Gravitational Plane Wave

$$\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = 0$$

Consider solutions of the form $\bar{h}_{\mu\nu} = \underbrace{A_{\mu\nu}}_{\text{const}} e^{i k_\alpha x^\alpha}$

$$= A_{\mu\nu} e^{i k_i x^i} e^{-i\omega t}$$

where $\omega \equiv k^0$

Then $\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = \underbrace{(i k_\alpha)(i k^\alpha)}\bar{h}_{\mu\nu}$

So $k_\alpha k^\alpha = 0$ $\quad \vec{k}$ is null.

How many degrees of freedom?

Looks like 10. $A_{\mu\nu}$ has 10 components.

Most of these d.o.f. represent gauge waves!

But Lorentz gauge $\bar{h}_{\mu\nu,}{}^{\nu} = 0 \Rightarrow \boxed{A_{\mu\nu} k^\nu = 0}$

4 eqs.

leaves 6.

But wait, there's more

Lorentz gauge unique only up to

$$x^\mu \to x^\mu + \varepsilon V^\mu \quad \text{where} \quad V_{\alpha,\beta}{}^{\beta} = 0$$

So suppose we choose $V_\alpha = \underbrace{B_\alpha}_{\text{const}} e^{ik_\mu x^\mu}$

satisfies $V_{\alpha,\ell}{}^{\ell} = 0$

Then $\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \eta_{\mu\nu} V^\alpha{}_{,\alpha} - V_{\mu,\nu} - V_{\nu,\mu}$

or $A_{\mu\nu} \to A_{\mu\nu} + k_\mu B_\nu + k_\nu B_\mu - k_\alpha B^\alpha \eta_{\mu\nu}$

Can choose B_μ however we want 4 conditions

$\Rightarrow$ choose B_μ s.t. $\left.\begin{array}{l} A^\alpha{}_\alpha = 0 \\ A_{0i} = 0 \end{array}\right\}$ Transverse Traceless (TT) gauge

Then $A_{\mu\nu} k^\nu = 0 \Rightarrow A_{00} k^0 + \cancel{A_{0i}}^{0} k^i = 0$ $\Rightarrow \boxed{A_{00} = 0}$

$\Rightarrow \cancel{A_{i0} k^0}^{0} + A_{ij} k^j = 0$ $\Rightarrow \boxed{A_{ij} k^j = 0}$

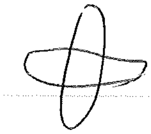
Only 2 degrees of freedom left.

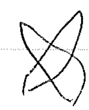
$\Rightarrow$ 2 polarizations

if $k_i = k\hat{z}$ Then $A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

i.e. $ds^2 = -dt^2 + (1 + A_+ e^{ik_\alpha x^\alpha}) dx^2 + (1 - A_+ e^{ik_\alpha x^\alpha}) dy^2$

$+ 2 A_\times e^{ik_\alpha x^\alpha} dx dy + dz^2$

For A_+ , test particles stretched/squeezed along x, y 

For $A_\times$, test " " along diagonal 

Static point mass

Static $\frac{\partial}{\partial t} \to 0$ so $\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = \boxed{\bar{h}_{\mu\nu,i}{}^{i} = -16\pi T_{\mu\nu}}$ Poisson Eq.

$$\Rightarrow \quad \bar{h}_{\mu\nu} = 4\int \frac{T_{\mu\nu}(\underset{\sim}{x}')\, d^3x'}{|\underset{\sim}{x}-\underset{\sim}{x}'|}$$

Point mass at origin $\quad T_{\mu\nu} = \rho U_\mu U_\nu \qquad U^\alpha = (1,0,0,0)$ to lowest order

$$\Rightarrow \quad T_{00} = \rho \quad , T_{\alpha i} = 0$$

$\rho = M\,\delta(\underset{\sim}{x})$ point mass

$$\Rightarrow \quad \bar{h}_{00} = \frac{4M}{|x|} = \frac{4M}{r} \quad , \bar{h}_{\alpha i} = 0$$

Now $\bar{h} = \bar{h}^{\alpha}{}_{\alpha} = -\bar{h}_{00} = \frac{-4M}{r}$ So $h = \frac{4M}{r}$

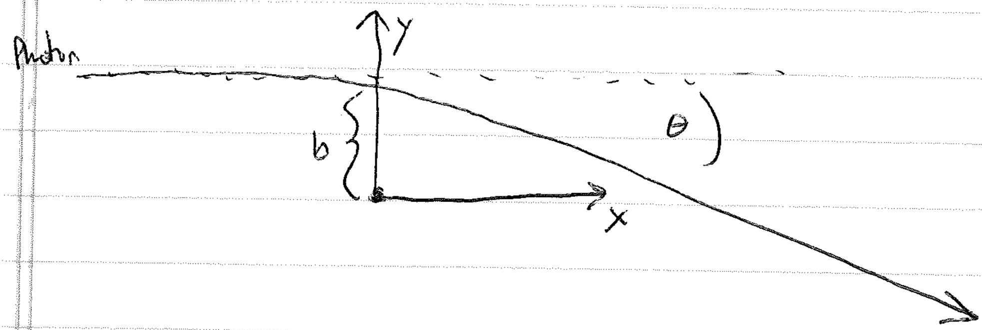
$$h_{00} = \bar{h}_{00} + \tfrac{1}{2}\eta_{00} h$$
$$= \frac{4M}{r} + \frac{1}{2}(-1)\frac{4M}{r} = \frac{2M}{r}$$

