a 4-velocity field u^a and

functions ρ, p such that $T_{ab} = (\rho + p)u_a u_b + p g_{ab}$.

ρ, p = energy density, pressure measured by observer co-moving with fluid

One can show: 1) $T_{ab}u^a u^b = \rho$

2) $\nabla^a T_{ab} = 0 \quad\Leftrightarrow\quad u^a\nabla_a\rho + (\rho+p)\nabla_a u^a = 0$

$\wedge\;\; (\rho+p)u^b\nabla_b u^a = -(g_{ab} + u^a u_b)\nabla^b p$

These are GR's version of the Euler eqs. and mass conservation.

Note: $p = 0 \Rightarrow$ fluid moves on geodesics.

6 Curvature

6.1 Parallel transport

A connection gives us a notion of "a tensor that does not change along a curve"

Def.: Let $\boldsymbol{X}$ be tangent to a curve. A tensor is

"parallely transported/propagated along the curve" $:\Leftrightarrow \quad \nabla_{\boldsymbol{X}}\boldsymbol{T} = 0$

Comments:
- The tangent of a geodesic is parallely propagated along itself.
- $\nabla_{\boldsymbol{X}}\boldsymbol{T} = 0$ determines $\boldsymbol{T}$ uniquely along the curve:

  in coords. (x^μ) the curve is $x^\mu(t)$

  $$\Rightarrow \quad X^\sigma \nabla_\sigma T^\mu{}_\nu = X^\sigma \partial_\sigma T^\mu{}_\nu + \Gamma^\mu_{\rho\sigma} T^\rho{}_\nu X^\sigma - \Gamma^\rho_{\nu\sigma} T^\mu{}_\rho X^\sigma$$

  $$= \frac{d}{dt} T^\mu{}_\nu + \Gamma^\mu_{\rho\sigma} T^\rho{}_\nu X^\sigma - \Gamma^\rho_{\nu\sigma} T^\mu{}_\rho X^\sigma = 0$$

  ODE theory $\Rightarrow$ unique solution for all $T^\mu{}_\nu$
- $q \in \mathcal{M},\ q \neq p$: parallel transport $\boldsymbol{T}$ along a curve from p to q

  $\to$ isomorphism between tensors at p, q
- Euclidean space or Minkowski in Cartesian coords.

  $$\Rightarrow \quad \Gamma^\mu_{\nu\rho} = 0 \quad \Rightarrow \quad \frac{d}{dt} T^\mu{}_\nu = 0$$

  $\Rightarrow$ parallel trapo. leaves tensor components constant

  $\Rightarrow$ parallel trapo. is independent of the curve chosen!

  This is not the case in GR!

6.2 The Riemann tensor

Def.: The Riemann curvature tensor $R^a{}_{bcd}$ is defined such that

for VFs $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$: $R^a{}_{bcd} Z^b X^c Y^d = \big(\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z}\big)^a$ with

$$\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} = \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{Z} - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{Z} - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z}$$

Linearity in $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$: Let f be a function.

$$
\begin{aligned}
(1)\quad \boldsymbol{R}(f\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} &= \nabla_{f\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{Z} - \nabla_{\boldsymbol{Y}}\nabla_{f\boldsymbol{X}}\boldsymbol{Z} - \nabla_{[f\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z} \\
&= f\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{Z} - \nabla_{\boldsymbol{Y}}(f\nabla_{\boldsymbol{X}}\boldsymbol{Z}) - \nabla_{f[\boldsymbol{X},\boldsymbol{Y}]-\boldsymbol{Y}(f)\,\boldsymbol{X}}\boldsymbol{Z} \\
&= f\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{Z} - f\nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{Z} - \boldsymbol{Y}(f)\,\nabla_{\boldsymbol{X}}\boldsymbol{Z} - \nabla_{f[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z} + \nabla_{\boldsymbol{Y}(f)\,\boldsymbol{X}}\boldsymbol{Z} \\
&= f\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{Z} - f\nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{Z} \underline{- \boldsymbol{Y}(f)\,\nabla_{\boldsymbol{X}}\boldsymbol{Z}} - f\nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z} \underline{+ \boldsymbol{Y}(f)\,\nabla_{\boldsymbol{X}}\boldsymbol{Z}} \\
&= f\,\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z}
\end{aligned}
$$

(2) $\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} = -\boldsymbol{R}(\boldsymbol{Y},\boldsymbol{X})\boldsymbol{Z} \quad \Rightarrow$ linear in $\boldsymbol{Y}$ too

$$
\begin{aligned}
(3)\ \boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\,(f\boldsymbol{Z}) &= \nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}(f\boldsymbol{Z}) - \nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}(f\boldsymbol{Z}) - \nabla_{[\boldsymbol{X},\boldsymbol{Y}]}(f\boldsymbol{Z}) \\
&= \nabla_{\boldsymbol{X}}\big(f\,\nabla_{\boldsymbol{Y}}\boldsymbol{Z} + \boldsymbol{Y}(f)\,\boldsymbol{Z}\big) - \nabla_{\boldsymbol{Y}}\big(f\,\nabla_{\boldsymbol{X}}\boldsymbol{Z} + \boldsymbol{X}(f)\,\boldsymbol{Z}\big) - f\nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z} - [\boldsymbol{X},\boldsymbol{Y}](f)\,\boldsymbol{Z} \\
&= f\,\nabla_{\boldsymbol{X}}\nabla_{\boldsymbol{Y}}\boldsymbol{Z} + \underline{\underline{\boldsymbol{X}(f)\,\nabla_{\boldsymbol{Y}}\boldsymbol{Z}}} + \underline{\boldsymbol{Y}(f)\,\nabla_{\boldsymbol{X}}\boldsymbol{Z}} + \underline{\boldsymbol{X}\big(\boldsymbol{Y}(f)\big)\,\boldsymbol{Z}} \\
&\quad -f\nabla_{\boldsymbol{Y}}\nabla_{\boldsymbol{X}}\boldsymbol{Z} - \underline{\boldsymbol{Y}(f)\,\nabla_{\boldsymbol{X}}\boldsymbol{Z}} - \underline{\underline{\boldsymbol{X}(f)\,\nabla_{\boldsymbol{Y}}\boldsymbol{Z}}} - \underline{\boldsymbol{Y}\big(\boldsymbol{X}(f)\big)\,\boldsymbol{Z}} \\
&\quad -f\,\nabla_{[\boldsymbol{X},\boldsymbol{Y}]}\boldsymbol{Z} - \underline{[\boldsymbol{X},\boldsymbol{Y}](f)\,\boldsymbol{Z}} \\
&= f\,\boldsymbol{R}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{Z} \qquad \square
\end{aligned}
$$

Coord. basis $\left\{\mathbf{e}_\mu = \dfrac{\partial}{\partial x^\mu}\right\} \quad \Rightarrow \quad [\mathbf{e}_\mu, \mathbf{e}_\nu] = 0\,; \quad \nabla_\mu := \nabla_{\mathbf{e}_\mu}$

$$
\begin{aligned}
\Rightarrow \boldsymbol{R}(\mathbf{e}_\rho,\mathbf{e}_\sigma)\mathbf{e}_\nu &= \nabla_\rho\nabla_\sigma\mathbf{e}_\nu - \nabla_\sigma\nabla_\rho\mathbf{e}_\nu \\
&= \nabla_\rho(\Gamma^\tau_{\nu\sigma}\mathbf{e}_\tau) - \nabla_\sigma(\Gamma^\tau_{\nu\rho}\mathbf{e}_\tau) \\
&= \partial_\rho\Gamma^\mu_{\nu\sigma}\mathbf{e}_\mu + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho}\mathbf{e}_\mu - \partial_\sigma\Gamma^\mu_{\nu\rho}\mathbf{e}_\mu - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma}\mathbf{e}_\mu
\end{aligned}
$$

$$\Rightarrow \boxed{R^\mu{}_{\nu\rho\sigma} = \partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho} - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma}} \qquad (*)$$

Comment: $R^\mu{}_{\nu\rho\sigma} = 0$ in Minkowski or Euclidean:

We can choose coords. such that $\Gamma^\mu_{\nu\rho} = 0$ everywhere.

Def.: "Ricci tensor" $\boxed{R_{ab} := R^c{}_{acb}}$

Comments: 2^{nd} cov. derivs. of functions commute $\Leftrightarrow$ no torsion

2^{nd} cov. derivs. of tensors do not commute even if torsion $= 0$

e.g. one can show: $(\nabla_c \nabla_d - \nabla_d \nabla_c) Z^a = R^a{}_{bcd} Z^b$ "Ricci identity"

6.3 Parallel transport and curvature

Let $\boldsymbol{X}, \boldsymbol{Y}$ be VFs with: lin. indep. everywhere and $[\boldsymbol{X}, \boldsymbol{Y}] = 0$; let torsion $= 0$

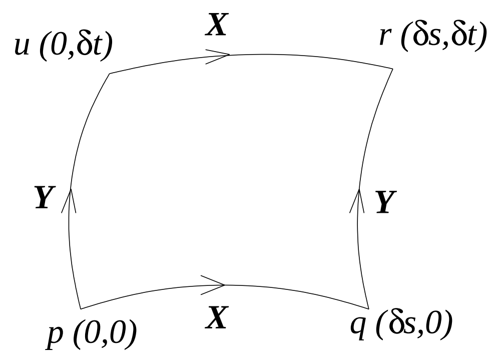
$\Rightarrow$ we can choose coords. $(s, t, \ldots)$ such that $\boldsymbol{X} = \dfrac{\partial}{\partial s}, \quad \boldsymbol{Y} = \dfrac{\partial}{\partial t}$

Let $p, q, r, u \in \mathcal{M}$ along integral curves of $\boldsymbol{X}, \boldsymbol{Y}$ with coords.

$(0, \ldots, 0)$, $(\delta s, 0, \ldots)$, $(\delta s, \delta t, 0, \ldots)$, $(0, \delta t, 0, \ldots)$

Let $\boldsymbol{Z}_p \in \mathcal{T}_p(\mathcal{M})$, parallel trapo $\boldsymbol{Z}$ along $pqrup$

to get $\boldsymbol{Z}'_p \in \mathcal{T}_p(\mathcal{M})$



$$\Rightarrow \boxed{\lim_{\delta \to 0} \frac{(\boldsymbol{Z}'_p - \boldsymbol{Z}_p)^a}{\delta s\, \delta t} = (R^a{}_{bcd} Z^b Y^c X^d)_p}$$

<u>Proof:</u>

Let $\boldsymbol{Z}_p \in \mathcal{T}_p(\mathcal{M})$, (x^μ) be normal coords. at p.

s, t are now "only" parameters along the curves.

We assume $\delta s, \delta t$ are small and $\delta t = a\, \delta s$ for $a = \text{const}$.

(1) $p \to q$: curve with tangent $\boldsymbol{X}$ and parameter s

parallel transport $\nabla_{\boldsymbol{X}} \boldsymbol{Z} = 0$

$$\Rightarrow X^\sigma \nabla_\sigma Z^\mu = X^\sigma \frac{\partial}{\partial x^\sigma} Z^\mu + \Gamma^\mu_{\rho\sigma} Z^\rho X^\sigma = \frac{dZ^\mu}{ds} + \Gamma^\mu_{\rho\sigma} Z^\rho X^\sigma = 0$$

$$\Rightarrow \frac{dZ^\mu}{ds} = -\Gamma^\mu_{\rho\sigma} Z^\rho X^\sigma$$

$$\Rightarrow \frac{d^2 Z^\mu}{ds^2} = -X^\lambda \partial_\lambda (\Gamma^\mu_{\rho\sigma} Z^\rho X^\sigma) \qquad \Big| \; \boldsymbol{X} = X^\mu \frac{\partial}{\partial x^\mu} = \frac{d}{ds}$$

Taylor expand around p and use $\Gamma^\mu_{\rho\sigma}\big|_p = 0$

$$\Rightarrow Z^\mu_q - Z^\mu_p = \left(\frac{dZ^\mu}{ds}\right)_p \delta s + \frac{1}{2}\left(\frac{d^2 Z^\mu}{ds^2}\right)_p \delta s^2 + \mathcal{O}(\delta s^3)$$

$$= \underline{-\frac{1}{2}\big(X^\lambda Z^\rho X^\sigma \partial_\lambda \Gamma^\mu_{\rho\sigma}\big)_p \delta s^2} + \mathcal{O}(\delta s^3)$$

(2) $q \to r$:

$$Z^\mu_r - Z^\mu_q = \left(\frac{dZ^\mu}{dt}\right)_q \delta t + \frac{1}{2}\left(\frac{d^2 Z^\mu}{dt^2}\right)_q \delta t^2 + \mathcal{O}(\delta t^3)$$

$$= -\underbrace{\big(\Gamma^\mu_{\rho\sigma} Z^\rho Y^\sigma\big)_q}\delta t - \frac{1}{2}\big[Y^\lambda \partial_\lambda (\Gamma^\mu_{\rho\sigma} Z^\rho Y^\sigma)\big]_q \delta t^2 + \mathcal{O}(\delta t^3)$$

$$= \Big[(\Gamma^\mu_{\rho\sigma} Z^\rho Y^\sigma)_p + \big(X^\lambda \partial_\lambda (\Gamma^\mu_{\rho\sigma} Z^\rho Y^\sigma)\big)_p \delta s + \mathcal{O}(\delta s^2)\Big]\,\delta t$$

$$= \Big[0 + (Z^\rho Y^\sigma X^\lambda \partial_\lambda \Gamma^\mu_{\rho\sigma})_p \delta s + \mathcal{O}(\delta s^2)\Big]\delta t$$

$$\Rightarrow Z^\mu_r - Z^\mu_q = \underline{-\Big[(Z^\rho Y^\sigma X^\lambda \partial_\lambda \Gamma^\mu_{\rho\sigma})_p \delta s + \mathcal{O}(\delta s^2)\Big]\delta t}$$

$$\underline{-\frac{1}{2}\Big[\underbrace{\big(Y^\lambda \partial_\lambda (\Gamma^\mu_{\rho\sigma} Z^\rho Y^\sigma)\big)_p}_{= (Z^\rho Y^\sigma Y^\lambda \partial_\lambda \Gamma^\mu_{\rho\sigma})_p} + \mathcal{O}(\delta s)\Big]\delta t^2} + \mathcal{O}(\delta t^3)$$

$$\Rightarrow \big(Z^\mu_r - Z^\mu_p\big)_{pqr} = -\frac{1}{2}(\partial_\lambda \Gamma^\mu_{\rho\sigma})\Big[Z^\rho\big(X^\sigma X^\lambda \delta s^2 + Y^\sigma Y^\lambda \delta t^2 + 2 Y^\sigma X^\lambda \delta s\, \delta t\big)\Big]_p + \mathcal{O}(\delta t^3)$$

We obtain $(Z^\mu_r - Z^\mu_p)_{pur}$ by simply interchanging $\boldsymbol{X} \leftrightarrow \boldsymbol{Y}$, $s \leftrightarrow t$

$$\Rightarrow \big(Z^\mu_r - Z^\mu_p\big)_{pur} = -\frac{1}{2}(\partial_\lambda \Gamma^\mu_{\rho\sigma})\Big[Z^\rho\big(Y^\sigma Y^\lambda \delta t^2 + X^\sigma X^\lambda \delta s^2 + 2 X^\sigma Y^\lambda \delta t\, \delta s\big)\Big]_p + \mathcal{O}(\delta s^3)$$

$$\Rightarrow Z'^{\mu}_p - Z^{\mu}_p = (Z^{\mu}_r - Z^{\mu}_p)_{pqr} - (Z^{\mu}_r - Z^{\mu}_p)_{pur} = -\big[(Y^\sigma X^\lambda - X^\sigma Y^\lambda)(\partial_\lambda \Gamma^{\mu}_{\rho\sigma})\big]_p Z^\rho\,\delta s\,\delta t + \mathcal{O}(\delta^3)$$

$$= \Big[X^\sigma Y^\lambda Z^\rho \underbrace{(\partial_\lambda \Gamma^{\mu}_{\rho\sigma} - \partial_\sigma \Gamma^{\mu}_{\rho\lambda})}_{\stackrel{(*)}{=}\; R^{\mu}{}_{\rho\lambda\sigma} \text{ in normal coords: } (\Gamma^{\alpha}_{\beta\gamma})_p = 0}\Big]_p + \mathcal{O}(\delta^3) = \big(R^{\mu}{}_{\rho\lambda\sigma} Z^\rho Y^\lambda X^\sigma\big)_p + \mathcal{O}(\delta^3) \qquad \square$$

Conclusion: Curvature measures the change of vectors under parallel transport along closed curves or, equivalently, the path (in)dependence of par. trapo.

