Killing vectors

IF $\boxed{\mathcal{L}_{\vec{V}} \overset{\approx}{g} = 0}$, then $\vec{V}$ is called a Killing vector field.

metric

Generates symmetries i.e. $\overset{\approx}{g}$ doesn't change along $\vec{V}$

Note: $(\mathcal{L}_{\vec{V}} g)_{\alpha\beta} = V^\gamma \underbrace{\nabla_\gamma g_{\alpha\beta}}_{0} + g_{\lambda\beta} \nabla_\alpha V^\lambda + g_{\alpha\lambda} \nabla_\beta V^\lambda$

$= \boxed{\nabla_\alpha V_\beta + \nabla_\beta V_\alpha = 0}$ or $V_{(\alpha;\beta)} = 0$

Killing's equation

Example: $ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$

$\frac{\partial}{\partial \phi}$ is a Killing vector i.e. metric independent of ϕ

Constants of the motion

Suppose $\vec{u}$ is tangent of geodesic. $\vec{V}$ is Killing field

$\nabla_{\vec{u}} \vec{u} = 0$

Then $\nabla_{\vec{u}} (\vec{u} \cdot \vec{V}) = u^\alpha \nabla_\alpha (u^\beta V_\beta) = u^\alpha u^\beta \nabla_\alpha V_\beta$

$= 0$ because $\nabla_\alpha V_\beta$ is antisymmetric

So $\boxed{\vec{u} \cdot \vec{V} \text{ is constant along geodesic}}$

Killing vectors in Flat spacetime

$$V_{\alpha,\beta} + V_{\beta,\alpha} = 0$$

$\Gamma = 0$

$g_{\alpha\beta} = \eta_{\alpha\beta}$

$\downarrow$

10 equations

$\Rightarrow$ 10 KVs

(i) 4 constant KVs $\Rightarrow$ 4 translations

$$V^{\mu}_{(\alpha)} = a^{\mu}_{(\alpha)} = \text{const}$$

component of KV

which KV

(ii) $V^{i}_{(k)} = \varepsilon^{i}{}_{km} x^{m}$, $V^{0}_{(k)} = 0$ $\Rightarrow$ 3 Rotations

example $\vec{V}_{(x)} = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$

(iii) $V^{\mu(k)} = 2\eta^{\mu[0} x^{k]}$ $\Rightarrow$ 3 boosts

example $\vec{V}^{(x)} = -x\frac{\partial}{\partial t} - t\frac{\partial}{\partial x}$

(Why Boost? in coord transform language

$$x^{\alpha'} = x^{\alpha} + \varepsilon V^{\alpha}$$

$$t' = t - \varepsilon x$$

$$x' = x - \varepsilon t$$

$$y' = y$$

$$z' = z$$
)

Conserved Quantities

IF $\vec{V}$ is a KV, $J^\mu \equiv T^{\mu\nu} V_\nu$ is a conserved current

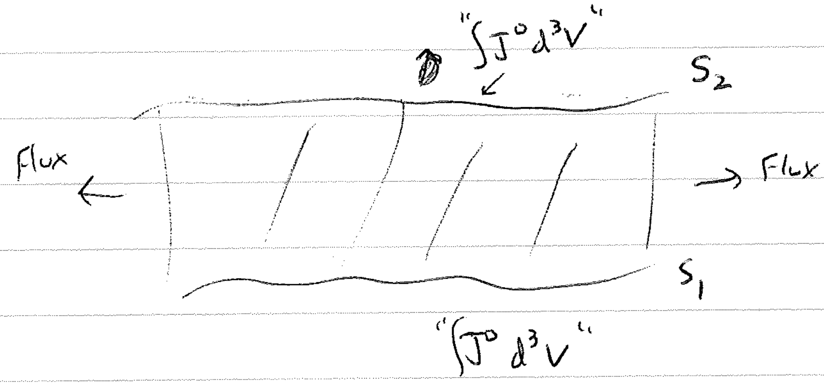
Why? $J^\mu{}_{;\mu} = (T^{\mu\nu} V_\nu)_{;\mu}$

$$= \underbrace{\cancel{T^{\mu\nu}{}_{;\mu}}}_{D\cdot T=0}\,{}^{0}\, V_\nu + \underbrace{\cancel{T^{\mu\nu}}}_{\text{Symm}}\,{}^{0}\underbrace{V_{\nu;\mu}}_{\text{antisymm}}$$

$$\Rightarrow \boxed{J^\mu{}_{;\mu} = 0}$$

Also get Conserved Charge

$$0 = \int J^\alpha{}_{;\alpha} \sqrt{g}\, d^4x \overset{\text{stokes/Gauss}}{=} \int J^\alpha d^3\Sigma_\alpha$$



