Spherical Symmetry

Birkhoff's Theorem:

A spherically symmetric vacuum metric is <u>static</u>.
+ The geometry is <u>unique</u> up to 1 constant

<u>Proof</u> Choose coords θ, ϕ adapted to spherical symmetry
$\Rightarrow$ usual coords on 2-sphere.

Let other coords be $a + b$ let $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$

So $$ds^2 = A(a,b)\,da^2 + B(a,b)\,db^2 + 2C(a,b)\,da\,db + D(a,b)\,d\Omega^2$$

metric indep of θ, ϕ except through $d\Omega^2$.

Now choose new coord $r = \sqrt{D(a,b)}$ instead of b

$$\Rightarrow ds^2 = E(a,r)\,da^2 + F(a,r)\,dr^2 + 2G(a,r)\,da\,dr + r^2 d\Omega^2$$

(Notice that area of const-r spherical surfaces are $4\pi r^2$
$ds^2 = r^2 d\Omega^2$)

Now eliminate $da\,dr$ term:

Choose new coord $t(a,r)$ such that $dt = H(a,r)\left(da - \frac{G(a,r)}{E(a,r)}dr\right)$

Can solve if $\frac{\partial}{\partial r}\frac{\partial}{\partial a}t = \frac{\partial}{\partial a}\frac{\partial}{\partial r}t \Rightarrow \frac{\partial}{\partial r}H = -\frac{\partial}{\partial a}\left(HG/E\right)$

$$\Rightarrow \left(\frac{\partial}{\partial r} + \frac{G}{E}\frac{\partial}{\partial a}\right)\ln H = -\frac{\partial}{\partial a}\frac{G}{E}$$

Can solve.

So $\quad ds^2 = I(t,r)\,dt^2 + J(t,r)\,dr^2 + r^2 d\Omega^2$

Only difference btwn $t+r$ is r is in front of $d\Omega^2$

Must choose either I or J negative so metric is $(-++)$

Choose $\quad \boxed{ds^2 = -e^{2\Phi(r,t)}dt^2 + e^{2\lambda(r,t)}dr^2 + r^2 d\Omega^2}$

For this metric

$$G_{\mu\nu} = 0 \quad \overset{tt}{\Rightarrow} \quad 2e^{-2\lambda}\frac{\lambda_{,r}}{r} + \frac{(1-e^{-2\lambda})}{r^2} = 0 \quad ①$$

$$tr \qquad 2e^{-(\Phi+\lambda)}\frac{\lambda_{,t}}{r} = 0 \quad ②$$

$$rr \qquad (e^{-2\lambda}-1)\frac{1}{r^2} + 2e^{-2\lambda}\frac{\Phi_{,r}}{r} = 0 \quad ③$$

$$\theta\theta,\phi\phi \qquad e^{-2\lambda}\left(\Phi_{,rr} + \Phi_{,r}^2 - \Phi_{,r}\lambda_{,r} + \frac{\Phi_{,r}}{r} - \frac{\lambda_{,r}}{r}\right) - e^{-2\phi}\left(\lambda_{,tt} + \lambda_{,t}^2 - \Phi_{,t}\lambda_{,t}\right) = 0 \quad ④$$

② $\Rightarrow \lambda_{,t} = 0 \qquad \lambda = \lambda(r)$ only

① $\Rightarrow \dfrac{2e^{-2\lambda}}{(1-e^{-2\lambda})}\,d\lambda = -\dfrac{dr}{r}$

$$\Rightarrow \ln(1-e^{-2\lambda}) = -\ln(r) + \text{const}$$

Call the constant $2M$ will see why later

$$\Rightarrow \ln(1-e^{-2\lambda}) = \ln\left(\frac{2M}{r}\right)$$

$$\Rightarrow \boxed{e^{2\lambda} = \left(1-\frac{2M}{r}\right)^{-1}}$$

① + ③ $\Rightarrow$ $\Phi_{,r} + \lambda_{,r} = 0$

so $\Phi = -\lambda + g(t)$

or $e^{2\Phi} = e^{-2\lambda} e^{2g(t)}$

choose new time coord. $dt' = dt\, e^{g(t)}$

then $e^{2\Phi} = e^{-2\lambda}$

$$\boxed{ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2}$$

Schwarzschild metric

(④ satisfied if ①, ②, ③ satisfied by $G^{\mu\nu}{}_{;\nu} = 0$)

What is M?

