**Mathematical Tripos Part III** Dr U. Sperhake

**GENERAL RELATIVITY: Examples 1** Michaelmas 2015

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**1.** A lab is freely falling in the gravitational field of the Earth (described by a Newtonian point mass). There are two particles in the lab, both of mass $m$, separated by a distance $\eta$ in the vertical direction and initially at rest relative to the lab. One of them carries an electric charge $q$, the other one is neutral. There is a constant homogeneous vertical electric field $E$ in the lab. Find the equation of motion for the separation of the two particles, including both electrical and tidal effects due to the inhomogeneity of the gravitational field. Estimate the change in the separation after a time $\sim L$, where $L$ is the height of the elevator. Is the outcome of the experiment compatible with the Einstein equivalence principle?

**2.** Let $\hat{T} : \mathcal{T}_p(\mathcal{M}) \to \mathcal{T}^*_p(\mathcal{M})$ be a linear map. Show that one can define a tensor $T$ of type $(0, 2)$ as the map $T : (X, Y) \mapsto (\hat{T}(Y))(X)$. Similarly show that a linear map $\mathcal{T}_p(\mathcal{M}) \to \mathcal{T}_p(\mathcal{M})$ defines a tensor of type $(1, 1)$. What tensor $\delta$ arises from the identity map?

**3.** Let $V^{ab}$ be an arbitrary $\binom{2}{0}$ tensor, and let $S_{ab}$, $A_{ab}$ be symmetric and antisymmetric $\binom{0}{2}$ tensors, i.e., $S_{ab} = S_{ba}$, $A_{ab} = -A_{ba}$. Show that $V^{ab}S_{ab} = V^{(ab)}S_{ab}$ and $V^{ab}A_{ab} = V^{[ab]}A_{ab}$.

**4.** You are given a $\binom{2}{0}$ tensor $K$. Working first in some basis devise a criterion to test whether it is the *direct product* of two vectors $A$, $B$, i.e., $K^{ab} = A^a B^b$. (You can, but do not need to, use determinants.) Can you express the test in a manifestly basis-invariant manner?
Show that the general $\binom{2}{0}$ tensor in $n$ dimensions cannot be written as a direct product, but can be expressed as a sum of many direct products.

**5.** Let $\mathcal{M}$ be a manifold and $f : \mathcal{M} \to \boldsymbol{R}$ be a smooth function such that $\mathrm{d}f = 0$ at some point $p \in \mathcal{M}$. Let $\{x^\mu\}$ be a coordinate chart defined in a neighbourhood of $p$. Define

$$F_{\mu\nu} = \frac{\partial^2 f}{\partial x^\mu \partial x^\nu} \,.$$

By considering the transformation law for components show that $F_{\mu\nu}$ defines a $\binom{0}{2}$ tensor, the *Hessian* of $f$ at $p$. Construct also a coordinate-free definition and demonstrate its tensorial properties.

**6.** Let $g_{ab}$ be a $\binom{0}{2}$ tensor. In a basis, one can regard the components $g_{\mu\nu}$ as elements of an $n \times n$ matrix, so that one may define the determinant $g = \det(g_{\mu\nu})$. How does $g$ transform under a change of basis?

**7.** Let $\{e_\mu\}$ be a basis for vectors and set

$$[e_\mu, e_\nu] = \gamma^\rho{}_{\mu\nu}\, e_\rho.$$

(The $\gamma^\rho{}_{\mu\nu}$ are the *commutator components*.) Let $\{\omega^\mu\}$ be the dual basis of covectors. Let $\{x^\alpha\}$ be coordinates (here we use early Greek indices $\alpha$, $\beta$, ... to denote coordinate basis components and distinguish them from components in the generic bases denoted by late Greek indices $\mu$, $\nu$, ...). Expanding $e_\mu$ and $\omega^\mu$ in the coordinate basis gives

$$e_\mu = e_\mu{}^\alpha \frac{\partial}{\partial x^\alpha}, \qquad \omega^\mu = \omega^\mu{}_\alpha \, \mathrm{d}x^\alpha \,,$$

where $e_\mu{}^\alpha \omega^\nu{}_\alpha = \delta_\mu{}^\nu$. Show first that

$$e_\mu{}^\alpha \frac{\partial e_\nu{}^\beta}{\partial x^\alpha} - e_\nu{}^\alpha \frac{\partial e_\mu{}^\beta}{\partial x^\alpha} = \gamma^\rho{}_{\mu\nu}\, e_\rho{}^\beta \,,$$

and deduce that

$$e_\mu{}^\alpha e_\nu{}^\beta \frac{\partial \omega^\sigma{}_\beta}{\partial x^\alpha} - e_\nu{}^\alpha e_\mu{}^\beta \frac{\partial \omega^\sigma{}_\beta}{\partial x^\alpha} = -\gamma^\sigma{}_{\mu\nu} \,,$$

and finally that

$$\frac{\partial \omega^\sigma{}_\delta}{\partial x^\gamma} - \frac{\partial \omega^\sigma{}_\gamma}{\partial x^\delta} = -\gamma^\sigma{}_{\mu\nu}\, \omega^\mu{}_\gamma\, \omega^\nu{}_\delta \,. \qquad (\dagger)$$

In certain circumstances there may exist coordinates $\{y^\mu\}$ such that

$$\omega^\mu = \mathrm{d}y^\mu, \qquad e_\mu = \frac{\partial}{\partial y^\mu} \,.$$

(We would then say that the bases are *coordinate induced*.) Show that if the bases are coordinate induced then $[e_\mu, e_\nu] = 0$, $\forall \mu$, $\nu$. Use ($\dagger$) to show also the converse, i.e. if $[e_\mu, e_\nu] = 0$ $\forall \mu$, $\nu$ then the basis is coordinate induced. Deduce that bases being coordinate induced is equivalent to vanishing commutator components.

**8.** In inertial frame coordinates, the metric of Minkowski spacetime is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

(a) Show that if we replace $(x, y, z)$ with spherical polar coordinates $(r, \theta, \phi)$ defined by

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{r}, \quad \tan\phi = \frac{y}{x} \,,$$

then the metric takes the form

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

(b) Find the components of the metric and inverse metric in "rotating coordinates" defined by

$$\tilde{t} = t, \quad \tilde{x} = \sqrt{x^2 + y^2}\, \cos(\phi - \omega t), \quad \tilde{y} = \sqrt{x^2 + y^2}\, \sin(\phi - \omega t), \quad \tilde{z} = z \,,$$

where $\tan\phi = y/x$.

**9.** The *Schwarzschild metric* in Schwarzschild coordinates $(t, r, \theta, \phi)$ is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \,.$$

Write out the components of the geodesic equation for this metric and hence determine the Christoffel symbols. Note that the Lagrangian from which geodesics are obtained is independent of $t$ and $\phi$. What consequence does this have?

**10.** Obtain the form of the general timelike geodesic in a two-dimensional spacetime with metric

$$ds^2 = t^{-2}(-dt^2 + dx^2) \,.$$

*Hint:* You should use the symmetries of the Lagrangian, and you will probably find the following indefinite integrals useful:

$$\int \frac{dt}{t\sqrt{1 + C^2 t^2}} = \frac{1}{2} \ln\left(\frac{\sqrt{1 + C^2 t^2} - 1}{\sqrt{1 + C^2 t^2} + 1}\right) , \qquad \int \frac{ds}{\sinh^2 s} = -\coth s \,,$$