IF J vanishes at spatial ∞

then $$\int_{S_1} J^\alpha d^3\Sigma_\alpha = \int_{S_2} J^\alpha d^3\Sigma_\alpha$$

Conserved "currents / charges" in Flat spacetime

(i) $\vec{V} = \frac{\partial}{\partial t}$ then $J^\alpha = T^{\alpha\beta} V_\beta = -T^{\alpha 0}$

$$Q = \int J^0 d^3x = \int \rho \, d^3x \quad \underline{\text{energy}}$$

$\vec{V} = \frac{\partial}{\partial x^i}$ $\quad J^\alpha = -T^{\alpha i}$ $\quad Q = \int J^0 d^3x = \int T^{0i} d^3x$

momentum

(ii) $\vec{V} = \cancel{x}\; y\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$ $\quad J^\alpha = xT^{\alpha y} - yT^{\alpha x}$

$$Q = \int (xT^{0y} - yT^{0x}) d^3x = \text{"}J_z\text{"}$$

$\underline{\text{angular momentum}}$

(iii) boosts

$\vec{V} = -x\frac{\partial}{\partial t} - t\frac{\partial}{\partial x}$ $\quad J^\alpha = xT^{\alpha 0} - tT^{\alpha x}$

$$Q = \int (xT^{00} - tT^{0x}) d^3x$$

what is this? define $X_{cm} = \dfrac{\int xT^{00}}{\int T^{00}}$ $\quad V^x_{cm} = \dfrac{\int T^{0x}}{\int T^{00}} = \dfrac{P^x}{E}$

So $Q = X_{cm} E - t P^x$

$= E(X_{cm} - V^x_{cm} t)$

$= \text{const}$ $\quad \Rightarrow \boxed{\text{Uniform motion of CM of system}}$

Linearized Gravity

– How to handle weak grav. fields.

Assume $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ $\quad |h_{\alpha\beta}| \ll 1$

- Drop all h^2 terms (for solar system, $h \sim \frac{M}{R} \sim 10^{-6}$)
- Think of h as tensor field in flat space
- Raise / lower indices with η (but note $g^{\alpha\beta} = -h^{\alpha\beta} + \eta^{\alpha\beta}$)

Then: $\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}\eta^{\mu\nu}(h_{\alpha\nu,\beta} + h_{\beta\nu,\alpha} - h_{\alpha\beta,\nu})$ coord basis

$$= \frac{1}{2} h_{\alpha}{}^{\mu}{}_{,\beta} + h_{\beta}{}^{\mu}{}_{,\alpha} - h_{\alpha\beta}{}^{,\mu}$$ (dropped 2nd order terms)

And: $R_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu}$ [drop $\Gamma \cdot \Gamma$ terms]

$$= \frac{1}{2}\left(h_{\mu}{}^{\alpha}{}_{,\nu\alpha} + h_{\nu}{}^{\alpha}{}_{,\mu\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h_{,\mu\nu}\right)$$

where $h \equiv h^{\alpha}{}_{\alpha}$
$= \eta^{\alpha\beta} h_{\alpha\beta}$

Equations simpler if we define $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$

Then $G_{\mu\nu} = 8\pi T_{\mu\nu} \Rightarrow$ $$\boxed{-\bar{h}_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu}\bar{h}_{\alpha\beta,}{}^{\alpha\beta} + \bar{h}_{\mu\alpha,\nu}{}^{\alpha} + \bar{h}_{\nu\alpha,\mu}{}^{\alpha} = 16\pi T_{\mu\nu}}$$

Linearized Gravity

Clarification

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

but $g^{\alpha\beta} \neq \eta^{\alpha\beta} + h^{\alpha\beta}$

instead, $g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$

So that $g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_{\gamma}$

In other words, $h^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\nu} h_{\mu\nu}$ use η to raise/lower h

$g^{\alpha\beta} \neq \eta^{\alpha\mu}\eta^{\beta\nu} g_{\mu\nu}$ use g to raise/lower g

2 types of coord xFormation:

① Global: Lorentz

$$X^{\mu'} = \Lambda^{\mu'}{}_{\mu} X^{\mu}$$

Then

$$g_{\bar{\alpha}\bar{\beta}} = (\eta_{\alpha\beta} + h_{\alpha\beta}) \Lambda^{\alpha}{}_{\bar{\alpha}} \Lambda^{\beta}{}_{\bar{\beta}}$$

$$= \eta_{\bar{\alpha}\bar{\beta}} + h_{\bar{\alpha}\bar{\beta}}$$

So $h_{\bar{\alpha}\bar{\beta}} = \Lambda^{\alpha}{}_{\bar{\alpha}} \Lambda^{\beta}{}_{\bar{\beta}} h_{\alpha\beta}$ like tensor in Flat spacetime