Let $r \to \infty$ so $\frac{2M}{r} \ll 1$ $\Rightarrow \left(1-\frac{2M}{r}\right)^{-1} \sim 1 + \frac{2M}{r}$ so $ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1+\frac{2M}{r}\right)dr^2 + r^2 d\Omega^2$

weak gravity metric for pt. mass

Schwarzschild solution describes:
- Any spherically symmetric vacuum spacetime
- Metric outside spherical matter distribution
- Spherical black hole

Let's study particle orbits in Schwarzschild

Assume $\theta = \pi/2$ WLOG

$\vec{P} = \frac{\partial}{\partial \lambda}$ = 4-momentum of particle let $\lambda = \tau/m$

$P^\theta = 0$

Can use geodesic eqn. Instead, easier to use KVs and conserved quantities

In Schwarzschild, $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ are KVs

So $\vec{P} \cdot \frac{\partial}{\partial t} = P_0$ = const, call it $-E$

$\vec{P} \cdot \frac{\partial}{\partial \phi} = P_\phi$ = const, call it L

$$\vec{P} \cdot \vec{P} = -m^2 = P^{0^2} g_{00} + P^{r^2} g_{rr} + \cancel{P^{\theta^2}} g_{\theta\theta} + P^{\phi^2} g_{\phi\phi}$$

$$= \frac{-E^2}{1-\frac{2M}{r}} + \left(\frac{dr}{d\lambda}\right)^2 \frac{1}{1-\frac{2M}{r}} + \frac{L^2}{r^2}$$

$g^{00} = -\left(1-\frac{2M}{r}\right)^{-1}$

$g^{rr} = \left(1-\frac{2M}{r}\right)$

$g^{\phi\phi} = \frac{1}{r^2 \sin^2\theta} = \frac{1}{r^2}$ at $\theta = \pi/2$

$$\Rightarrow \frac{1}{m^2}\left(\frac{dr}{d\lambda}\right)^2 = \frac{E^2}{m^2} - \left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{m^2 r^2}\right)$$ massive particle

$$\Rightarrow \left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1-\frac{2M}{r}\right)\frac{L^2}{r^2}$$ Photon

First

Assume massive particle. Let $\tilde{E} = \frac{E}{m}$ $\tilde{L} = \frac{L}{m}$

$$\frac{d}{d\tau} = \frac{1}{m}\frac{d}{d\lambda}$$

$$\Rightarrow \quad \left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right) \qquad ①$$

also $$\frac{d\phi}{d\tau} = \frac{p^\phi}{m} = \frac{g^{\phi\phi}L}{m} = \frac{\tilde{L}}{r^2} \qquad ②$$

also $$\frac{dt}{d\tau} = \frac{p^t}{m} = \frac{g^{00}E}{m} = \frac{\tilde{E}}{1-\frac{2M}{r}} \qquad ③$$

τ = proper time measured by particle

t = coord. time = time measured by observer at ∞.

Note: Time dilation / Redshift

Consider observer w/ 4-velocity $\vec{u}$.

Energy of particle measured by observer is $-\vec{u}\cdot\vec{p}$

suppose observer is stationary $u^i = 0$

then $-\vec{u}\cdot\vec{p} = -u^0 p_0$

but $\vec{u}\cdot\vec{u} = -1 = u^0 u^0 g_{00} \Rightarrow u^0 = \left(1-\frac{2M}{r}\right)^{-1/2}$

