Mathematical Tripos Part III Dr U. Sperhake

GENERAL RELATIVITY: Examples 3 Michaelmas 2015

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1. Let $\phi : \mathcal{M} \to \mathcal{N}$ be a diffeomorphism. Let ∇ be a covariant derivative on $\mathcal{M}$. The push-forward of ∇ is a covariant derivative $\tilde{\nabla}$ on $\mathcal{N}$ defined by

$$\tilde{\nabla}_X T = \phi_* \left(\nabla_{\phi^*(X)} \left(\phi^*(T) \right) \right) ,$$

where X is a vector field and T a tensor field on $\mathcal{N}$. (In words: pull-back X and T to $\mathcal{M}$, evaluate the covariant derivative there and push-forward te result to $\mathcal{N}$.)

(a) Check that this satisfies the properties of a covariant derivative. (b) Show that the Riemann tensor of $\tilde{\nabla}$ is the push-forward of the Riemann tensor of ∇. (c) Let ∇ be the Levi-Civita connection defined by a metric g on $\mathcal{M}$. Show that $\tilde{\nabla}$ is the Levi-Civita connection defined by the metric $\phi_*(g)$ on $\mathcal{N}$.

2. (a) Use the Leibniz rule to derive the formula for the Lie derivative of a covector ω along the vector X valid in any coordinate basis:

$$(\mathcal{L}_X \omega)_\mu = X^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu \, .$$

(Hint: consider $(\mathcal{L}_X \omega)(Y)$ for a vector field Y.)

(b) Use normal coordinates to argue that one can replace partial derivatives with covariant derivatives to obtain the basis-independent result (where ∇ is the Levi-Civita connection)

$$(\mathcal{L}_X \omega)_a = X^b \nabla_b \omega_a + \omega_b \nabla_a X^b \, .$$

(c) Show that the Lie derivative of a metric tensor is given in a coordinate basis by

$$(\mathcal{L}_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu X^\rho + g_{\rho\nu} \partial_\mu X^\rho \, .$$

(d) Show that this can be written in the basis-independent form

$$(\mathcal{L}_X g)_{ab} = \nabla_a X_b + \nabla_b X_a \, .$$

3. (a) Let X and Y be two vector fields. Show that

$$\mathcal{L}_X(\mathcal{L}_Y Q) - \mathcal{L}_Y(\mathcal{L}_X Q) = \mathcal{L}_{[X,Y]} Q \, ,$$

where Q is either a function or a vector field. Deduce that the result holds if Q is a tensor field.

(b) Demonstrate that if a Riemannian or Lorentzian manifold has two "independent" isometries then it has a third, and define what is meant by "independent" here.

(c) Consider the unit sphere with metric

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \, .$$

Show that

$$\frac{\partial}{\partial \phi} \quad \text{and} \quad \sin\phi \, \frac{\partial}{\partial \theta} + \cot\theta \, \cos\phi \, \frac{\partial}{\partial \phi}$$

are Killing vectors. What is the third? Are there any more?

4. Let K^a be a Killing vector field and T_{ab} the energy momentum tensor. Let $J^a = T^a{}_b K^b$. Show that $\nabla_a J^a = 0$; J^a is a *conserved current.*

5. (a) Show that a Killing vector field K^a satisfies the equation

$$\nabla_a \nabla_b K^c = R^c{}_{bad} K^d \,.$$

[*Hint:* use the identity $R^a{}_{[bcd]} = 0$.]

(b) Deduce that in Minkowski spacetime, the components of Killing covectors are linear functions of the coordinates.

6. Consider Minkowski spacetime in an inertial frame, so the metric is $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$.

(a) Let K^a be a Killing vector field. Write down Killing's equation in the inertial frame coordinates.

(b) Using the result of the previous problem, show that the general solution can be written in terms of a constant antisymmetric matrix $a_{\mu\nu}$ and a constant covector b_μ.

(c) Identify the isometries corresponding to Killing fields with (i) $a_{\mu\nu} = 0$, (ii) $a_{0i} = 0$ and $b_\mu = 0$, (iii) $a_{ij} = 0$ and $b_\mu = 0$, where i, j take values from 1 to 3.

(d) Identify the conserved quantities along a timelike geodesic corresponding to cases (i) – (iii).

7. Consider the energy-momentum tensor describing a point mass at the origin: in "almost inertial" coordinates it is $T_{00}(t, \mathbf{x}) = M\delta^3(\mathbf{x})$, $T_{0i} = T_{ij} = 0$. Determine the linearized gravitational field produced by this energy momentum tensor, assuming it to be independent of t. For what values of $R = |\mathbf{x}|$ is the linear approximation valid?

8. (a) In "almost inertial" coordinates the energy momentum tensor of a straight *cosmic string* aligned along the z-axis is

$$T_{\mu\nu} = \mu\, \delta(x)\, \delta(y)\, \mathrm{diag}(1, 0, 0, -1)\,,$$

where μ is a small positive constant. Terms of order μ^2 are to be ignored. Look for a time-independent solution of the linearized Einstein equation, finding $h_{11} = h_{22} = -\lambda$ as the only non-zero components of the metric perturbation tensor, where $\lambda \equiv 8\mu \log(r/r_0)$, $r = \sqrt{x^2 + y^2}$, and r_0 is an arbitrary length.