$$h_{ij} = \bar{h}_{ij} + \tfrac{1}{2}\eta_{ij} h = \frac{2M}{r}\eta_{ij}$$

$$\Rightarrow \quad h_{00} = h_{xx} = h_{yy} = h_{zz} = \frac{2M}{r}$$

$$\Rightarrow \quad \boxed{ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1+\frac{2M}{r}\right)\underbrace{\left(dx^2+dy^2+dz^2\right)}_{dr^2 + r^2(d\theta^2+\sin^2\theta\, d\phi^2)}}$$

Light bending around sun



Assume $p^z = 0$, $|p^y| \ll |p^x|, |p^t|$

$$\Rightarrow \quad \vec{p}\cdot\vec{p} = 0 = -p^{0^2}\left(1-\frac{2M}{r}\right) + p^{x^2}\left(1+\frac{2M}{r}\right)$$

$$\Rightarrow \quad p^x = p^0\sqrt{\frac{1-\frac{2M}{r}}{1+2M/r}} \qquad = p^0 + \mathcal{O}\left(\frac{M}{r}\right)$$

For the pt mass metric, $\Gamma^i_{00} = \frac{Mx^i}{r^3}$ $\quad \Gamma^j_{k\ell} = \frac{M}{r^3}\left(x^j\delta_{k\ell} - x_k\delta^j_\ell - x_\ell\delta^j_k\right)$

$$\Gamma^j_{0k} = 0 \qquad j \neq k$$

$$\text{So } \frac{dp^y}{d\lambda} = -\Gamma^y_{\alpha\beta}p^\alpha p^\beta = -\Gamma^y_{00}p^{0^2} - \Gamma^y_{xx}p^{x^2} - 2\Gamma^y_{x0}p^x p^0 \; \cancel{+\Gamma p^y \text{ terms}} \to 0$$

$$= -\frac{My}{r^3}p^{0^2} - \frac{My}{r^3}p^{x^2} \qquad = \frac{-2My}{r^3}p^{x^2} \qquad = \frac{-2My}{(x^2+y^2)^{3/2}}p^{x^2}$$

$$p^x = \frac{dx}{d\lambda} \qquad \Rightarrow \quad \frac{dp^y}{d\lambda} = \frac{-2Mb}{(x^2+b^2)^{3/2}}\, p^x \frac{dx}{d\lambda} \quad (y \approx b) \qquad \Rightarrow p^y = \int_{-\infty}^{\infty}\frac{-2Mb}{(x^2+b^2)^{3/2}}\, p^x\, dx$$

$$\Rightarrow p^y = -2Mb\,p^x\int_{-\infty}^{\infty}\frac{dx}{(x^2+b^2)^{3/2}} \qquad = \frac{-4M}{b}p^x \qquad \frac{p^y}{p^x} = \tan\theta = \frac{4M}{b} \sim 1.75'' \text{ for sun}$$

Spherical Symmetry

Birkhoff's Theorem:

A spherically symmetric vacuum metric is <u>static</u>.
+ The geometry is <u>unique</u> up to 1 constant

<u>Proof</u> Choose coords θ, ϕ adapted to spherical symmetry
$\Rightarrow$ usual coords on 2-sphere.

Let other coords be $a + b$ let $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$

So $$ds^2 = A(a,b)\,da^2 + B(a,b)\,db^2 + 2C(a,b)\,da\,db + D(a,b)\,d\Omega^2$$

metric indep of θ, ϕ except through $d\Omega^2$.

Now choose new coord $r = \sqrt{D(a,b)}$ instead of b

$$\Rightarrow ds^2 = E(a,r)\,da^2 + F(a,r)\,dr^2 + 2G(a,r)\,da\,dr + r^2 d\Omega^2$$

(notice that area of const-r spherical surfaces are $4\pi r^2$
$ds^2 = r^2 d\Omega^2$)

Now eliminate $da\,dr$ term:

Choose new coord $t(a,r)$ such that $dt = H(a,r)\left(da - \dfrac{G(a,r)}{E(a,r)}\,dr\right)$

Can solve if $\frac{\partial}{\partial r}\frac{\partial}{\partial a} t = \frac{\partial}{\partial a}\frac{\partial}{\partial r} t \Rightarrow \frac{\partial}{\partial r} H = -\frac{\partial}{\partial a}\left(HG/E\right)$

$$\Rightarrow \left(\frac{\partial}{\partial r} + \frac{G}{E}\frac{\partial}{\partial a}\right)\ln H = -\frac{\partial}{\partial a}\frac{G}{E}$$

Can solve.