$\Rightarrow -\vec{u}\cdot\vec{p} = (-p_0)\left(1-\frac{2M}{r}\right)^{-1/2} = E\left(1-\frac{2M}{r}\right)^{-1/2}$

so the constant E is not exactly the measured energy. Grav. redshift

Special Case: Radial infall

$$\frac{d\phi}{d\tau} = 0 \quad \Rightarrow \quad \tilde{L} = 0$$

so $$\frac{dt}{d\tau} = \frac{\tilde{E}}{1-\frac{2M}{r}}$$

$$\frac{dr}{d\tau} = -\left(\tilde{E}^2 - 1 + \frac{2M}{r}\right)^{1/2}$$

Assume particle falls from rest from $r = R$

$$\Rightarrow \quad \frac{dr}{d\tau} = 0 \text{ at } r = R \quad \Rightarrow \quad \tilde{E}^2 = 1 - \frac{2M}{R}$$

so $$\frac{dr}{d\tau} = -\left(\frac{2M}{r} - \frac{2M}{R}\right)^{1/2}$$ (Note: Need $r \leq R$)

or $$d\tau = \frac{-dr}{\left(\frac{2M}{r} - \frac{2M}{R}\right)^{1/2}}$$

$$\Rightarrow \quad \tau = \left(\frac{R^3}{8M}\right)^{1/2}\left[2\left(\frac{r}{R} - \frac{r^2}{R^2}\right)^{1/2} + \cos^{-1}\left(\frac{2r}{R} - 1\right)\right]$$

$\tau = 0$ at $r = R$

Can write simpler by defining $\eta = \cos^{-1}\left(\frac{2r}{R} - 1\right)$ $\quad 0 \leq \eta \leq \pi$

Then $$\tau = \left(\frac{R^3}{8M}\right)^{1/2}(\eta + \sin\eta)$$

$$r = \frac{R}{2}(1 + \cos\eta)$$

$$t = \int \frac{\tilde{E}\, d\tau}{1-\frac{2M}{r}} = \int \tilde{E}\, d\eta \frac{d\tau}{d\eta} \frac{1}{1-\frac{2M}{r(\eta)}}$$

You get $$\frac{t}{2M} = \ln \left| \frac{\left(\frac{R}{2M}-1\right)^{1/2} + \tan \eta/2}{\left(R/2M-1\right)^{1/2} - \tan \eta/2} \right|$$

$$+ \left(\frac{R}{2M}-1\right)^{1/2}\left(\eta + \frac{R}{4M}\left(\eta+\sin\eta\right)\right)$$

How much time to reach $r=2M$?

① Proper time: $r=2M$ means $2M = \frac{R}{2}(1+\cos\eta)$

$\Rightarrow \frac{4M}{R} - 1 = \cos\eta$ ($R > 2M$ so $-1 < \cos\eta < 1$ $\underline{\underline{OK}}$)

$$\tau = \left(\frac{R^3}{8M}\right)^{1/2}(\eta+\sin\eta) \Rightarrow \text{get Finite proper time}$$

② Coord time t

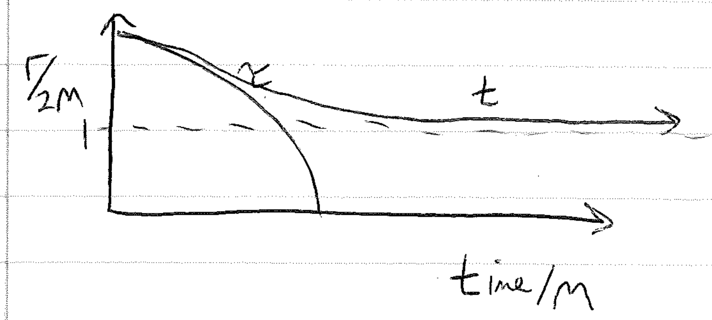
$$\cos\eta = \frac{4M}{R} - 1 \Rightarrow \tan\left(\frac{\eta}{2}\right) = \left(R/2M - 1\right)^{1/2}$$

So $t \to \infty$

Observer at ∞ sees particle cross $r=2M$ at $t \to \infty$!

Particle crosses $r=2M$ in Finite proper time,

hits $r=0$ at proper time $\tau = \pi\left(\frac{R^3}{8M}\right)^{1/2}$!



What's wrong?

- Particle does fall past $r=2M$
- Not seen by observer at ∞

both are true.

$r=2M$ called <u>Event horizon</u>

problem is t is a bad coord. at $r=2M$

<u>coord. singularity</u> like north pole, $\theta=0$

more on $r=2M$ later