6.4 Symmetries of the Riemann tensor

(i) $R^a{}_{bcd} = -R^a{}_{bdc} \quad \Leftrightarrow \quad R^a{}_{b(cd)} = 0$ by def.

Torsion $= 0$, let $p \in \mathcal{M}$, (x^μ) normal coords. Then:

(ii) $\Gamma^{\mu}_{\nu\rho} = 0$ at p, $\Gamma^{\mu}_{[\nu\rho]} = 0$ everywhere

$\Rightarrow R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho}$ | antisymmetrize on $\nu\rho\sigma$

$\Rightarrow R^{\mu}{}_{[\nu\rho\sigma]} = 0 \quad \Rightarrow \quad R^a{}_{[bcd]} = 0$

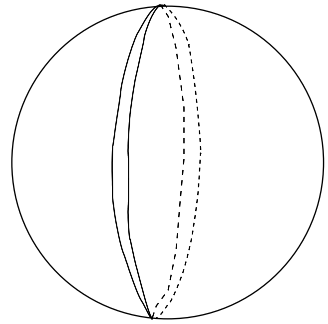
(iii) $\nabla_\tau R^{\mu}{}_{\nu\rho\sigma} = \partial_\tau R^{\mu}{}_{\nu\rho\sigma}$ | "$\partial R = \partial\partial\Gamma - \Gamma\,\partial\Gamma = \partial\partial\Gamma$"

$= \partial_\tau \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\tau \partial_\sigma \Gamma^{\mu}_{\nu\rho}$ | antisymmetrize on $\rho\sigma\tau$

$\Rightarrow R^{\mu}{}_{\nu[\rho\sigma;\tau]} = 0$ "Bianchi identity"

$\Rightarrow R^a{}_{b[cd;e]} = 0$ tensorial equation !

6.5 Geodesic deviation



Goal: quantify relative acceleration of geodesics

Def.: Let $(\mathcal{M}, \Gamma)$ be a manifold with connection.

"1-parameter family of geodesics" := a map

$\gamma : I \times I' \to \mathcal{M}$ with $I,\ I' \subset \mathbb{R}$, open and

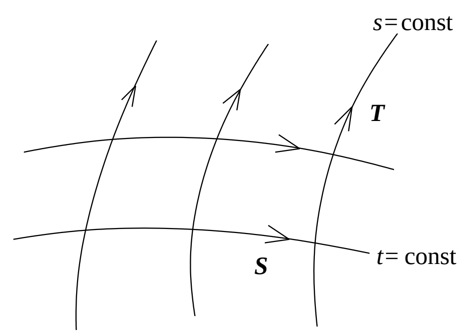
(i) for fixed s, $\gamma(s,t)$ is a geodesic with affine par. t

(ii) locally, $(s,t) \mapsto \gamma(s,t)$ is smooth, 1-to-1 has smooth inverse

$\Rightarrow$ the family of geodesics forms a 2-dim. surface $\Sigma \subset \mathcal{M}$

Let $\boldsymbol{T}$ be the tangent vector to $\gamma(s = \text{const}, t)$ and $\boldsymbol{S}$ to $\gamma(s, t = \text{const})$

In coords. (x^μ): $S^\mu = \dfrac{\partial x^\mu}{\partial s}$



$\Rightarrow x^\mu(s + \delta s, t) = x^\mu(s,t) + \delta s\, S^\mu(s,t) + \mathcal{O}(\delta s^2)$

$\Rightarrow \delta s\, \boldsymbol{S}$ points from one geodesic to a nearby one: "deviation vector"

$\Rightarrow$ "relative velocity" of nearby geodesics: $\nabla_{\boldsymbol{T}}(\delta s\, \boldsymbol{S}) = \delta s\, \nabla_{\boldsymbol{T}} \boldsymbol{S}$

$\Rightarrow$ "relative acceleration" of nearby geodesics: $\delta s\, \nabla_{\boldsymbol{T}} \nabla_{\boldsymbol{T}} \boldsymbol{S}$

Geodesic deviation: $\boxed{\nabla_{\boldsymbol{T}} \nabla_{\boldsymbol{T}} \boldsymbol{S} = \boldsymbol{R}(\boldsymbol{T}, \boldsymbol{S})\boldsymbol{T}}$

$$\Leftrightarrow \quad T^c \nabla_c (T^b \nabla_b S^a) = R^a{}_{bcd} T^b T^c S^d$$

Proof: Use coords. (s,t) on Σ and extend to $(s,t,\ldots)$ in nbhd. of Σ

$$\Rightarrow \boldsymbol{S} = \frac{\partial}{\partial s}, \quad \boldsymbol{T} = \frac{\partial}{\partial t} \quad \Rightarrow \quad [\boldsymbol{S}, \boldsymbol{T}] = 0$$

No torsion $\Rightarrow \nabla_{\boldsymbol{T}} \boldsymbol{S} - \nabla_{\boldsymbol{S}} \boldsymbol{T} = [\boldsymbol{T}, \boldsymbol{S}] = 0$

$$\Rightarrow \nabla_{\boldsymbol{T}} \nabla_{\boldsymbol{T}} \boldsymbol{S} = \nabla_{\boldsymbol{T}} \nabla_{\boldsymbol{S}} \boldsymbol{T} = \nabla_{\boldsymbol{S}} \underbrace{\nabla_{\boldsymbol{T}} \boldsymbol{T}}_{= 0 \ \text{geodesic!}} + \boldsymbol{R}(\boldsymbol{T}, \boldsymbol{S})\boldsymbol{T} \qquad \square$$

Comments:
- $R^a{}_{bcd}$ measures geodesic deviation; manifestation of curvature.
- $R^a{}_{bcd} = 0 \Leftrightarrow$ relative acceleration $= 0$ for all families of geodesics.
- Tidal forces arise from geodesic deviation.

6.6 Curvature of the Levi-Civita connection

From now on: A manifold is assumed to have a metric and

and the connection is the Levi-Civita one unless stated otherwise.

Note: $R_{abcd} = g_{ae} R^e{}_{bcd}$

Def.: "Ricci scalar" $R := g^{ab} R_{ab}$

"Einstein tensor" $G_{ab} := R_{ab} - \frac{1}{2} R g_{ab}$

Propositions: (1) $R_{abcd} = R_{cdab}$ $\big(\Rightarrow R_{(ab)cd} = R_{cd(ab)} = 0 \quad \Rightarrow \quad R_{bacd} = -R_{abcd} \big)$

(2) $R_{ab} = R_{ba}$

(3) $\nabla^a G_{ab} = 0$ "contracted Bianchi identities"

Proof: (1) Let $p \in \mathcal{M}$, use normal coords. at $p \quad \Rightarrow \quad \partial_\mu g_{\nu\rho} = 0$

$$\Rightarrow 0 = \partial_\mu \delta^\nu{}_\rho = \partial_\mu (g^{\nu\sigma} g_{\sigma\rho}) = g_{\sigma\rho} \partial_\mu g^{\nu\sigma} \qquad \Big| \cdot g^{\rho\tau}$$

$$\Rightarrow \partial_\mu g^{\nu\tau} = 0$$

$$\Rightarrow \partial_\rho \Gamma^\tau_{\nu\sigma} = \frac{1}{2} g^{\tau\mu} \big(\partial_\rho \partial_\sigma g_{\mu\nu} + \partial_\rho \partial_\nu g_{\sigma\mu} - \partial_\rho \partial_\mu g_{\nu\sigma} \big)$$

$$\Rightarrow R_{\mu\nu\rho\sigma} = \frac{1}{2} \big(\partial_\rho \partial_\nu g_{\sigma\mu} + \partial_\sigma \partial_\mu g_{\nu\rho} - \partial_\sigma \partial_\nu g_{\rho\mu} - \partial_\rho \partial_\mu g_{\nu\sigma} \big) + \underbrace{\text{“}\Gamma\Gamma - \Gamma\Gamma\text{”}}_{=0}$$

$$= R_{\rho\sigma\mu\nu} \quad \text{because } g_{\alpha\beta} \text{ symmetric, } \partial_\alpha \partial_\beta \text{ commute}$$

(2) $R_{ab} = g^{cd} R_{dacb} = g^{cd} R_{cbda} = R_{ba}$

(3) Example sheet.

6.7 Einstein's equation

Postulates of GR

(1) Spacetime is a 4-dim. Lorentzian manifold with metric and Levi-Civita connection.

(2) Free particles follow timelike or null geodesics.

(3) Energy, momentum and stress of matter is described by a symmetric, conserved tensor T_{ab} : $\nabla^a T_{ab} = 0\,.$

(4) Curvature is related to matter by the Einstein eqs.

$$\boxed{G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = \frac{8\pi G}{c^4} T_{ab}} \;; \quad G = \text{Newton's constant}$$

Comments:

(i) Simplest relation between curvature and energy-momentum is linear.

$\rightarrow$ Einstein's first guess: $R_{ab} = \kappa\, T_{ab}\,; \quad \kappa = \text{const}$

But: $\nabla^a G_{ab} = \nabla^a R_{ab} - \frac{1}{2} g_{ab} \nabla^a R = 0 - \frac{1}{2} g_{ab} \nabla^a R \quad \Big| \text{ because } \nabla^a T_{ab} = 0$

$$\stackrel{!}{=} 0 \quad \Rightarrow \quad \nabla^a R = 0 \quad \Rightarrow \quad \nabla^a T = 0$$

not satisfactory since $T = 0$ outside and $T \neq 0$ inside matter

Solution: replace R_{ab} with G_{ab} $\leftarrow$ "contracted Bianchi Id."

κ follows from Newtonian limit; cf. below.

(ii) Vacuum $\Rightarrow G_{ab} = R_{ab} - \frac{1}{2} g_{ab}\, R = 0 \quad \Big| \cdot g^{ab}$

$$\Rightarrow R = 0 \quad \Rightarrow \quad R_{ab} = 0$$

(iii) The geodesic postulate can be shown to follow from $\nabla^a T_{ab} = 0$

(iv) $G_{ab} = \frac{8\pi G}{c^4} T_{ab}$ are 10 coupled, non-linear PDEs $\rightarrow$ tough to solve

Theorem: (Lovelock 1972) Let H_{ab} be a symmetric tensor with

(i) in any chart $H_{\mu\nu} = H_{\mu\nu}(g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \partial_\sigma \partial_\rho g_{\mu\nu})$ at every $p \in \mathcal{M}$

(ii) $\nabla^a H_{ab} = 0$

(iii) $H_{\mu\nu}$ linear in $\partial_\sigma \partial_\rho g_{\mu\nu}$

$\Rightarrow \exists_{\alpha,\beta \in \mathbb{R}} \;\; H_{ab} = \alpha G_{ab} + \beta g_{ab}$

$\Rightarrow$ we can modify Einstein's eq.: $G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$

$\rightarrow$ Cosmological constant: Λ ; $|\Lambda|^{-1/2} \approx 10^9$ light years (from observations);

Λ can be regarded as a perfect fluid with $\rho = -p = \dfrac{\Lambda\, c^4}{8\pi G}$: "dark energy"

6.8 Units

In GR we often set $G = 1$, $c = 1$

$G = 6.67 \times 10^{-11} \dfrac{\mathrm{m}^3}{\mathrm{kg\,s}^2}$, $\quad c = 3 \times 10^8 \dfrac{\mathrm{m}}{\mathrm{s}}$

$\Rightarrow \quad 1\ \mathrm{s} = 3 \times 10^8\ \mathrm{m}$

$\wedge \quad 1\ \mathrm{kg} = 0.74 \times 10^{-27}\ \mathrm{m}$

E.g. $M_\odot \approx 1.48$ km

7 Diffeomorphisms and Lie derivative

7.1 Maps between manifolds

Def.: Let $\mathcal{M}, \mathcal{N}$ be differentiable manifolds of dimension m, n respectively.

A function $\phi : \mathcal{M} \to \mathcal{N}$ is smooth

$:\Leftrightarrow$ $\psi_A \circ \phi \circ \psi_\alpha^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth $\forall$ charts ψ_α on $\mathcal{M}$, ψ_A on $\mathcal{N}$

Def.: Let $\phi : \mathcal{M} \to \mathcal{N}$, $f : \mathcal{N} \to \mathbb{R}$ be smooth. The "pull-back of f by ϕ" is

$\phi^*(f) : \mathcal{M} \to \mathbb{R}\,,\quad p \mapsto \phi^*(f)\,(p) := f\big(\phi(p)\big)$

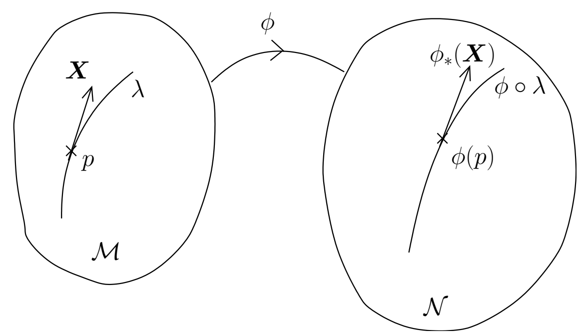
Def.: The "push-forward of a curve $\lambda : I \subset \mathbb{R} \to \mathcal{M}$" is

$\phi \circ \lambda : I \subset \mathbb{R} \to \mathcal{N}\,,\quad t \mapsto \phi\big(\lambda(t)\big)$

Def.: Let $p \in \mathcal{M}\,,\ \boldsymbol{X} \in \mathcal{T}_p(\mathcal{M})$ be the tangent vector of $\lambda : I \subset \mathbb{R} \to \mathcal{M}$

The "push-forward of $\boldsymbol{X}$ by ϕ" is

$\phi_*(\boldsymbol{X}) \in \mathcal{T}_{\phi(p)}(\mathcal{N})$ defined as tangent vector of $\phi \circ \lambda$



Lemma: Let $\boldsymbol{X} \in \mathcal{T}_p(\mathcal{M})$, $f : \mathcal{N} \to \mathbb{R}$

$\Rightarrow \big(\phi_*(\boldsymbol{X})\big)(f) = \boldsymbol{X}\big(\phi^*(f)\big)$

Proof: Let $\lambda(0) = p$

$$\Rightarrow \big(\phi_*(\boldsymbol{X})\big)(f)\Big|_{\phi(p)} = \left[\frac{d}{dt}\big(f \circ (\phi \circ \lambda)\big)(t)\right]\Bigg|_{t=0} = \left[\frac{d}{dt}\big((f \circ \phi) \circ \lambda\big)(t)\right]\Bigg|_{t=0} = \boldsymbol{X}\ \big(\phi^*(f)\big)\Big|_p$$

Def.: Let $\phi : \mathcal{M} \to \mathcal{N}$ be smooth, $p \in \mathcal{M}$, $\boldsymbol{\eta} \in \mathcal{T}^*_{\phi(p)}(\mathcal{N})$.