So $\quad ds^2 = I(t,r)\,dt^2 + J(t,r)\,dr^2 + r^2 d\Omega^2$

Only difference btwn $t + r$ is r is in front of $d\Omega^2$

Must choose either I or J negative so metric is $\left(^{-}{}_{++}\right)$

choose $\quad \boxed{ds^2 = -e^{2\Phi(r,t)}dt^2 + e^{2\lambda(r,t)}dr^2 + r^2 d\Omega^2}$

For this metric

$$G_{\mu\nu} = 0 \quad \overset{tt}{\Rightarrow} \quad \frac{2e^{-2\lambda}\lambda_{,r}}{r} + \frac{(1-e^{-2\lambda})}{r^2} = 0 \quad ①$$

$$tr \qquad \frac{2e^{-(\Phi+\lambda)}\lambda_{,t}}{r} = 0 \quad ②$$

$$rr \qquad \left(e^{-2\lambda}-1\right)\frac{1}{r^2} + 2e^{-2\lambda}\frac{\Phi_{,r}}{r} = 0 \quad ③$$

$$\theta\theta, \phi\phi \qquad e^{-2\lambda}\left(\Phi_{,rr} + \Phi_{,r}^2 - \Phi_{,r}\lambda_{,r} + \frac{\Phi_{,r}}{r} - \frac{\lambda_{,r}}{r}\right) - e^{-2\Phi}\left(\lambda_{,tt} + \lambda_{,t}^2 - \Phi_{,t}\lambda_{,t}\right) = 0 \quad ④$$

② $\Rightarrow \quad \lambda_{,t} = 0 \qquad \lambda = \lambda(r)$ only

① $\Rightarrow \quad \dfrac{2e^{-2\lambda}}{(1-e^{-2\lambda})}\,d\lambda = -\dfrac{dr}{r}$

$$\Rightarrow \quad \ln\left(1-e^{-2\lambda}\right) = -\ln(r) + \text{const}$$

Call the constant $2M$ will see why later

$$\Rightarrow \ln(1-e^{-2\lambda}) = \ln\left(\frac{2M}{r}\right)$$

$$\Rightarrow \boxed{e^{2\lambda} = \left(1-\frac{2M}{r}\right)^{-1}}$$

① + ③ $\Rightarrow$ $\Phi_{,r} + \lambda_{,r} = 0$

so $\Phi = -\lambda + g(t)$

or $e^{2\Phi} = e^{-2\lambda} e^{2g(t)}$

choose new time coord. $dt' = dt\, e^{g(t)}$

then $e^{2\Phi} = e^{-2\lambda}$

$$\boxed{ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2}$$

Schwarzschild metric

(④ satisfied if ①, ②, ③ satisfied by $G^{\mu\nu}{}_{;\nu} = 0$)

What is M?

Let $r \to \infty$ so $\frac{2M}{r} \ll 1$ $\Rightarrow \left(1-\frac{2M}{r}\right)^{-1} \sim 1 + \frac{2M}{r}$ so $ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1+\frac{2M}{r}\right)dr^2 + r^2 d\Omega^2$

weak gravity metric for pt. mass

– Any spherically symmetric vacuum spacetime

Schwarzschild solution describes: – Metric outside spherical matter distribution

– Spherical black hole

Let's study particle orbits in Schwarzschild

Assume $\theta = \pi/2$ WLOG

$\vec{P} = \frac{\partial}{\partial\lambda}$ = 4-momentum of particle let $\lambda = \tau/m$

$P^\theta = 0$

Can use geodesic eqn. Instead, easier to use KVs and conserved quantities

In Schwarzschild, $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial\phi}$ are KVs

So $\vec{P}\cdot\frac{\partial}{\partial t} = P_0$ = const, call it $-E$

$\vec{P}\cdot\frac{\partial}{\partial\phi} = P_\phi$ = const, call it L

$$\vec{P}\cdot\vec{P} = -m^2 = {P^0}^2 g_{00} + {P^r}^2 g_{rr} + \cancel{{P^\theta}^2}^{\,0} g_{\theta\theta} + {P^\phi}^2 g_{\phi\phi}$$

$$= \frac{-E^2}{1-\frac{2M}{r}} + \left(\frac{dr}{d\lambda}\right)^2 \frac{1}{1-\frac{2M}{r}} + \frac{L^2}{r^2}$$

$g^{00} = -\left(1-\frac{2M}{r}\right)^{-1}$

$g^{rr} = \left(1-\frac{2M}{r}\right)$

$g^{\phi\phi} = \frac{1}{r^2\sin^2\theta} = \frac{1}{r^2}$ at $\theta = \pi/2$

$$\Rightarrow \frac{1}{m^2}\left(\frac{dr}{d\lambda}\right)^2 = \frac{E^2}{m^2} - \left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{m^2r^2}\right)$$ massive particle

$$\Rightarrow \left(\frac{dr}{d\lambda}\right)^2 = E^2 - \left(1-\frac{2M}{r}\right)\frac{L^2}{r^2}$$ Photon