(b) Show that the perturbed metric can be written in cylindrical polar coordinates as

$$ds^2 = -dt^2 + dz^2 + (1 - \lambda)(dr^2 + r^2 d\phi^2)\,.$$

(c) Perform a change of radial coordinate given by $(1 - \lambda)r^2 = (1 - 8\mu)\bar{r}^2$ to obtain

$$ds^2 = -dt^2 + dz^2 + d\bar{r}^2 + (1 - 8\mu)\bar{r}^2 d\phi^2\,,$$

and change the angular coordinate to obtain

$$ds^2 = -dt^2 + dz^2 + d\bar{r}^2 + \bar{r}^2 d\bar{\phi}^2\,.$$

Is this Minkowski spacetime? Show intuitively how a distant object may give rise to double images.

9. Consider a large thin shell of mass M and radius R which rotates slowly about the z axis (in "almost inertial" coordinates) with angular velocity Ω, so that terms of order $\mathcal{O}(R^2\Omega^2)$ can be neglected. Introduce a shell density $\rho = M\,\delta(r-R)/(4\pi R^2)$, where $r^2 = x^2+y^2+z^2$, and a 4-velocity $u^\mu = (1,\, -\Omega\, y,\, \Omega\, x,\, 0)$. The energy momentum tensor has components $T^{\mu\nu} = \rho u^\mu u^\nu$. We can regard this source as a superposition of two sources, one for which only T_{00} is nonzero, and one for which only T_{0i} is nonzero.

(a) Solve the linearized Einstein equations sourced by T_{00}. Show that the result agrees with Newtonian theory.

(b) Consider the perturbations sourced by T_{0i}. Argue that the only non-vanishing components are h_{0i}, which satisfy $\nabla^2 h_{0i} = -16\pi T_{0i}$. Consider the combination $h_{01} + ih_{02}$ and work in spherical polar coordinates. You should find that the RHS of the linearized Einstein equation is proportional to $\sin\theta\, e^{i\phi}$, i.e., to a spherical harmonic with $l = m = 1$. Since a general solution to the Laplace equation can be expanded as a sum over spherical harmonics, this implies that the solution must be of the form $h_{01} + ih_{02} = f(r)\,\sin\theta\, e^{i\phi}$. Hence obtain the solution

$$h_{0i} = \begin{cases} \omega\,(y,\, -x,\, 0) & r < R \\ \omega \frac{R^3}{r^3}\,(y,\, -x,\, 0) & r > R \end{cases}$$

where $\omega = 4M\Omega/(3R)$. Note that this decays as $1/r^2$ at large r. This is a general result: rotation of the source affects $h_{\mu\nu}$ at $\mathcal{O}(1/r^2)$, subleading compared to the $\mathcal{O}(1/r)$ contribution arising from the energy density.

10. Let $A = (a_{ij})$ be an $n\times n$ matrix, $\det A$ its determinant and $\hat{\epsilon}_{i_1 i_2 \ldots i_n}$ the Levi-Civita or totally antisymmetric symbol defined by

$$\hat{\epsilon}_{i_1\ldots i_n} = \begin{cases} 0 & \text{if} \geq 2 \text{ of the indices } i_1,\, \ldots,\, i_n \text{ are equal} \\ (-1)^p & p = \text{number of exchanges of indices in } (1,\, 2,\, \ldots,\, n) \leftrightarrow (i_1,\, i_2,\, \ldots,\, i_n) \end{cases}$$

(a) Show that

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \hat{\epsilon}_{i_1 i_2 \ldots i_n} a_{1i_1} a_{2i_2} \ldots a_{ni_n}\,, \qquad (*)$$

where we sum over the repeated indices $i_1, \ldots i_n$ on the right hand side.

(b) Conclude that

$$\det A = \frac{1}{n!}\hat{\epsilon}_{j_1 j_2 \ldots j_n}\, \hat{\epsilon}_{i_1 i_2 \ldots i_n}\, a_{j_1 i_1}\, a_{j_2 i_2}\, \ldots\, a_{j_n i_n}\,,$$

and

$$\hat{\epsilon}_{j_1 j_2 \ldots j_n}\, \det A = \hat{\epsilon}_{i_1 i_2 \ldots i_n}\, a_{j_1 i_1}\, a_{j_2 i_2}\, \ldots\, a_{j_n i_n}\,.$$

11. Let $\mathcal{M}$ be an n-dimensional manifold with metric $g_{\mu\nu}$ and f^μ denote a dual basis. The Levi-Civita tensor ϵ on $\mathcal{M}$ is given in terms of the Levi-Civita symbol $\hat{\epsilon}$ defined in the previous exercise by

$$\epsilon_{\mu_1 \ldots \mu_n} = \sqrt{|g|}\hat{\epsilon}_{\mu_1 \ldots \mu_n} \,,$$

where $g = \det g_{\mu\nu}$ is the determinant of the metric on $\mathcal{M}$. Show that (unlike $\hat{\epsilon}$), the Levi-Civita tensor ϵ transforms like a tensor under a change of basis

$$f^\mu \to \bar{f}^{\bar{\alpha}} = A^{\bar{\alpha}}{}_\mu f^\mu \,.$$