The "pull-back of $\boldsymbol{\eta}$ by ϕ" is

$\phi^*(\boldsymbol{\eta}) \in \mathcal{T}^*_p(\mathcal{M})\,,\quad \big(\phi^*(\boldsymbol{\eta})\big)(\boldsymbol{X}) = \boldsymbol{\eta}\big(\phi_*(\boldsymbol{X})\big) \quad \forall \boldsymbol{X} \in \mathcal{T}_p(\mathcal{M})$

Lemma: Let $f : \mathcal{N} \to \mathbb{R} \quad \Rightarrow \quad \mathbf{d}f \in \mathcal{T}^*_{\phi(p)}(\mathcal{N})$.

Then $\phi^*(\mathbf{d}f) \in \mathcal{T}^*_p(\mathcal{M})$ is $\phi^*(\mathbf{d}f) = \mathbf{d}\big(\phi^*(f)\big)$

Proof: Let $\boldsymbol{X} \in \mathcal{T}_p(\mathcal{M})$

$$\Rightarrow \big(\phi^*(\mathbf{d}f)\big)\,(\boldsymbol{X}) = \mathbf{d}f\big(\phi_*(\boldsymbol{X})\big) = \big(\phi_*(\boldsymbol{X})\big)\,(f) = \boldsymbol{X}\big(\phi^*(f)\big) = \big[\mathbf{d}\big(\phi^*(f)\big)\big]\,(\boldsymbol{X})$$

Components

Let x^μ be coords. on $\mathcal{M}$, y^α coords. on $\mathcal{N}$, $\mu = 1 \ldots \dim(\mathcal{M})$, $\alpha = 1 \ldots \dim(\mathcal{N})$

$\Rightarrow \phi : \mathcal{M} \to \mathcal{N}$ defines a map $x^\mu \mapsto y^\alpha(x^\mu)$

One can show: for a vector $\boldsymbol{X} \in \mathcal{T}_p(\mathcal{M})$: $\big(\phi_*(\boldsymbol{X})\big)^\alpha = \left.\dfrac{\partial y^\alpha}{\partial x^\mu}\right|_p X^\mu$

for a 1-form $\boldsymbol{\eta} \in \mathcal{T}^*_{\phi(p)}(\mathcal{N})$: $\big(\phi^*(\boldsymbol{\eta})\big)_\mu = \left.\dfrac{\partial y^\alpha}{\partial x^\mu}\right|_p \eta_\alpha$

Comments:

- $p \in \mathcal{M}$ was arbitrary

  $\Rightarrow$ push-forward applies to vector fields, pull-back to covector fields

- pull-back of $\binom{0}{s}$ tensor $\boldsymbol{S}$:

  $\big(\phi^*(\boldsymbol{S})\big)\,(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_s) := \boldsymbol{S}\big(\phi_*(\boldsymbol{X}_1), \ldots, \phi_*(\boldsymbol{X}_s)\big) \quad \forall \boldsymbol{X}_1, \ldots, \boldsymbol{X}_s \in \mathcal{T}_p(\mathcal{M})$

  push-forward of $\binom{r}{0}$ tensor $\boldsymbol{T}$:

  $\big(\phi_*(\boldsymbol{T})\big)\,(\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_r) := \boldsymbol{T}\big(\phi^*(\boldsymbol{\eta}_1), \ldots, \phi^*(\boldsymbol{\eta}_r)\big) \quad \forall \boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_r \in \mathcal{T}^*_{\phi(p)}(\mathcal{N})$

  Components: $\big(\phi^*(S)\big)_{\mu_1\ldots\mu_s} = \left.\dfrac{\partial y^{\alpha_1}}{\partial x^{\mu_1}}\right|_p \cdot \ldots \cdot \left.\dfrac{\partial y^{\alpha_s}}{\partial x^{\mu_s}}\right|_p S_{\alpha_1\ldots\alpha_s}$

  $\big(\phi_*(T)\big)^{\alpha_1\ldots\alpha_r} = \left.\dfrac{\partial y^{\alpha_1}}{\partial x^{\mu_1}}\right|_p \cdot \ldots \cdot \left.\dfrac{\partial y^{\alpha_r}}{\partial x^{\mu_r}}\right|_p T^{\mu_1\ldots\mu_r}$

Example: Let $\mathcal{M} = S^2$ (unit sphere), $\mathcal{N} = \mathbb{R}^3$, $x^\mu = (\theta, \phi)$ spherical coords. on S^2

$\phi : \mathcal{M} \to \mathcal{N}$, $p(\theta, \phi) \mapsto y^\alpha = (\sin\theta\,\cos\phi,\, \sin\theta\,\sin\phi,\, \cos\theta) \in \mathbb{R}^3$.

Let $\boldsymbol{g}$ be the Euclidean metric on $\mathbb{R}^3$, $g_{\alpha\beta} = \delta_{\alpha\beta}$ in coords. $(x,\, y,\, z)$

$\Rightarrow \ldots \Rightarrow$ The pull-back of $\boldsymbol{g}$ onto S^2 is: $(\phi^* g)_{\mu\nu} = \mathrm{diag}(1,\, \sin^2\theta)$.

7.2 Diffeomorphisms, Lie derivative

Def.: $\phi : \mathcal{M} \to \mathcal{N}$ is a "diffeomorphism" (dfm.)

$:\Leftrightarrow$ ϕ is 1-to-1, onto, smooth, and has a

smooth inverse. $\mathcal{M}, \mathcal{N}$ must have the same dimension.

with a dfm., we have:

Def.: Let $\phi : \mathcal{M} \to \mathcal{N}$ be a dfm., $\boldsymbol{T}$ a $\binom{r}{s}$ tensor on $\mathcal{M}$.

The "push-forward of $\boldsymbol{T}$ under ϕ is the $\binom{r}{s}$ tensor on $\mathcal{N}$:

$$\phi_*(\boldsymbol{T})(\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_r, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_s)$$

$$:= \boldsymbol{T}\big(\phi^*(\boldsymbol{\eta}_1), \ldots, \phi^*(\boldsymbol{\eta}_r), (\phi^{-1})_*(\boldsymbol{X}_1), \ldots, (\phi^{-1})_*(\boldsymbol{X}_s)\big)$$

$$\forall \boldsymbol{\eta}_i \in \mathcal{T}^*_{\phi(p)}(\mathcal{N}), \quad \boldsymbol{X}_i \in \mathcal{T}_{\phi(p)}(\mathcal{N})$$

One can show:

1) Push-forward commutes with contraction and outer product.
2) Components for $\binom{1}{1}$ tensor in coord. basis:

$$\big[\big(\phi_*(\boldsymbol{T})\big)^\mu{}_\nu\big]_{\phi(p)} = \left.\frac{\partial y^\mu}{\partial x^\rho}\right|_p \left.\frac{\partial x^\sigma}{\partial y^\nu}\right|_p (T^\rho{}_\sigma)_p \qquad (*)$$

generalizes obviously for $\binom{r}{s}$ tensors

Comments: 1) pull-back of $\binom{r}{s}$ tensors can be defined likewise $\Rightarrow \phi^* = (\phi^{-1})_*$

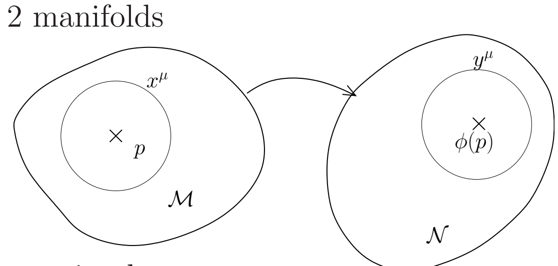
2) We took "active" viewpoint: $\phi : p \mapsto \phi(p)$, 2 manifolds

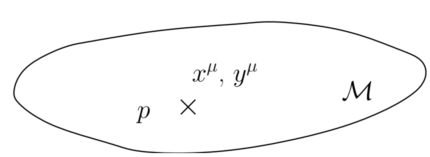
"passive interpretation":

pull coords. y^μ back from $\mathcal{N}$ to $\mathcal{M}$

$\Rightarrow$ simply 2 coord. charts x^μ, y^μ on $\mathcal{M}$

$\Rightarrow$ $(*)$ becomes the ordinary tensor transformation law





Def.: Let $\phi : \mathcal{M} \to \mathcal{N}$ be a dfm., ∇ a covariant deriv. on $\mathcal{M}$,

$\boldsymbol{X}$ a vector, $\boldsymbol{T}$ a tensor on $\mathcal{N}$.

$\Rightarrow$ The push-forward of ∇ is the cov. deriv. $\tilde{\nabla}$ on $\mathcal{N}$ defined by

$\tilde{\nabla}_{\boldsymbol{X}}\boldsymbol{T} := \phi_*\big[\nabla_{\phi^*(\boldsymbol{X})}\big(\phi^*(\boldsymbol{T})\big)\big]$

One can show (Example sheet 3):

(1) $\tilde{\nabla}$ satisfies the properties of a cov. deriv.

(2) The Riemann tensor of $\tilde{\nabla}$ is the push-forward of Riemann(∇)

(3) Let ∇ be the cov. deriv. of the Levi-Civita connection of $\boldsymbol{g}$ on $\mathcal{M}$

$\Rightarrow$ $\tilde{\nabla}$ is that of the Levi-Civita connection of $\phi_*(\boldsymbol{g})$ on $\mathcal{N}$

Diffeomorphism invariance

We defined a spacetime as a pair $(\mathcal{M}, \boldsymbol{g})$.

Let's add matter fields $\boldsymbol{F}, \ldots \quad \to \quad (\mathcal{M}, \boldsymbol{g}, \boldsymbol{F}, \ldots)$

2 models $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{F}, \ldots)$, $(\mathcal{M}', \boldsymbol{g}', \boldsymbol{F}', \ldots)$ are taken to be equivalent if

$\exists$ dfm. $\phi : \mathcal{M} \to \mathcal{M}'$ which carries $\boldsymbol{g}$, $\boldsymbol{F}$, … to $\boldsymbol{g}'$, $\boldsymbol{F}'$, … :

$$\boldsymbol{g}' = \phi_*\boldsymbol{g}, \quad \boldsymbol{F}' = \phi_*\boldsymbol{F}, \ \ldots$$

active-passive equivalence $\Rightarrow$ the models just differ by a coord. trafo.

$\Rightarrow$ A spacetime is really an equivalence class of all equivalent $(\mathcal{M}', \boldsymbol{g}', \boldsymbol{F}', \ldots)$

Consequences: 1) Einstein's eqs. will not predict all 10 metric components!

2) Physical statements in GR must be diffeomorphism invariant.

3) This is the gauge freedom of GR.

Examples: 1) "Two geodesics intersect at $x^\mu = (\ldots)$" is not gauge invariant

2) Consider a geodesic intersected exactly once by each of two other geodesics.

$\Rightarrow$ The proper time along the geodesic between the intersections is gauge invariant.

Lie derivatives, symmetries

Push-forward and pull-back provide a way to compare tensors at different $p,\ q \in \mathcal{M}$

Def.: A dfm. $\phi : \mathcal{M} \to \mathcal{M}$ is a "symmetry transformation of a tensor field $\boldsymbol{T}$"

$:\Leftrightarrow \phi_*(\boldsymbol{T}) = \boldsymbol{T}$ everywhere.

"isometry" := a symmetry trafo. of the metric

Def.: Let $\boldsymbol{X}$ be a VF on a manifold $\mathcal{M}$. Let $\phi_t : \mathcal{M} \to \mathcal{M},\ \ p \mapsto q$ such that

$q :=$ point a parameter distance t along the integral curve of $\boldsymbol{X}$ through p

For small enough t, ϕ_t can be shown to be a dfm.

Comments: 1) ϕ_0 is the identity map; $\phi_s \circ \phi_t = \phi_{t+s}$; $\phi_{-t} = (\phi_t)^{-1}$.

2) If ϕ_t is a dfm. $\forall t \in \mathbb{R} \Rightarrow$ the ϕ_t form a 1-par. Abelian group

Then we can define $\forall p \in \mathcal{M}$ the curve

$\lambda_p : \mathbb{R} \to \mathcal{M}\,,\ \ t \mapsto \phi_t(p)$.

Doing this $\forall p \in \mathcal{M}$ defines a VF:

$\boldsymbol{X}$:= tangent vectors of these curves.

3) The push-forward $(\phi_t)_*$ allows us to compare tensors at different points.

$\to$ **Def.:** The "Lie derivative of a tensor $\boldsymbol{T}$ along a VF $\boldsymbol{X}$ at $p \in \mathcal{M}$" is

$$(\mathcal{L}_{\boldsymbol{X}}\boldsymbol{T})_p = \lim_{t\to 0} \frac{[(\phi_{-t})_*\boldsymbol{T}]_p - \boldsymbol{T}_p}{t}$$

Comments: • $\mathcal{L}_{\boldsymbol{X}}$ maps $\binom{r}{s}$ tensor fields to $\binom{r}{s}$ tensor fields

• α, β const. $\Rightarrow$ $\mathcal{L}_{\boldsymbol{X}}(\alpha\boldsymbol{S} + \beta\boldsymbol{T}) = \alpha\mathcal{L}_{\boldsymbol{X}}\boldsymbol{S} + \beta\mathcal{L}_{\boldsymbol{X}}\boldsymbol{T}$

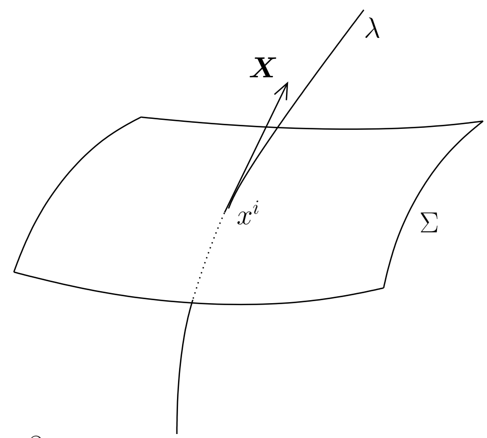
Adapted coordinates

1) Let Σ be an $n-1$ dim. hypersurface of $\mathcal{M}$,

   $\boldsymbol{X}$ a VF that is nowhere tangent to Σ.

2) Let $x^i\,,\ i = 1\ldots n-1$ be coords. on Σ. Assign to $q \in \mathcal{M}$ coords. $(t,\, x^i)$ such that q is a parameter distance t along the integral curve of $\boldsymbol{X}$ through x^i on Σ.

   $\rightarrow$ coord. chart for sufficiently small t



Note: Int. curves of $\boldsymbol{X}$ have fixed (x^i) and parameter t: $\boldsymbol{X} = \dfrac{\partial}{\partial t}$.