② Infinitesimal

$$X^{\mu}_{New}(P) = X^{\mu}_{old}(P) + \varepsilon V^{\mu}(P)$$

where V^{μ} is a vector field

We know that $g_{\mu\nu}^{New} = g_{\mu\nu}^{old} - \varepsilon \mathcal{L}_{\vec{V}}\, g_{\mu\nu}^{old}$

$$= g_{\mu\nu}^{old} - \varepsilon \left[V^{\alpha} g_{\mu\nu,\alpha} + g_{\alpha\nu} V^{\alpha}{}_{,\mu} + g_{\alpha\mu} V^{\alpha}{}_{,\nu} \right]$$

So $\boxed{h_{\mu\nu}^{New} = h_{\mu\nu}^{old} - V_{\mu,\nu} - V_{\nu,\mu} + \mathcal{O}(h \cdot \varepsilon)}$

Gauge XFormation

like $A_{\mu} \to A_{\mu} + \psi_{,\mu}$ in E+M

$$\text{Now}\quad h^{new} = h_{old} - 2V^{\alpha}{}_{,\alpha}$$

So $$\bar{h}^{new}_{\alpha\beta} = h^{new}_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta} h^{new}$$

$$= h^{old}_{\alpha\beta} - V_{\alpha,\beta} - V_{\beta,\alpha} - \tfrac{1}{2}\eta_{\alpha\beta}(h^{old} - 2V^{\gamma}{}_{,\gamma})$$

$$= \bar{h}^{old}_{\alpha\beta} - V_{\alpha,\beta} - V_{\beta,\alpha} + \eta_{\alpha\beta} V^{\gamma}{}_{,\gamma}$$

Note that $$\bar{h}^{new}_{\alpha\beta}{}^{,\beta} = \bar{h}^{old}_{\alpha\beta}{}^{,\beta} - V_{\alpha,\beta}{}^{\beta} - \underbrace{V_{\beta,\alpha}{}^{\beta} + \eta_{\alpha\beta} V^{\gamma}{}_{,\gamma}{}^{\beta}}_{\text{cancel}}$$

$V^{\gamma}{}_{,\gamma\alpha}$

So can choose gauge xformation so that

$$\boxed{\bar{h}^{new}_{\alpha\beta,}{}^{\beta} = 0}$$

Lorentz Gauge

by choosing $\vec{V}$ s.t. $V_{\alpha,\beta}{}^{\beta} = \bar{h}^{old}_{\alpha\beta}{}^{,\beta}$

wave equation

Unique up to $V_{\alpha,\beta}{}^{\beta} = 0$ homogeneous solution.

So in Lorentz Gauge

$$\boxed{\begin{gathered}\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = -16\pi T_{\mu\nu} \\ \bar{h}_{\mu\nu}{}^{,\nu} = 0\end{gathered}}$$

Field eqs. in Lorentz Gauge

$$\left(\begin{gathered}\text{like}\quad \Box A^{\mu} = 4\pi J^{\mu} \\ A^{\alpha}{}_{,\alpha} = 0\end{gathered}\right)$$

In curvilinear coords, $\eta_{\mu\nu} \rightarrow g_{\mu\nu}^{\text{Flat}}$

$h_{\mu\nu}$ is tensor with respect to

So
$$\boxed{\begin{gathered}\bar{h}_{\mu\nu;\alpha}{}^{\alpha} = -16\pi T_{\mu\nu} \\ \bar{h}_{\mu\nu}{}^{;\nu} = 0\end{gathered}}$$

New phenomenon

In vacuum, $\bar{h}_{\mu\nu;\alpha}{}^{\alpha} = 0$ $\qquad \bar{h}_{\mu\nu}{}^{;\nu} = 0$

admits wave solutions

example $\quad \bar{h}_{xx} = \bar{h}_{xx}(t-z)$, $\bar{h}_{xy} = \bar{h}_{xy}(t-z)$, $\bar{h}_{yy} = \bar{h}_{yy}(t-z)$

$\bar{h}_{\mu 0} = \bar{h}_{\mu z} = 0$

Gravitational waves $\quad$ not present in Newtonian gravity.

In general, solution is $\quad \bar{h}_{\mu\nu}(t, \underset{\sim}{x}) = 4 \cdot \int \frac{T_{\mu\nu}(t - |\underset{\sim}{x} - \underset{\sim}{x}'|, \underset{\sim}{x}')\, d^3x'}{|\underset{\sim}{x} - \underset{\sim}{x}'|}$

retarded time solution