The dfm. ϕ_t sends point p with $(t_p,\, x^i)$ to q with $y^\mu = (t_p + t,\, x^i)$.

$$\Rightarrow \frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu{}_\nu$$

Now consider an $\binom{r}{s}$ tensor $\boldsymbol{T}$ in these coords.:

$$\Rightarrow \left[\left((\phi_t)_* \boldsymbol{T}\right)^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}\right]_{\phi_t(p)} = \frac{\partial y^{\mu_1}}{\partial x^{\rho_1}} \cdots \frac{\partial y^{\mu_r}}{\partial x^{\rho_r}} \frac{\partial x^{\sigma_1}}{\partial y^{\nu_1}} \cdots \frac{\partial x^{\sigma_s}}{\partial y^{\nu_s}} \left[T^{\rho_1\ldots\rho_r}{}_{\sigma_1\ldots\sigma_s}\right]_p$$

$$= \left[T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}\right]_p$$

$$\Rightarrow \left[\left((\phi_{\pm t})_* \boldsymbol{T}\right)^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}\right]_p = \left[T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}\right]_{\phi_{\mp t}(p)}$$

$\Rightarrow$ at p with $(t_p,\, x^i)$:

$$(\mathcal{L}_{\boldsymbol{X}} \boldsymbol{T})^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} = \lim_{t\to 0} \frac{1}{t} \left[T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}(t_p + t, x^i) - T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}(t_p, x^i)\right]$$

$$= \left[\frac{\partial}{\partial t} T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}(t, x^i)\right]_{(t_p, x^i)} \qquad \text{in this chart!}$$

It follows: Leibniz rule: $\mathcal{L}_{\boldsymbol{X}}(\boldsymbol{S} \otimes \boldsymbol{T}) = (\mathcal{L}_{\boldsymbol{X}} \boldsymbol{S}) \otimes \boldsymbol{T} + \boldsymbol{S} \otimes (\mathcal{L}_{\boldsymbol{X}} \boldsymbol{T})$;

$\mathcal{L}_{\boldsymbol{X}}$ commutes with contraction

We still need a chart independent expression:

1) In this chart: $\mathcal{L}_{\boldsymbol{X}} f \stackrel{*}{=} \frac{\partial}{\partial t} f$ for function f, $\boldsymbol{X}(f) \stackrel{*}{=} \frac{\partial}{\partial t} f$

$\Rightarrow \mathcal{L}_{\boldsymbol{X}} f = \boldsymbol{X}(f)$ in any basis!

2) In our chart: $(\mathcal{L}_{\boldsymbol{X}} \boldsymbol{Y})^\mu = \frac{\partial Y^\mu}{\partial t}$ for VF $\boldsymbol{Y}$;

$$[\boldsymbol{X}, \boldsymbol{Y}]^\mu = \frac{\partial Y^\mu}{\partial t} \quad \text{(because } X^\mu = \delta^\mu{}_0\text{)}$$

$\Rightarrow \mathcal{L}_{\boldsymbol{X}} \boldsymbol{Y} = [\boldsymbol{X}, \boldsymbol{Y}]$ in any basis!

Comment: $\mathcal{L}_{\boldsymbol{X}} \boldsymbol{T}$ depends on $\boldsymbol{X}_p$ and its derivative

$\Rightarrow \mathcal{L}$, $\mathcal{L}\boldsymbol{T}$ are not tensors

cf. covariant deriv.: $\nabla_{\boldsymbol{X}} \boldsymbol{T}$ depends only on $\boldsymbol{X}_p$; also linear in $\boldsymbol{X}_p$

$\Rightarrow \nabla \boldsymbol{T}$ is a tensor

One can show:

1) For 1-form $\boldsymbol{\omega}$: $(\mathcal{L}_{\boldsymbol{X}} \boldsymbol{\omega})_\mu = X^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu$,

$$(\mathcal{L}_{\boldsymbol{X}} \boldsymbol{\omega})_a = X^b \nabla_b \omega_a + \omega_b \nabla_a X^b$$

2) For a tensor $\boldsymbol{T}$: $(\mathcal{L}_{\boldsymbol{X}} \boldsymbol{T})^{\alpha\ldots}{}_{\beta\ldots} = X^\gamma \partial_\gamma T^{\alpha\ldots}{}_{\beta\ldots} - (\partial_\gamma X^\alpha) T^{\gamma\ldots}{}_{\beta\ldots} - \ldots + (\partial_\beta X^\gamma) T^{\alpha\ldots}{}_{\gamma\ldots} + \ldots$

$$(\mathcal{L}_{\boldsymbol{X}} \boldsymbol{T})^{a\ldots}{}_{b\ldots} = X^c \nabla_c T^{a\ldots}{}_{b\ldots} - (\nabla_c X^a) T^{c\ldots}{}_{b\ldots} - \ldots + (\nabla_b X^c) T^{a\ldots}{}_{c\ldots} + \ldots$$

3) For metric: $(\mathcal{L}_{\boldsymbol{X}} \boldsymbol{g})_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu X^\rho + g_{\rho\nu} \partial_\mu X^\rho$

$$= g_{\mu\rho} \nabla_\nu X^\rho + g_{\rho\nu} \nabla_\mu X^\rho \quad \text{(for Levi-Civita connection)}$$

Killing's equation: Let ϕ_t be an isometry $\forall_{t\in\mathbb{R}} \Rightarrow \mathcal{L}_{\boldsymbol{X}} \boldsymbol{g} = 0$

$$\boxed{\nabla_a X_b + \nabla_b X_a = 0} \qquad \text{solutions } \boldsymbol{X} \text{ are “Killing vectors”}$$

Note: 1) If $\exists$ chart with one coord. z on which $g_{\mu\nu}$ do not depend

$$\Rightarrow \frac{\partial}{\partial z} \text{ is Killing VF}$$

2) Conversely, if $\exists$ a Killing VF

$\Rightarrow$ we can choose coords. such that $g_{\mu\nu}$ does not depend on one of them

Lemma: Let $\boldsymbol{X}$ be a Killing field and $\boldsymbol{V}$ a VF tangent to an affinely parametrized geodesic.

$$\begin{aligned}\frac{d}{d\tau}(X_a V^a) &= \boldsymbol{V}(X_a V^a) = \nabla_{\boldsymbol{V}}(X_a V^a) = V^b \nabla_b (X_a V^a) \\ &= \underbrace{V^a V^b}_{\text{symm.}} \underbrace{\nabla_b X_a}_{\text{antisymm.}} + X_a V^b \nabla_b V^a = 0\end{aligned}$$

$\Rightarrow X_a V^a$ const. along geodesic.

One can show: T_{ab} = energy-momentum tensor, X^a = Killing VF, $J^a := T^a{}_b X^b$

$\Rightarrow \nabla_a J^a = 0$ "conserved current"

8 Linearized Theory

8.1 The linearized Einstein eqs.

Consider small deviations from Minkowski in Cart. coords.

"Background": Manifold $\mathcal{M} = \mathbb{R}^4$, $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$

"Perturbation": $h_{\mu\nu} = \mathcal{O}(\epsilon) \ll 1 \quad \Rightarrow \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

regard $h_{\mu\nu}$ as a tensor field on Minkowski background

2 metrics: $\eta_{\mu\nu}$ and the "physical metric" $g_{\mu\nu}$.

inverse metric: $g^{\mu\nu} = \eta^{\mu\nu} + k^{\mu\nu}$

$$\Rightarrow g^{\mu\nu} g_{\nu\rho} = \delta^\mu{}_\rho + k^{\mu\nu}\eta_{\nu\rho} + \eta^{\mu\nu} h_{\nu\rho} + \underbrace{k^{\mu\nu} h_{\nu\rho}}_{=\mathcal{O}(\epsilon^2)\to 0} \stackrel{!}{=} \delta^\mu{}_\rho$$

$$\Rightarrow k^{\mu\nu} = -\eta^{\mu\rho}\eta^{\nu\sigma} h_{\rho\sigma} =: -h^{\mu\nu} = \mathcal{O}(\epsilon)$$

To $\mathcal{O}(\epsilon)$: $\Gamma^\mu_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}(\partial_\rho h_{\sigma\nu} + \partial_\nu h_{\rho\sigma} - \partial_\sigma h_{\nu\rho})\,,$

$$R_{\mu\nu\rho\sigma} = \eta_{\mu\tau}\big(\partial_\rho \Gamma^\tau_{\nu\sigma} - \partial_\sigma \Gamma^\tau_{\nu\rho}\big) \qquad \Big| \ \Gamma\cdot\Gamma = \mathcal{O}(\epsilon^2)$$

$$= \tfrac{1}{2}\big(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\sigma\partial_\nu h_{\mu\rho}\big)$$

$$R_{\mu\nu} = \partial^\rho\partial_{(\mu} h_{\nu)\rho} - \frac{1}{2}\partial^\rho\partial_\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h \qquad \Big| \ h := h^\mu{}_\mu\,, \quad \partial^\mu := g^{\mu\rho}\partial_\rho$$

$$G_{\mu\nu} = \partial^\rho\partial_{(\mu} h_{\nu)\rho} - \frac{1}{2}\partial^\rho\partial_\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h - \frac{1}{2}\eta_{\mu\nu}(\partial^\rho\partial^\sigma h_{\rho\sigma} - \partial^\rho\partial_\rho h) \stackrel{!}{=} 8\pi T_{\mu\nu}$$

$$\Rightarrow T_{\mu\nu} \ll 1$$

Def.: "trace-reversed pert." $\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} h\eta_{\mu\nu} \quad \Leftrightarrow \quad h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu}\,,$

$$\bar{h} = \bar{h}^\mu{}_\mu = -h$$

$$\Rightarrow \ldots \Rightarrow G_{\mu\nu} = -\frac{1}{2}\partial^\rho\partial_\rho \bar{h}_{\mu\nu} + \partial^\rho\partial_{(\mu}\bar{h}_{\nu)\rho} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho\partial^\sigma \bar{h}_{\rho\sigma} = 8\pi T_{\mu\nu}$$

Gauge symmetry

Let $(\mathcal{M}, \boldsymbol{g}, \boldsymbol{T})$ be a spacetime, $\phi : \mathcal{M} \to \mathcal{M}'$ a diffeomorphism.

$\Rightarrow \quad \big(\mathcal{M}', \phi_*(\boldsymbol{g}), \phi_*(\boldsymbol{T})\big)$ is a physically equivalent spacetime.

We want $\eta_{\mu\nu}$ to remain the background metric $\Rightarrow \phi \sim \mathcal{O}(\epsilon)$

Consider dfm. ϕ_t defined by integral curves of VF $\boldsymbol{X} \Rightarrow t = \mathcal{O}(\epsilon)$

$\Rightarrow$ With $\xi^\mu = t\, X^\mu = \mathcal{O}(\epsilon)$ for any tensor $\boldsymbol{T}$:

$$(\phi_{-t})_*(\boldsymbol{T}) = \boldsymbol{T} + t\,\mathcal{L}_{\boldsymbol{X}}\boldsymbol{T} + \mathcal{O}(t^2) = \boldsymbol{T} + \mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{T} + \mathcal{O}(\epsilon^2)$$

energy momentum tensor: $T_{\mu\nu} \overset{!}{=} \mathcal{O}(\epsilon) \Rightarrow \big((\phi_{-t})_* T\big)_{\mu\nu} = T_{\mu\nu} + \mathcal{O}(\epsilon^2)$

metric: $(\phi_{-t})_*(\boldsymbol{g}) = \boldsymbol{g} + \mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{g} + \ldots = \boldsymbol{\eta} + \boldsymbol{h} + \mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{\eta} + \mathcal{O}(\epsilon^2)$

$\Rightarrow h_{\mu\nu}$ and $h_{\mu\nu} + (\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{\eta})_{\mu\nu}$ are physically equivalent perturbations

$\Rightarrow$ gauge symmetry: $\boxed{h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu}\,, \quad \xi_\mu = \mathcal{O}(\epsilon)$

Now choose ξ_μ such that $\partial^\nu \partial_\nu \xi_\mu = -\partial^\nu \bar{h}_{\mu\nu}$

$\Rightarrow \quad \partial^\nu \bar{h}_{\mu\nu} \to \ldots = \partial^\nu \bar{h}_{\mu\nu} + \partial^\nu \partial_\nu \xi_\mu = 0$

$\Rightarrow G_{\mu\nu} = -\dfrac{1}{2}\partial^\rho \partial_\rho \bar{h}_{\mu\nu}$

$\Rightarrow$ lin. Einstein eqs.: $\boxed{\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}}$ "Lorentz gauge"

8.2 Newtonian limit

Newtonian gravity: $\vec{\nabla}^2 \Phi = 4\pi\rho\,; \quad c = G = 1\,, \quad \Phi \sim v^2 \ll 1\,, \quad \epsilon := \dfrac{v^2}{c^2} = v^2$

$\Rightarrow$ matter sources weak: $\rho \sim \mathcal{O}(\epsilon)$

energy momentum tensor

for Newtonian matter: $T_{00} = \rho + \mathcal{O}(\epsilon^2)$

$$T_{0i} \sim T_{00}\, v_i \sim \mathcal{O}(\epsilon^{3/2})$$

$$T_{ij} \sim T_{00}\, v_i v_j \sim \mathcal{O}(\epsilon^2)$$

E.g. perfect fluid: $T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}$

$$P \sim \rho \frac{v^2}{c^2}\,, \quad \frac{P}{\rho} \approx 10^{-5} \text{ in sun}$$

$$\text{In SR: } u^\mu = \left[\frac{1}{\sqrt{1-v^2}}, \frac{v^i}{\sqrt{1-v^2}}\right]\;; \quad v^2 = v_i v^i$$

In Newt. gravity temporal changes in Φ are caused by motion of sources

$$\Rightarrow \frac{\partial}{\partial t} \sim v\frac{\partial}{\partial x^i} = \mathcal{O}(\epsilon^{1/2})\frac{\partial}{\partial x^i}\,, \quad i = 1\dots 3$$

$$\Rightarrow \Box \bar{h}_{\mu\nu} = \partial^\rho\partial_\rho \bar{h}_{\mu\nu} = \partial^i\partial_i \bar{h}_{\mu\nu} = \vec{\nabla}^2 \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$$

$$\Rightarrow \vec{\nabla}^2 \bar{h}_{00} = -16\pi T_{00} = -16\pi\rho + \mathcal{O}(\epsilon^2)\,, \quad \bar{h}_{0i} = \mathcal{O}(\epsilon^{3/2})\,, \quad \bar{h}_{ij} = \mathcal{O}(\epsilon^2)$$

Newton's law with $\bar{h}_{00} = -4\Phi$

$$\Rightarrow \bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = 4\Phi + \mathcal{O}(\epsilon^2) = -h$$

$$\Rightarrow h_{00} = \bar{h}_{00} - \frac{1}{2}\eta_{00}\bar{h} = -2\Phi\,, \quad h_{ij} = \bar{h}_{ij} - \frac{1}{2}\eta_{ij}\bar{h} = -2\Phi\delta_{ij}$$

or $\boxed{ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2 + dy^2 + dz^2)}$ cf. Sec. 1.4

Geodesics in the weak field

Lagrangian:
$$L = -g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \quad \big(= G^2 \text{ in Sec. 3.3}\big)$$
$$= (1+2\Phi)\dot{t}^2 - \delta_{ij}(1-2\Phi)\dot{x}^i\dot{x}^j \overset{!}{=} 1 \quad \text{(proper time)}$$

$$\Rightarrow \dot{t}^2 = (1+2\Phi)^{-1}\big[1 + \delta_{ij}\dot{x}^i\dot{x}^j + \mathcal{O}(\epsilon^2)\big]$$

$$\Rightarrow \dot{t} = 1 - \Phi + \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j + \mathcal{O}(\epsilon^2)$$

EL-eq. for x^k:
$$\frac{d}{d\tau}\big[-2\delta_{jk}(1-2\Phi)\dot{x}^j\big] = \frac{\partial L}{\partial x^k} = 2\partial_k\Phi\dot{t}^2 + 2\partial_k\Phi\delta_{ij}\dot{x}^i\dot{x}^j + \mathcal{O}(\epsilon^2)$$
$$= 2\partial_k\Phi + \mathcal{O}(\epsilon^2)$$
$$\Rightarrow -2\delta_{jk}\ddot{x}^j + \mathcal{O}(\epsilon^2) = 2\partial_k\Phi$$
$$\Rightarrow \frac{d^2x^k}{d\tau^2} = \frac{d^2x^k}{dt^2} = -\partial_k\Phi \quad \text{test body in Newt. gravity}$$

8.3 Gravitational waves

weak field but now: vacuum; no longer "$\partial_t \ll \partial_x$"

$\Rightarrow \Box \bar{h}_{\mu\nu} = (\partial_t^2 - \vec{\nabla}^2)\bar{h}_{\mu\nu} = 0$

Plane wave solution: $\bar{h}_{\mu\nu} = H_{\mu\nu} e^{ik_\rho x^\rho}$; $H_{\mu\nu} = \text{const}$

(i) $\Box \bar{h}_{\mu\nu} = 0 \quad \Rightarrow \quad k_\mu k^\mu = 0 \quad \rightarrow$ speed of light

(ii) Lorentz gauge: $\partial^\nu \bar{h}_{\mu\nu} = 0 \quad \Rightarrow \quad k^\mu H_{\mu\nu} = 0$ "transverse"

E.g. wave in z-dir.: $k^\mu = \omega\,(1,\,0,\,0,\,1) \quad \Rightarrow \quad H_{\mu 0} + H_{\mu 3} = 0$

Remaining gauge freedom: take $\xi_\mu = X_\mu e^{ik_\rho x^\rho} \Rightarrow \partial^\nu \partial_\nu \xi_\mu = 0$

$\Rightarrow \ldots \Rightarrow \quad H_{\mu\nu} \to H_{\mu\nu} + i(k_\mu X_\nu + k_\nu X_\mu - \eta_{\mu\nu} k^\rho X_\rho)$

$\Rightarrow \ldots \Rightarrow \quad \exists X_\mu : \quad H_{0\mu} = 0\,, \quad H^\mu{}_\mu = 0$ "traceless"

In this gauge: 1) $h = 0 \Rightarrow h_{\mu\nu} = \bar{h}_{\mu\nu}$

2) plane wave in z-dir.: $H_{0\mu} = H_{3\mu} = H^\mu{}_\mu = 0$

$$\Rightarrow \quad H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Effect on particles

Consider particle at rest in background Lorentz frame: $u_0^\alpha = (1,\,0,\,0,\,0)$

geodesic eq.: $\displaystyle \frac{d}{d\tau} u^\alpha + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu = \dot{u}^\alpha + \Gamma^\alpha_{00} = 0$

$\displaystyle \Gamma^\alpha_{00} = \frac{1}{2} \eta^{\alpha\beta} (\partial_0 h_{\beta 0} + \partial_0 h_{0\beta} - \partial_\beta h_{00}) = 0$ since $H_{0\mu} = 0$

$\Rightarrow u^\alpha = (1,\,0,\,0,\,0)$ always

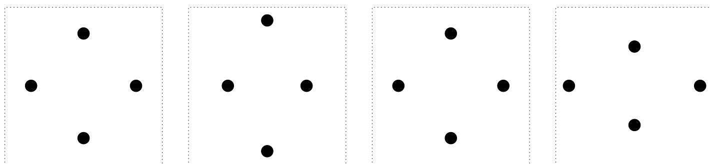
$\Rightarrow$ particle stays at $x^\mu = \text{const}$ in this gauge

Proper separation: $ds^2 = -dt^2 + (1+h_+)dx^2 + (1-h_+)dy^2 + 2h_\times dx\,dy + dz^2$

Case 1: $H_\times = 0\,,\ H_+ \neq 0 \quad \Rightarrow \quad h_+$ oscillates

2 particles at $(-\delta,\,0,\,0),\ (\delta,\,0,\,0) \quad \Rightarrow \quad ds^2 = (1+h_+)\,4\delta^2$

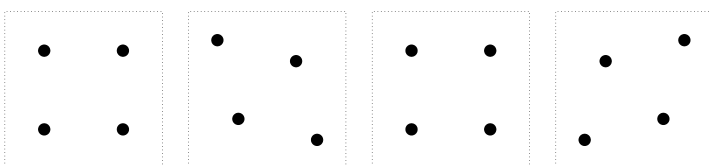
2 particles at $(0,\,-\delta,\,0),\ (0,\,\delta,\,0) \quad \Rightarrow \quad ds^2 = (1-h_+)\,4\delta^2$



Case 2: $H_+ = 0\,,\ H_\times \neq 0$

2 particles at $(-\delta,\,-\delta,\,0)\,/\sqrt{2}\,,\quad (\delta,\,\delta,\,0)\,/\sqrt{2} \quad \Rightarrow \quad ds^2 = (1+h_\times)\,4\delta^2$

2 particles at $(\delta,\,-\delta,\,0)\,/\sqrt{2}\,,\quad (-\delta,\,\delta,\,0)\,/\sqrt{2} \quad \Rightarrow \quad ds^2 = (1-h_\times)\,4\delta^2$



8.4 The field far from a source

weak field with matter: $\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$

Green's function: $\bar{h}_{\mu\nu}(t,\vec{x}) = 4 \displaystyle\int \frac{T_{\mu\nu}(t-|\vec{x}-\vec{y}|,\vec{y})}{|\vec{x}-\vec{y}|} d^3y\,, \qquad |\vec{x}|^2 = x_1^2 + x_2^2 + x_3^2$

Assume matter has compact support inside radius d

$\Rightarrow$ far from the source: $r := |\vec{x}| \gg d \geq |\vec{y}|$

$\Rightarrow \ldots \Rightarrow |\vec{x}-\vec{y}| = r - \hat{\vec{x}}\cdot\vec{y} + \mathcal{O}\left(\frac{d}{r}\right)\ ; \qquad \hat{\vec{x}} := \dfrac{\vec{x}}{r}$

$\Rightarrow T_{\mu\nu}\big(t-|\vec{x}-\vec{y}|,\vec{y}\big) = T_{\mu\nu}(t-r,\vec{y}) - \hat{\vec{x}}\cdot\vec{y}\,\big(\partial_0 T_{\mu\nu}\big)(t-r,\vec{y}) \qquad$ Taylor

Assume $v \ll c \ \Rightarrow \partial_0 T_{\mu\nu} \sim T_{\mu\nu}\dfrac{v}{d} \ll \dfrac{T_{\mu\nu}}{d}$

$$\Rightarrow \bar{h}_{\mu\nu}(t,\vec{x}) \approx \frac{4}{r}\int T_{\mu\nu}(t-r,\vec{y})\,d^3y \qquad (*)$$

Lorentz gauge: $\partial^\nu \bar{h}_{\mu\nu} = 0$

$\Rightarrow \partial_0 \bar{h}_{0i} = \partial_j \bar{h}_{ji}\,,\quad \partial_0 \bar{h}_{00} = \partial_i \bar{h}_{oi}\,;$ sum over $i,\, j = 1 \ldots 3$

$\Rightarrow$ Strategy: calculate $\bar{h}_{ij}\,, \to \bar{h}_{0i} \to \bar{h}_{00}$

$$\int T^{ij} d^3y = \int \underbrace{\partial_k (T^{ik}\, y^j)}_{\text{surface term } \to 0} - (\partial_k T^{ik})\, y^j\, d^3y$$

$$= \int (\partial_0 T^{i0})\, y^j\, d^3y \qquad \text{since } \partial_\mu T^{i\mu} = 0$$

$$\Rightarrow \int T^{(ij)} d^3y = \partial_0 \int T^{0(i} y^{j)}\, d^3y = \partial_0 \int \frac{1}{2} \underbrace{\partial_k (T^{0k} y^i\, y^j)}_{\to 0} - \frac{1}{2} (\partial_k T^{0k})\, y^i\, y^j\, d^3y$$

$$= \frac{1}{2} \partial_0 \partial_0 \int T^{00} y^i\, y^j\, d^3y \qquad \Big|\ \partial_\mu T^{0\mu} = 0$$

$$\Rightarrow \boxed{\bar{h}_{ij}(t, \vec{x}) = \frac{2}{r} \ddot{I}_{ij}(t - r)\,; \quad I_{ij}(t - r) = \int T_{00}(t - r, \vec{y})\, y^i\, y^j\, d^3y}$$

$I_{ij} =$ "Quadrupole tensor"

Next: $\bar{h}_{0i}$

$$\partial_0 \bar{h}_{0i} = \partial_j \bar{h}_{ji} = \partial_j \left(\frac{2}{r} \ddot{I}_{ij}(t - r) \right) \qquad \Big|\ \partial_j r = \frac{x^j}{r}$$

$$\Rightarrow \bar{h}_{0i} = \partial_j \left(\frac{2}{r} \dot{I}_{ij}(t - r) \right) + C_i = \underbrace{-2 \frac{\hat{x}_j}{r^2} \dot{I}_{ij}}_{\mathcal{O}(\frac{1}{r^2}) \to 0} - 2 \frac{\hat{x}_j}{r} \ddot{I}_{ij} + C_i$$

Const. of integration: $(*) \Rightarrow C_i = \frac{4}{r} \int T_{0i}(0, \vec{y})\, d^3y =: -\frac{4}{r} P_i$ "Momentum"

$$\partial_0 P_i(t - r) = -\int \partial_0 T_{0i}\, d^3y = \underbrace{-\int \partial_j T_{ji}\, d^3y}_{\text{surface term}} = 0$$

$\Rightarrow$ P_i conserved at leading order

$P_i = C_i = 0$ in ctr. of mass frame

Finally: $\bar{h}_{00}$

$\partial_0 \bar{h}_{00} = \partial_i \bar{h}_{0i}$

$$\Rightarrow \bar{h}_{00} = \partial_i \left(-2\frac{\hat{x}_j}{r} \dot{I}_{ij}(t-r) \right) + C_0 = \frac{2\hat{x}_i \hat{x}_j}{r} \ddot{I}_{ij}(t-r) + C_0 + \mathcal{O}(\tfrac{1}{r^2})$$

$$C_0 = \frac{4}{r} \int T_{00}(0, \vec{y})\, d^3y =: \frac{4}{r} E \qquad \text{“Energy”}$$

$$\partial_0 E(t-r) = \partial_0 \int T_{00}\, d^3y = \underbrace{\int (\partial_i T_{i0})\, d^3y}_{\text{surface term}} = 0$$

$\Rightarrow E$ conserved at 1st order

At higher order: E, P_i not conserved!

8.5 Energy in gravitational waves

Consider 2nd order pert. theory, vacuum

Notation: $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta^{(1)} g_{\mu\nu} + \delta^{(2)} g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + h^{(2)}_{\mu\nu}$

E.g.: $g^{\mu\nu} = \eta^{\mu\nu} + \delta^{(1)} g^{\mu\nu} + \delta^{(2)} g^{\mu\nu}$

$$\Rightarrow g^{\mu\rho} g_{\rho\nu} = \delta^\mu{}_\nu + \underbrace{\left[h^\mu{}_\nu + \delta^{(1)} g^{\mu\rho}\, \eta_{\rho\nu} \right]}_{\sim\epsilon} + \underbrace{\left[\delta^{(2)} g^{\mu\rho}\, \eta_{\rho\nu} + h^{(2)\,\mu}{}_\nu + \delta^{(1)} g^{\mu\rho}\, h_{\rho\nu} \right]}_{\sim\epsilon^2}$$

with $h^{(2)\,\mu}{}_\nu = \eta^{\mu\rho} h^{(2)}_{\rho\nu}$

$$\Rightarrow \delta^{(1)} g^{\mu\nu} = -h^{\mu\rho} =: g^{(1)\,\mu\rho}[h]$$

$$\delta^{(2)} g^{\mu\nu} = -h^{(2)\,\mu\rho} + h^{\mu\sigma} h_\sigma{}^\nu =: \underbrace{g^{(1)\,\mu\nu}[h^{(2)}]}_{\text{linear in } h^{(2)}} + \underbrace{g^{(2)\,\mu\nu}[h]}_{\text{quadratic in } h}$$

Generic pattern in pert. theory: $\delta^{(1)} S^\mu{}_\nu = S^{(1)\,\mu}{}_\nu[h]$

$$\delta^{(2)} S^\mu{}_\nu = \underset{\sim\sim\sim}{S^{(1)\,\mu}{}_\nu[h^{(2)}]} + S^{(2)\,\mu}{}_\nu[h]$$

Einstein equations

$$G_{\mu\nu} = \bar{G}_{\mu\nu} + \delta^{(1)}G_{\mu\nu} + \delta^{(2)}G_{\mu\nu}$$
$$= 0 + G^{(1)}_{\mu\nu}[h] + G^{(1)}_{\mu\nu}[h^{(2)}] + G^{(2)}_{\mu\nu}[h]$$

$$G^{(2)}_{\mu\nu}[h] = R^{(2)}_{\mu\nu}[h] - \frac{1}{2}R^{(1)}[h]\, h_{\mu\nu} - \frac{1}{2}R^{(2)}[h]\,\eta_{\mu\nu}$$

In vacuum: $G^{(1)}_{\mu\nu}[h] = R^{(1)}_{\mu\nu}[h] = 0$ as before

$$\boxed{\begin{aligned} &G^{(1)}_{\mu\nu}[h^{(2)}] = 8\pi t_{\mu\nu}[h] \\ &t_{\mu\nu} = -\frac{1}{8\pi}G^{(2)}_{\mu\nu}[h] = -\frac{1}{8\pi}\big(R^{(2)}_{\mu\nu}[h] - \frac{1}{2}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}[h]\,\eta_{\mu\nu}\big)\end{aligned}}$$

Contracted Bianchi Identities: $g^{\mu\rho}\nabla_\rho G_{\mu\nu} = 0$

at ϵ: $\partial^\mu G^{(1)}_{\mu\nu}[h] = 0 \overset{!}{\Rightarrow} \partial^\mu G^{(1)}_{\mu\nu}[h^{(2)}] = 0$ | Bianchi Ids. true for $\eta_{\mu\nu} + h^{(2)}_{\mu\nu}$!

at ϵ^2: Einstein eqs.: $\bar{G}_{\mu\nu} = 0\,,\quad \delta^{(1)}G_{\mu\nu} = 0$

$$\Rightarrow \ldots \Rightarrow \partial^\mu\big(G^{(2)}_{\mu\nu}[h]\big) = 0$$

$$\Rightarrow \boxed{\partial^\mu t_{\mu\nu} = 0};$$ like energy-momentum tensor!

$\Rightarrow$ regard $t_{\mu\nu}$ as energy momentum of grav. field.

Problem: $t_{\mu\nu}$ gauge dependent

"global solution": integrate over all space $\rightarrow$ ADM mass...

"local approximation": use "large" 4-volume $V \sim a^4$ as follows:

Def.: "average" $\langle X_{\mu\nu}\rangle := \int_V X_{\mu\nu}(x)\, W(x)\, d^4x$

weight $W(x) \geq 0\,,\quad \int_V W\, d^4x = 1\,,\quad W(x) \to 0$ on ∂V smoothly

$$\Rightarrow \langle \partial_\rho X_{\mu\nu}\rangle = \int_V (\partial_\rho X_{\mu\nu})\, W\, d^4x = -\int_V X_{\mu\nu}(\partial_\rho W)\, d^4x$$

Let $X_{\mu\nu}$ oscillate with wavelength λ $\Rightarrow$ $\partial_\rho X_{\mu\nu} \sim \dfrac{X_{\mu\nu}}{\lambda}$

Also: $\partial_\rho W \sim \dfrac{W}{a}\,, \quad a \gg \lambda$

$\Rightarrow \quad \langle \partial_\rho X_{\mu\nu} \rangle \sim \dfrac{X_{\mu\nu}}{a} \ll \dfrac{X_{\mu\nu}}{\lambda} \sim \partial_\rho X_{\mu\nu}$

$\Rightarrow$ neglect total derivs. in $\langle\ .\ \rangle$

$\Rightarrow$ " $\langle A\,\partial B \rangle = \langle \partial(AB) \rangle - \langle (\partial A)\, B \rangle \approx -\langle (\partial A)\, B \rangle$ "

$\Rightarrow \ldots \Rightarrow$ (i) $\langle \eta^{\mu\nu} R^{(2)}_{\mu\nu}[h] \rangle = 0$

(ii) $\langle t_{\mu\nu} \rangle = \dfrac{1}{32\pi} \langle \partial_\mu \bar{h}_{\rho\sigma}\, \partial_\nu \bar{h}^{\rho\sigma} - \dfrac{1}{2} \partial_\mu \bar{h}\, \partial_\nu \bar{h} - 2\partial_\sigma \bar{h}^{\rho\sigma}\, \partial_{(\mu} \bar{h}_{\nu)\rho} \rangle$

(iii) $\langle t_{\mu\nu} \rangle$ is gauge invariant

8.6 The quadrupole formula

Energy flux in gravitational waves: $-\langle t_{0i} \rangle$

consider sphere far from source: $r \gg d\,;\ \ \hat{x}_i = \dfrac{x^i}{r}$

$\Rightarrow$ power $\langle p \rangle = -\displaystyle\int r^2 \langle t_{0i} \rangle \hat{x}^i\, d\Omega\,; \quad d\Omega := \sin\theta\, d\theta\, d\phi$

Lorentz gauge: $\partial^\nu \bar{h}_{\nu\mu} = 0$

$$\Rightarrow \langle t_{0i} \rangle = \frac{1}{32\pi} \Big\langle \partial_0 \bar{h}_{\rho\sigma}\, \partial_i \bar{h}^{\rho\sigma} - \frac{1}{2} \partial_0 \bar{h}\, \partial_i \bar{h} \Big\rangle$$

$$= \frac{1}{32\pi} \Big\langle \underbrace{\partial_0 \bar{h}_{jk}\, \partial_i \bar{h}_{jk}}_{①} - \underbrace{2\partial_0 \bar{h}_{0j}\, \partial_i \bar{h}_{0j}}_{②} + \underbrace{\partial_0 \bar{h}_{00}\, \partial_i \bar{h}_{00}}_{③} - \underbrace{\frac{1}{2} \partial_0 \bar{h}\, \partial_i \bar{h}}_{④} \Big\rangle$$

Take $\bar{h}_{\rho\sigma}$ from Sec. 8.4, order $\mathcal{O}(1/r)$, do some δ_{ij} algebra (cf. [5])

$$\Rightarrow \ldots \Rightarrow \quad \boxed{\langle p \rangle_t = \frac{1}{5} \big\langle \dddot{Q}_{ij} \dddot{Q}_{ij} \big\rangle_{t-r}\,; \quad Q_{ij} := I_{ij} - \frac{1}{3} I_{kk}\, \delta_{ij}}$$

valid for: wave zone $r \gg d\,,\ \ \lambda \gg d\ \ (\Leftrightarrow v \ll c)$

Examples:

1) binary $M_1 = M_2 = M \quad \Rightarrow \ldots \Rightarrow \langle p \rangle \sim \left(\frac{M}{d}\right)^{5/2} \rightarrow$ black holes, neutron stars

2) $\bar{h}_{ij} \sim \frac{M^2}{dr} \sim \mathcal{O}(10^{-21})$ when the signal reaches the earth

9 Differential forms

Consider curve $\lambda(t)$, vector $\frac{d}{dt}$, 1-form $\boldsymbol{\omega}$

$$\Rightarrow \int_\lambda \boldsymbol{\omega} := \int_\lambda \boldsymbol{\omega}\Big(\frac{d}{dt}\Big)\, dt = \int_\lambda \omega_\mu \, dx^\mu$$

Goal: generalize to areas, ...

Note: in 3 dims. $\vec{V} \times \vec{W}$ is an antisymmetric area element

9.1 p-forms

Def.: "p-form" := totally antisymmetric $\binom{0}{p}$ tensor

0-form: function, 1-form: covector

Def.: Let $\boldsymbol{\eta}$ be a p-form, $\boldsymbol{\omega}$ a q-form

$$(\eta \wedge \omega)_{a_1 \ldots a_p b_1 \ldots b_q} := \frac{(p+q)!}{p!\, q!} \eta_{[a_1 \ldots a_p} \omega_{b_1 \ldots b_q]}$$

$$\Leftrightarrow \boldsymbol{\eta} \wedge \boldsymbol{\omega} = \frac{(p+q)!}{p!\, q!} \underset{\substack{\uparrow \\ \text{totally antisymm. operator}}}{\mathcal{A}}[\boldsymbol{\eta} \otimes \boldsymbol{\omega}]$$

e.g. $\eta_a \wedge \omega_b = \eta_a \omega_b - \eta_b \omega_a$

one can show: 1) $\boldsymbol{\eta} \wedge \boldsymbol{\omega} = (-1)^{pq} \boldsymbol{\omega} \wedge \boldsymbol{\eta}\,; \quad \boldsymbol{\eta} \wedge \boldsymbol{\eta} = 0$ if p odd

2) $(\boldsymbol{\eta} \wedge \boldsymbol{\omega}) \wedge \boldsymbol{\chi} = \boldsymbol{\eta} \wedge (\boldsymbol{\omega} \wedge \boldsymbol{\chi})$

Basis: Dual basis $\{\mathbf{f}^\mu\}$ $\Rightarrow$ The set of p-forms

$\mathbf{f}^{\mu_1} \wedge \ldots \wedge \mathbf{f}^{\mu_p} = p!\big(\mathbf{f}^{[\mu_1} \otimes \ldots \otimes \mathbf{f}^{\mu_p]}\big)$ is a basis for p-forms:

$$\boldsymbol{\eta} = \frac{1}{p!}\eta_{\mu_1\ldots\mu_p}\mathbf{f}^{\mu_1} \wedge \ldots \wedge \mathbf{f}^{\mu_p}$$

Def.: "Exterior derivative" of p-form $\boldsymbol{\eta} := p+1$ form

$$\begin{aligned}(\boldsymbol{d\eta})_{\mu_1\ldots\mu_{p+1}} &= (p+1)\,\partial_{[\mu_1}\eta_{\mu_2\ldots\mu_{p+1}]} \\ &= (p+1)\left[\nabla_{[\mu_1}\eta_{\mu_2\ldots\mu_{p+1}]} + \underbrace{\Gamma^\rho_{[\mu_2\mu_1}}_{\text{torsion}=0}\,\eta_{|\rho|\mu_3\ldots\mu_{p+1}]} + \ldots\right]\end{aligned}$$

$\Rightarrow \ldots \Rightarrow$ 1) $\boldsymbol{d}(\boldsymbol{d\eta}) = 0$

2) $\boldsymbol{d}(\boldsymbol{\eta} \wedge \boldsymbol{\omega}) = (\boldsymbol{d\eta}) \wedge \boldsymbol{\omega} + (-1)^p \boldsymbol{\eta} \wedge \boldsymbol{d\omega}$

3) $\boldsymbol{d}(\phi^*\boldsymbol{\eta}) = \phi^*\,\boldsymbol{d\eta}$ "$\boldsymbol{d}$, pullback commute"

Def.: A p-form $\boldsymbol{\eta}$ is "closed" $:\Leftrightarrow$ $\boldsymbol{d\eta} = 0$.

$\boldsymbol{\eta}$ is "exact" $:\Leftrightarrow$ $\exists (p-1)$ form $\boldsymbol{\omega}$: $\boldsymbol{\eta} = \boldsymbol{d\omega}$.

$\boldsymbol{\eta}$ exact $\Rightarrow$ $\boldsymbol{\eta}$ closed

Poincaré Lemma: $\boldsymbol{\eta}$ closed

$$\Rightarrow \quad \forall_{\text{points } r\in\mathcal{M}}\ \exists_{\text{neighbourhood } \mathcal{O} \text{ of } r,\ (p-1) \text{ form } \boldsymbol{\omega}} \ : \quad \boldsymbol{\eta} = \boldsymbol{d\omega} \text{ in } \mathcal{O}$$

9.2 Integration on manifolds

Lemma: Let $\boldsymbol{\omega}$ be a n-form, $\{\mathbf{f}^\mu\}$ basis, $\mathcal{N}$ n-dim. manifold

$\Rightarrow \exists$ func. h : $\boldsymbol{\omega} = h\,\mathbf{f}^1 \wedge \ldots \wedge \mathbf{f}^n$

Def.: "Orientation" of n-dim. manifold $\mathcal{N}$

:= a smooth nowhere vanishing n-form $\boldsymbol{\eta}$.

2 orientations $\boldsymbol{\eta}$, $\boldsymbol{\eta}'$ are "equivalent" $:\Leftrightarrow \exists$ func. $h > 0$: $\boldsymbol{\eta}' = h\boldsymbol{\eta}$

Def.: a coord. chart x^μ on $\mathcal{N}$ is "right-handed" (RH) relative to orientation $\boldsymbol{\eta}$

$:\Leftrightarrow \exists_{h>0}\ \boldsymbol{\eta} = h\,\mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^n$

Def.: "volume form" on $\mathcal{N}$: $\boldsymbol{\epsilon} := \sqrt{|g|}\,\mathbf{f}^1 \wedge \ldots \wedge \mathbf{f}^n$; $g := \det g_{\mu\nu}$

Def.: Let $\psi = x^\mu : \mathcal{O} \subset \mathcal{N} \to \mathbb{R}^n$ be a RH coord. chart, $\boldsymbol{\omega}$ a n-form

$$\boxed{\int_{\mathcal{O}} \boldsymbol{\omega} := \int_{\psi(\mathcal{O})\subset\mathbb{R}^n} \omega_{1\ldots n}\, dx^1 \ldots dx^n}$$

can be shown to chart independent

> 1 chart $\to$ add patches $\mathcal{O}_\alpha$

Example: scalar f : $\displaystyle\int_{\mathcal{O}} f\,\boldsymbol{\epsilon} = \int_{\psi(\mathcal{O})} f\sqrt{|g|}\,dx^1 \ldots dx^n$

Def.: a diffeomorphism $\phi : \mathcal{N} \to \mathcal{N}$ is "orientation preserving"

$:\Leftrightarrow \phi^*(\boldsymbol{\eta})$ is equivalent to $\boldsymbol{\eta}$ $\forall$ orientations $\boldsymbol{\eta}$

$$\Rightarrow \ldots \Rightarrow \int_{\mathcal{N}} \phi^*(\boldsymbol{\omega}) = \int_{\mathcal{N}} \boldsymbol{\omega}$$

9.3 Submanifolds, Stokes' theorem

Let $\mathcal{M}$, $\mathcal{N}$ be orientable manifolds of dim. $m < n$

Def.: "embedding": $\phi : \mathcal{M} \to \mathcal{N}$, ϕ smooth, 1-to-1 and

$\forall_{p\in\mathcal{M}}\ \exists_{\text{nbhhd. } \mathcal{O}}$: $\phi^{-1} : \phi[\mathcal{O}] \to \mathcal{M}$ is smooth.

$m = n - 1 \quad \Rightarrow \quad \phi[\mathcal{M}]$ is a "hypersurface"

Def.: Let $\phi[\mathcal{M}]$ be m-dim., $\boldsymbol{\eta}$ a m-form on $\mathcal{N}$

$$\Rightarrow \int_{\phi[\mathcal{M}]} \boldsymbol{\eta} := \int_{\mathcal{M}} \phi^*(\boldsymbol{\eta})\,; \qquad \boldsymbol{\eta} = \boldsymbol{d\omega} \;\;\Rightarrow\;\; \int_{\phi[\mathcal{M}]} \boldsymbol{d\omega} = \int_{\mathcal{M}} \boldsymbol{d}(\phi^*\boldsymbol{\omega}) \qquad (*)$$

Def.: $\frac{1}{2}\mathbb{R}^n := \left\{(x^1,\ldots,x^n) \in \mathbb{R}^n \middle|\, x^1 \le 0\right\}$

$\mathcal{N}$ = "manifold with boundary": like manifold, but charts $\psi_\alpha : \mathcal{O}_\alpha \to \frac{1}{2}\mathbb{R}^n$

"boundary" $:= \partial\mathcal{N} := \left\{p \in \mathcal{N} \;\middle|\; x^1(p) = 0\right\}$ is $n-1$ dim.

$(x^2,\ldots,x^n)$ is right-handed on $\partial\mathcal{N}$ $\;:\Leftrightarrow\;$ $(x^1,\ldots,x^n)$ is RH on $\mathcal{N}$

Stokes' Theorem:

For a n-dim. orientable mfld. $\mathcal{N}$ with boundary $\partial\mathcal{N}$ and $(n-1)$-form $\boldsymbol{\eta}$

$$\boxed{\int_{\mathcal{N}} \boldsymbol{d\eta} = \int_{\partial\mathcal{N}} \boldsymbol{\eta}}$$

where the rhs. is defined through $(*)$ with $\phi : \partial\mathcal{N} \to \mathcal{N}\,,\;\; p \mapsto p$ $\qquad (**)$

Def.: a) $\boldsymbol{X} \in \mathcal{T}_p(\mathcal{N})$ is "tangent to $\phi[\mathcal{M}]$

$:\Leftrightarrow$ $\exists$ curve in $\phi[\mathcal{M}]$ with tangent $\boldsymbol{X}$

b) $\tilde{\boldsymbol{n}} \in \mathcal{T}^*_p(\mathcal{N})$ is "normal" to $\phi[\mathcal{M}]$

$:\Leftrightarrow \tilde{\boldsymbol{n}}(\boldsymbol{X}) = 0 \;\; \forall \boldsymbol{X}$ tangent to $\phi[\mathcal{M}]$

Def.: Let Σ be a hypersurface of a Lorentzian mfld., $\tilde{\boldsymbol{n}}$ its normal field.

Σ is "timelike" ("spacelike", "null") $:\Leftrightarrow$ $\tilde{\boldsymbol{n}}$ is spacelike (timelike, null)

On $\partial\mathcal{N}$: $x^1 = 0 \;\Rightarrow\;$ $\mathbf{d}x^1$ is *outgoing* normal to $\partial\mathcal{N}$

$$\Rightarrow \;\; \tilde{\boldsymbol{n}} = \frac{\mathbf{d}x^1}{\sqrt{\pm \boldsymbol{g}(\mathbf{d}x^1, \mathbf{d}x^1)}} = \text{unit normal}$$

Divergence Theorem:

Let $\partial\mathcal{N}$ be time or spacelike, $\boldsymbol{X}$ a VF on $\mathcal{N}$, $h_{\mu\nu} := \phi^* g_{\mu\nu}$, ϕ as in $(**)$

$$\Rightarrow \boxed{\int_{\mathcal{N}} \nabla_a X^a \sqrt{|g|}\, d^n x = \int_{\partial\mathcal{N}} \tilde{n}_a X^a \sqrt{|h|}\, d^{n-1} x}$$

10 The initial value problem

10.1 Extrinsic curvature

Let $\mathcal{N}$ be a manifold, Σ a hypersurface, $\boldsymbol{g}$ the metric (Riemannian or Lorentzian)

Unit normal to σ : $n_a n^a = \mp 1$; upper sign: $\boldsymbol{n}$ timelike

lower sign: $\boldsymbol{n}$ spacelike

Def.: "Projector" $\perp^a{}_b := \delta^a{}_b \pm n^a n_b$

Projection of tensor: $\perp T^{ab\ldots}{}_{cd\ldots} = \perp^a{}_e \perp^b{}_f \ldots \perp^g{}_c \perp^h{}_d \ldots T^{ef\ldots}{}_{gh\ldots}$

$\Rightarrow$ 1) $\perp^a{}_b n^b = 0\,,\quad \perp^a{}_c \perp^c{}_b = \perp^a{}_b$

2) $\forall \boldsymbol{X} \in \mathcal{T}_p(\mathcal{N})$: $\perp^a{}_b X^b$ tangent to Σ, $X^a = \perp^a{}_b X^b \mp n^a n_b X^b$

3) $\boldsymbol{X}, \boldsymbol{Y}$ tangent to $\Sigma \Rightarrow g_{ab} X^a Y^b = \perp_{ab} X^a Y^b$

$\Rightarrow \perp_{ab} =$ induced metric on Σ,

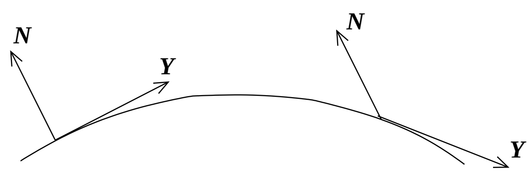
We write $\gamma_{ab} = \perp_{ab}$ "1st fundamental form"

Let $\boldsymbol{X}, \boldsymbol{Y}$ be tangent VFs to Σ, $\boldsymbol{N}$ normal VF

par. transport $\boldsymbol{N}$ along int. curve of $\boldsymbol{X}$: $X^b \nabla_b N^a = 0$

Does $\boldsymbol{N}$ remain normal to Σ? No!

$X^b \nabla_b (Y^a N_a) = N_a X^b \nabla_b Y^a$



Def.: Extend unit normal $\boldsymbol{n}$ in nbhd. of Σ with $n^a n_a = \mp 1$

"extrinsic curvature" $:= \boldsymbol{K} : \mathcal{T}_p(\mathcal{N}) \times \mathcal{T}_p(\mathcal{N}) \to \mathbb{R}\,,\quad \boldsymbol{X}, \boldsymbol{Y} \mapsto n_a \big(\nabla_{\perp \boldsymbol{X}} (\perp \boldsymbol{Y})\big)^a$

Note: sign convention

Lemma: $K_{ab} = -\perp^c{}_a \perp^d{}_b \nabla_c n_d$ indep. of extension

Proof: 1) $K_{ab}X^aY^b = n_a(\perp X)^c\nabla_c(\perp Y)^a = -(\perp X)^c(\perp Y)^a\nabla_c n_a \quad \Big| \quad n_a(\perp Y)^a = 0$

$$= -\perp^c{}_b X^b \perp^a{}_d Y^d \nabla_c n_a$$

$$\Rightarrow K_{bd} = -\perp^c{}_b \perp^a{}_d \nabla_c n_a$$

2) n'_a another extension $\rightarrow \quad m_a = n'_a - n_a = 0$ on Σ

$$\Rightarrow \text{On } \Sigma : X^aY^b(K_{ab} - K'_{ab}) = \perp^c{}_a \perp^d{}_b X^aY^b\nabla_c m_d$$

$$= (\perp X)^c\big[(\perp Y)^d\nabla_c m_d + \underbrace{m_d}_{=0}\nabla_c(\perp Y)^d\big]$$

$$= (\perp X)^c\nabla_c\big(m_d(\perp Y)^d\big) = 0 \quad \Big| \quad \text{deriv. inside } \Sigma$$

Comment: $n^b\nabla_c n_b = \dfrac{1}{2}\nabla_c(n_b n^b) = 0$

$$\Rightarrow K_{ab} = -\perp^c{}_a \perp^d{}_b \nabla_c n_d = -\perp^c{}_a(\delta^d{}_b \pm n^d n_b)\nabla_c n_d = -\perp^c{}_a \nabla_c n_b$$

<u>**Def.:**</u> Let $t : \mathcal{N} \to \mathbb{R}$ with $t = \text{const}$ and normal $\mathbf{d}t \neq 0$ on Σ

$$\Rightarrow \text{unit normal } \boldsymbol{n} = \mp\alpha\mathbf{d}t\,, \quad \alpha := \frac{1}{\sqrt{\mp \boldsymbol{g}^{-1}(\mathbf{d}t, \mathbf{d}t)}} = \text{“Lapse function”}$$

↑

n future pointing if timelike

<u>**Lemma:**</u> $K_{ab} = K_{ba}$

Proof: $\nabla_c n_d = \mp\nabla_c(\alpha\,\mathbf{d}t_d) = \mp\alpha\nabla_c\nabla_d t + (\nabla_c\alpha)\dfrac{n_d}{\alpha}$

$$\Rightarrow K_{ab} = +\perp^c{}_a\perp^d{}_b\,\alpha\,\nabla_c\nabla_d t + 0 \quad \text{is symmetric (torsion} = 0)$$

<u>**Def.:**</u> $K := K^b{}_b = g^{ab}K_{ab}$

10.2 The Gauss-Codazzi equations

Def.: Covariant deriv. D_a on Σ:

$$D_a T^{b_1 b_2 \dots}{}_{c_1 c_2 \dots} := \perp^d{}_a \perp^{b_1}{}_{e_1} \perp^{b_2}{}_{e_2} \dots \perp^{f_1}{}_{c_1} \perp^{f_2}{}_{c_2} \dots \nabla_d T^{e_1 e_2 \dots}{}_{f_1 f_2 \dots}$$

$\Rightarrow \dots \Rightarrow D$ is torsion free and Levi-Civita conn. of γ_{ab} on Σ if ∇ is that of g_{ab} on $\mathcal{N}$

$$D_a \gamma_{bc} = 0$$

D defines the Riemann tensor of γ_{ab}: $\mathcal{R}^a{}_{bcd}$

One can calculate the projections of $R^a{}_{bcd}$ from the Ricci Identity:

Gauss eq.:	$\perp R^a{}_{bcd} = \mathcal{R}^a{}_{bcd} \pm 2K^a{}_{[c} K_{d]b}$
Contracted Gauss:	$\perp R_{ab} \pm \perp^c{}_a n^d \perp^e{}_b n^f R_{cdef} = \mathcal{R}_{ab} \pm K K_{ab} \mp K_{ac} K^c{}_b$
Scalar Gauss:	$R \pm 2R_{cd} n^c n^d = \mathcal{R} \pm K^2 \mp K_{cd} K^{cd}$
Codazzi eq.:	$\perp^d{}_a \perp^e{}_b \perp^f{}_c n^g R_{defg} = -D_a K_{bc} + D_b K_{ac}$
Contr. Codazzi:	$\perp^c{}_b R_{cd} n^d = -D_a K^a{}_b + D_b K$

10.3 The constraint equations

From now on $\boldsymbol{n}$ timelike, "upper sign"

Project Einstein eqs.: $G_{ab} = 8\pi T_{ab}$

1) EM tensor: $\rho := T_{ab} n^a n^b\,, \quad j_a := -\perp^b{}_a T_{bc} n^c\,, \quad S_{ab} := \perp T_{ab}$

$$\Rightarrow \quad T_{ab} = \rho n_a n_b + j_a n_b + j_b n_a + S_{ab}\,, \quad T = T^b{}_b = -\rho + S$$

2) n-n proj.: $R_{ab} n^a n^b + \dfrac{1}{2} R = 8\pi\rho \quad \Big| \quad \leftarrow$ scalar Gauss

$$\Rightarrow \boxed{\mathcal{R} - K_{cd} K^{cd} + K^2 - 16\pi\rho = 0} \quad \text{“Hamiltonian constraint”}$$

3) n-$\perp$ proj.: $\perp^b{}_a n^c \left(R_{bc} - \dfrac{1}{2} g_{bc} R \right) = \perp^b{}_a n^c R_{bc} = -8\pi j_a \quad \Big| \quad \leftarrow$ contr. Codazzi

$$\Rightarrow \boxed{D_c K^c{}_a - D_a K - 8\pi j_a = 0} \quad \text{“momentum constraint”}$$

10.4 Foliations

Def.: "Cauchy surface" := spacelike hypersurface Σ in $\mathcal{N}$ such that

each timelike or null curve without endpoint intersects Σ exactly once.

$(\mathcal{N}, \boldsymbol{g})$ is "globally hyperbolic" $:\Leftrightarrow$ it admits a Cauchy surface

From now on: Let $(\mathcal{N}, \boldsymbol{g})$ be globally hyperbolic.

$\Rightarrow \ldots \Rightarrow$ $\exists$ smooth $\hat{t} : \mathcal{N} \to \mathbb{R}$, $\mathbf{d}\hat{t} \neq 0$ everywhere and hypersurfaces Σ are level

surfaces $\hat{t} = \text{const}$: $\forall_{t\in\mathbb{R}}$ $\Sigma_t = \{p \in \mathcal{N} : \hat{t}(p) = t\}$, $\Sigma_t \cap \Sigma_{t'} = \emptyset \Leftrightarrow t \neq t'$

We assume: Σ_t spacelike, $\mathcal{N} = \bigcup_{t\in\mathbb{R}} \Sigma_t$; this is called a "foliation" of $\mathcal{N}$.

From now on: use just t (no $\hat{t}$)

Def.: $\boldsymbol{m} := \alpha \boldsymbol{n}$ "normal evolution vector"

Note: $\boldsymbol{n} = -\alpha \mathbf{d}t$, $\boldsymbol{n} \cdot \boldsymbol{n} = -1$

$$\Rightarrow \quad \boldsymbol{m} \cdot \boldsymbol{m} = -\alpha^2 , \quad \langle \mathbf{d}t, \boldsymbol{m} \rangle = -\frac{1}{\alpha} \langle \boldsymbol{n}, \boldsymbol{m} \rangle = -\langle \boldsymbol{n}, \boldsymbol{n} \rangle = 1$$

$$\Rightarrow \quad \mathcal{L}_{\boldsymbol{m}} t = \boldsymbol{m}(t) = \langle \mathbf{d}t, \boldsymbol{m} \rangle = 1$$

$\Rightarrow$ Proper time along int. curve of $\boldsymbol{m}$ (cf. Sec. 3.2):

$$\tau = \int_{t_0}^{t} \sqrt{-\boldsymbol{g}(\boldsymbol{m}, \boldsymbol{m})}\, d\tilde{t} \quad \Rightarrow \quad \frac{d\tau}{dt} = \sqrt{-\boldsymbol{g}(\boldsymbol{m}, \boldsymbol{m})} = \alpha$$

Def.: "acceleration" $a_b := n^c \nabla_c n_b$

Lemma: $a_b = D_b \ln \alpha$

Recall: $K_{ab} = -\perp^c{}_a \nabla_c n_b = -\nabla_a n_b - n_a n^c \nabla_c n_b$

$$\Rightarrow \boxed{\nabla_a n_b = -K_{ab} - n_a a_b = -K_{ab} - n_a D_b \ln \alpha}$$

$$\Rightarrow \nabla_a m_b = \nabla_a (\alpha\, n_b) = n_b \nabla_a \alpha + \alpha \nabla_a n_b$$

$$\Rightarrow \boxed{\nabla_a m_b = n_b \nabla_a \alpha - \alpha K_{ab} - n_a\, D_b \alpha}$$

Lemma: 1) $\mathcal{L}_{\boldsymbol{m}}\gamma_{ab} = -2\alpha K_{ab}$

2) $\mathcal{L}_{\boldsymbol{n}}\gamma_{ab} = -2K_{ab}$

3) $\mathcal{L}_{\boldsymbol{m}}\gamma^a{}_b = \mathcal{L}_{\boldsymbol{m}}\perp^a{}_b = 0$

4) $\mathcal{L}_{\boldsymbol{n}}\gamma^a{}_b = n^a D_b \ln\alpha$

Corollary: Let $\boldsymbol{T}$ be a tangent tensor: $\perp\boldsymbol{T} = \boldsymbol{T}$

$$\Rightarrow \mathcal{L}_{\boldsymbol{m}}\boldsymbol{T} = \mathcal{L}_{\boldsymbol{m}}(\perp\boldsymbol{T}) = (\mathcal{L}_{\boldsymbol{m}}\perp)\boldsymbol{T} + \perp\mathcal{L}_{\boldsymbol{m}}\boldsymbol{T} \qquad \Big| \quad \mathcal{L}_{\boldsymbol{m}}\perp^a{}_b = \mathcal{L}_{\boldsymbol{m}}\gamma^a{}_b = 0$$

$\Rightarrow \mathcal{L}_{\boldsymbol{m}}\boldsymbol{T}$ is tangent to Σ

With these tools we can calculate the final projection: $\perp^e{}_a n^f \perp^g{}_b n^h R_{efgh}$

Starting point: Ricci Identity; cf. Sec.3.4.1 in [2]

$$\Rightarrow \ldots \Rightarrow \quad \perp^e{}_a\perp^g{}_b n^h R_{efgh} n^f = \frac{1}{\alpha}\mathcal{L}_{\boldsymbol{m}}K_{ab} + K_{ac}K^c{}_b + \frac{1}{\alpha}D_a D_b\alpha$$

with contracted Gauss eq.: $$\boxed{\perp R_{ab} = -\frac{1}{\alpha}\mathcal{L}_{\boldsymbol{m}}K_{ab} - \frac{1}{\alpha}D_a D_b\alpha + \mathcal{R}_{ab} + K\,K_{ab} - 2K_{ac}K^c{}_b} \qquad (*)$$

$\cdot\perp^{ab}$, use scalar Gauss: $$\boxed{R = \frac{2}{\alpha}\mathcal{L}_{\boldsymbol{m}}K - \frac{2}{\alpha}D_c D^c\alpha + \mathcal{R} + K^2 + K_{cd}K^{cd}}$$

10.5 The 3+1 equations

Einstein eqs.: $R_{ab} - \frac{1}{2}g_{ab}R = 8\pi T_{ab} \quad \Rightarrow \quad -R = 8\pi T$

$$\Rightarrow R_{ab} = 8\pi\left(T_{ab} - \frac{1}{2}g_{ab}T\right) \qquad \Big| \quad \perp\cdot$$

$$\Rightarrow \perp R_{ab} = 4\pi\big(2S_{ab} + (\rho - S)\gamma_{ab}\big)$$

$$\overset{(*)}{\Rightarrow} \quad \mathcal{L}_{\boldsymbol{m}}K_{ab} = -D_a D_b\alpha + \alpha\Big\{\mathcal{R}_{ab} + K\,K_{ab} - 2K_{ac}K^c{}_b + 4\pi\big[(S-\rho) - 2S_{ab}\big]\Big\}$$

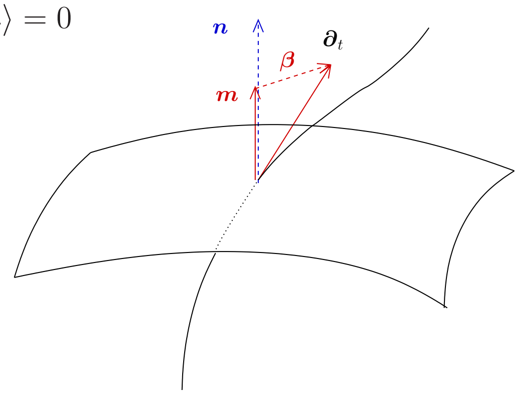
Open question: Relate $\mathcal{L}_{\boldsymbol{m}}$ to a time derivative $\dfrac{\partial}{\partial t}$

Adapted coordinates: $x^\alpha = (t, x^i)\,,\quad i = 1,\,2,\,3\,,\quad x^i$ label points in Σ_t

$\rightarrow$ basis $\boldsymbol{\partial}_t\,,\ \boldsymbol{\partial}_i\,;$ dual basis $\mathbf{d}t\,,\ \mathbf{d}x^i$

Integral curves of the $\boldsymbol{\partial}_i$ have $t = \text{const}$, i.e. are in Σ_t

What about $\boldsymbol{\partial}_t$? Clearly $\langle \mathbf{d}t, \boldsymbol{\partial}_t \rangle = 1 = \langle \mathbf{d}t, \boldsymbol{m} \rangle \quad\Rightarrow\quad \langle \mathbf{d}t, \boldsymbol{\partial}_t - \boldsymbol{m} \rangle = 0$



<u>Def.:</u> "shift vector" $\boldsymbol{\beta} := \boldsymbol{\partial}_t - \boldsymbol{m} \quad\Rightarrow\quad \langle \mathbf{d}t, \boldsymbol{\beta} \rangle = 0$

$\Rightarrow \boldsymbol{\partial}_t = \alpha \boldsymbol{n} + \boldsymbol{\beta}$

Curves $x^i = \text{const}$ are in general not normal to Σ_t.

$\boldsymbol{\beta}$ measures this deviation.

Metric components: $g_{00} = \boldsymbol{g}(\boldsymbol{\partial}_t, \boldsymbol{\partial}_t) = \ldots = -\alpha^2 + \boldsymbol{\beta}\cdot\boldsymbol{\beta}$ etc.

$$\Rightarrow \ldots \Rightarrow \quad g_{\alpha\beta} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \quad\Leftrightarrow\quad g^{\alpha\beta} = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^j \\ \alpha^{-2}\beta^i & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{pmatrix}$$

$$\det g_{\alpha\beta} = -\alpha^2 \det \gamma_{ij} \quad\Rightarrow\quad \sqrt{-g} = \alpha\sqrt{\gamma}$$

In adapted coords.: The 3+1 eqs. contain only tensors tangent to Σ_t

$\Rightarrow$ we can ignore time components

$\Rightarrow$ substitute $i,\ j,\ \ldots = 1,\ 2,\ 3$ for abstract indices

We have: $$\mathcal{L}_{\boldsymbol{m}}\gamma_{ij} = \mathcal{L}_{\boldsymbol{\partial}_t}\gamma_{ij} - \mathcal{L}_{\boldsymbol{\beta}}\gamma_{ij} = \frac{\partial}{\partial t}\gamma_{ij} - \beta^m\partial_m\gamma_{ij} - \gamma_{mj}\partial_i\beta^m - \gamma_{im}\partial_j\beta^m$$

$$\mathcal{L}_{\boldsymbol{m}}K_{ij} = \frac{\partial}{\partial t}K_{ij} - \beta^m\partial_m K_{ij} - K_{mj}\partial_i\beta^m - K_{im}\partial_j\beta^m$$

$$\Rightarrow \boxed{\begin{aligned} &\partial_t\gamma_{ij} = \mathcal{L}_{\boldsymbol{\beta}}\gamma_{ij} - 2\alpha K_{ij} \\ &\partial_t K_{ij} = \mathcal{L}_{\boldsymbol{\beta}}K_{ij} - D_iD_j\alpha + \alpha\Big\{\mathcal{R}_{ij} + K\,K_{ij} - 2K_{im}K^m{}_j + 4\pi\big[(S-\rho)\gamma_{ij} - 2S_{ij}\big]\Big\} \\ &\mathcal{R} + K^2 - K_{mn}K^{mn} - 16\pi\rho = 0 \\ &D_mK^m{}_i - D_iK - 8\pi j_i = 0 \end{aligned}}$$

Comments: 1) $\alpha,\ \beta^i$ freely specifiable! $\rightarrow$ gauge freedom

2) Bianchi Identities $\Rightarrow \ldots \Rightarrow$ constraints preserved under evolution

3) numerical relativity $\rightarrow$ need new variables

11 The Lagrangian formulation

Consider scalar field in curved spacetime: $S = \int_{\mathcal{M}} \left[-\frac{1}{2} g^{\alpha\beta} \nabla_\alpha \Phi \, \nabla_\beta \Phi - V(\Phi)\right] \sqrt{-g}\, d^4x$

Vary with respect to Φ; assume $\delta\Phi$ vanishes on $\partial\mathcal{M}$; use divergence theorem

$$\Rightarrow \delta S = S[\Phi + \delta\Phi] - S[\Phi]$$

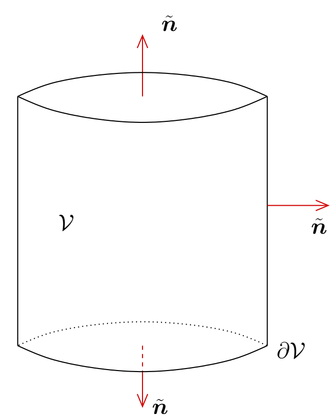
$$= \int_{\mathcal{M}} \big(- g^{\alpha\beta} \nabla_\alpha \Phi \, \nabla_\beta \delta\Phi - V'(\Phi)\, \delta\Phi \big) \sqrt{-g}\, d^4x$$

$$\int_{\mathcal{M}} \big[- \nabla_\alpha (\delta\Phi \, \nabla^\alpha \Phi) + \delta\Phi \, \nabla^\alpha \nabla_\alpha \Phi - V'(\Phi) \delta\Phi \big] \sqrt{-g}\, d^4x$$

$$\underbrace{\int_{\partial\mathcal{M}} -\delta\Phi \, \tilde{n}_\alpha \nabla^\alpha \Phi \sqrt{|h|} d^3x}_{=0} + \int_{\mathcal{M}} \big(\nabla^\alpha \nabla_\alpha \Phi - V'(\Phi) \big) \delta\Phi \, \sqrt{-g}\, d^4x$$

$\Rightarrow \nabla^\alpha \nabla_\alpha \Phi - V'(\Phi) = 0$ "Eqs. of motion"

Goal: same for GR



Sign convention: 1) Unit normal: $\tilde{\boldsymbol{n}}$ always outward; past, future, spacelike!

2) Extrinsic curvature: $\tilde{K}_{ab} = +\perp \nabla_a \tilde{n}_b$

Saves us case distinctions on spacelike boundaries.

The action in GR: $S_{\rm GR}[g, \phi] = \frac{1}{16\pi} \big(I_H[g] + I_B[g] - I_0 \big) + S_M[\phi, g]$

1) Hilbert term: $I_H = \int_{\mathcal{V}} R \sqrt{-g}\, d^4x$

2) boundary term: $I_B = 2 \oint_{\partial\mathcal{V}} \tilde{K} \sqrt{|\gamma|} d^3y$

3) constant term: $I_0 = 2 \oint_{\partial\mathcal{V}} \tilde{K}_0 \sqrt{|\gamma|}\, d^3y$

4) matter term: $S_M = \int_{\mathcal{V}} L(\phi, \phi_{,\alpha}; g_{\alpha\beta}) \sqrt{-g}\, d^4x$

It is convenient to vary $g^{\alpha\beta}$ instead of $g_{\alpha\beta}$: $g^{\alpha\mu} g_{\mu\beta} = \delta^\alpha{}_\beta \quad \Rightarrow \quad \delta g_{\alpha\beta} = -g_{\alpha\mu} g_{\beta\nu} \delta g^{\mu\nu}$

Lemma: $\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\alpha\beta}\, \delta g^{\alpha\beta}$

1) $$\delta I_H = \int_{\mathcal{V}} \delta\big(g^{\alpha\beta} R_{\alpha\beta}\sqrt{-g}\big)\, d^4x$$
$$= \int_{\mathcal{V}} R_{\alpha\beta}\sqrt{-g}\,\delta g^{\alpha\beta} + g^{\alpha\beta}\sqrt{-g}\,\delta R_{\alpha\beta} + R\,\delta\sqrt{-g}\, d^4x$$
$$= \int_{\mathcal{V}} \underbrace{\left(R_{\alpha\beta} - \frac{1}{2}R g_{\alpha\beta}\right)}_{\text{Einstein eqs.}} \delta g^{\alpha\beta}\sqrt{-g}\, d^4x + \underbrace{\int_{\mathcal{V}} g^{\alpha\beta}\,\delta R_{\alpha\beta}\,\sqrt{-g}\, d^4x}_{?}$$

In normal coords.: $$\delta R_{\alpha\beta} \overset{*}{=} \delta\big(\Gamma^{\mu}_{\alpha\beta,\mu} - \Gamma^{\mu}_{\alpha\mu,\beta}\big) \quad \Big|\ \Gamma = 0$$
$$\overset{*}{=} \delta\Gamma^{\mu}_{\alpha\beta,\mu} - \delta\Gamma^{\mu}_{\alpha\mu,\beta}$$
$$\overset{*}{=} \delta\Gamma^{\mu}_{\alpha\beta;\mu} - \delta\Gamma^{\mu}_{\alpha\mu;\beta} \quad \Big|\ \Gamma = 0\,,\quad \delta\Gamma = \text{tensor!}$$

tensorial eq. $\Rightarrow$ valid in any coords.!

$$\Rightarrow \int_{\mathcal{V}} g^{\alpha\beta}\,\delta R_{\alpha\beta}\,\sqrt{-g}\, d^4x = \int X^{\mu}{}_{;\mu}\sqrt{-g}\, d^4x\,; \quad X^{\mu} := g^{\alpha\beta}\,\delta\Gamma^{\mu}_{\alpha\beta} - g^{\alpha\mu}\,\delta\Gamma^{\beta}_{\alpha\beta}$$
$$= \oint_{\partial\mathcal{V}} X^{\mu}\tilde{n}_{\mu}\sqrt{|\gamma|}\, d^3y \quad \Big|\ \text{Divergence theorem}$$

On $\partial\mathcal{V}$: $\delta g_{\alpha\beta} = 0 = \delta g^{\alpha\beta}$

$$\Rightarrow \delta\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\big(\delta g_{\nu\alpha,\beta} + \delta g_{\nu\beta,\alpha} - \delta g_{\alpha\beta,\nu}\big)$$
$$\Rightarrow \ldots \Rightarrow X^{\mu} = g^{\mu\nu}\,\underbrace{g^{\alpha\beta}\big(\delta g_{\nu\alpha,\beta} - \delta g_{\alpha\beta,\nu}\big)}_{=X_\nu}$$
$$\Rightarrow \tilde{n}^{\mu} X_{\mu} = \tilde{n}^{\mu}(\gamma^{\alpha\beta} \mp \tilde{n}^{\alpha}\tilde{n}^{\beta})\big(\underbrace{\delta g_{\mu\beta,\alpha} - \delta g_{\alpha\beta,\mu}}_{\text{antisymm. in } \alpha,\mu}\big)$$
$$= \tilde{n}^{\mu}\,\underbrace{\gamma^{\alpha\beta}\big(\delta g_{\mu\beta,\alpha}}\,-\delta g_{\alpha\beta,\mu}\big)$$
$$= \text{deriv. of } \delta g_{\mu\beta} \text{ tangent to } \partial\mathcal{V} \quad \to 0$$

$$\Rightarrow \delta I_H = \int_{\mathcal{V}} G_{\alpha\beta}\,\delta g^{\alpha\beta}\,\sqrt{-g}\, d^4x - \oint_{\partial\mathcal{V}} \gamma^{\alpha\beta}\,\delta g_{\alpha\beta,\mu}\,\tilde{n}^{\mu}\,\sqrt{|\gamma|}\, d^3y \qquad (*)$$

2) $\tilde{K} = \gamma^{\alpha\beta}\tilde{K}_{\alpha\beta} = \gamma^{\alpha\beta}\nabla_\alpha \tilde{n}_\beta = \gamma^{\alpha\beta}(\partial_\alpha \tilde{n}_\beta - \Gamma^\mu_{\alpha\beta}\tilde{n}_\mu)$

$$\Rightarrow \delta\tilde{K} = -\gamma^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta}\,\tilde{n}_\mu = -\gamma^{\alpha\beta}\,\delta\Gamma_{\mu\alpha\beta}\tilde{n}^\mu$$

$$= -\frac{1}{2}\gamma^{\alpha\beta}\big(\delta g_{\mu\alpha,\beta} + \delta g_{\mu\beta,\alpha} - \delta g_{\alpha\beta,\mu}\big)\tilde{n}^\mu$$

$$= \frac{1}{2}\gamma^{\alpha\beta}\,\delta g_{\alpha\beta,\mu}\tilde{n}^\mu \qquad \Big|\ \text{tang. derivs of } g_{\alpha\beta} \text{ vanish on } \partial\mathcal{V}$$

$$\Rightarrow \delta I_B = \oint_{\partial\mathcal{V}} \gamma^{\alpha\beta}\,\delta g_{\alpha\beta,\mu}\tilde{n}^\mu|\gamma|^{1/2}\,d^3y \quad \text{cancels term in } (*)$$

3) I_0 depends on $g_{\alpha\beta}$ only through $\sqrt{|\gamma|}$

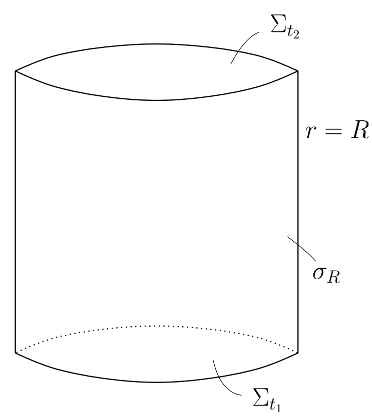
$\Rightarrow \delta I_0 = 0$ on $\partial\mathcal{V}$

$\Rightarrow$ no effect on eqs. of motion, but on numerical value of S_{GR}

Let $g_{\alpha\beta}$ be a solution of the vacuum eqs. $R_{\alpha\beta} = 0 \quad \Rightarrow \quad R = 0$

$$\Rightarrow S_{\mathrm{GR}} + \frac{1}{16\pi}I_0 = \frac{1}{16\pi}I_B = \frac{1}{8\pi}\oint K\,|\gamma|^{1/2}\,d^3y$$

evaluate on closed 3-cylinder for a flat spacetime:



on $\Sigma_{t_1}, \Sigma_{t_2}$: $\quad \tilde{K} = 0$

at $r = R$: $\quad \tilde{K} = \tilde{n}^\alpha{}_{;\alpha} = \ldots = \dfrac{2}{R}, \quad |\gamma|^{1/2} = R^2 \sin\theta$

$$\Rightarrow \oint_{\partial\mathcal{V}} \tilde{K}\,|\gamma|^{1/2}\,d^3y = 8\pi R(t_2 - t_1) \ \text{ diverges as } R \to \infty$$

This divergence persists in curved spacetimes

$\Rightarrow$ cured by I_0 with K_0 = curvature of $\partial\mathcal{V}$ embedded in flat spacetime

4) $\delta S_M = \displaystyle\int_{\mathcal{V}} \frac{\partial L}{\partial g^{\alpha\beta}}\,\delta g^{\alpha\beta}\,\sqrt{-g} + L\,\delta\sqrt{-g}\,d^4x = \int_{\mathcal{V}} \left(\frac{\partial L}{\partial g^{\alpha\beta}} - \frac{1}{2}Lg_{\alpha\beta}\right)\,\delta g^{\alpha\beta}\,\sqrt{-g}\,d^4x$

Def.: $T_{\alpha\beta} := -2\dfrac{\partial L}{\partial g^{\alpha\beta}} + L\,g_{\alpha\beta}$ Energy-momentum tensor

$$\Rightarrow \delta S_M = -\frac{1}{2}\int_{\mathcal{V}} T_{\alpha\beta}\,\delta g^{\alpha\beta}\,\sqrt{-g}\,d^4x$$

Conclusion: $\delta\left[\dfrac{1}{16\pi}\big(I_H + I_B - I_0\big) + S_M\right] = 0 \quad \Rightarrow \quad G_{\alpha\beta} = 8\pi T_{\alpha\beta}$

References

[1] S. M. Carroll. Lecture notes on general relativity, 1997. gr-qc/9712019.

[2] E. Gourgoulhon. 3+1 Formalism and Bases of Numerical Relativity. 2007. gr-qc/0703035.

[3] S. W. Hawking and G. F. R. Ellis. The Large Scale Structure of Space-Time. Cambridge University Press, 1973.

[4] J. Stewart. Advanced general relativity. Cambridge University Press, 1991.

[5] Harvey Reall's lecture notes on General Relativity: `http://www.damtp.cam.ac.uk/user/hsr1000/teaching.html`.

[6] Eric Poisson's Lecture Notes: `http://www.physics.uoguelph.ca/poisson/research/notes.html`.