Last time:

$$G_{\alpha\beta} = K\, T_{\alpha\beta}$$

— Fixes only 6 degrees of freedom of metric, $g_{\alpha\beta}$ because $G^{\alpha\beta}{}_{;\beta} = 0$

Think of $g_{\alpha\beta}$ like A_α in EM

$G_{\alpha\beta}$ like $F_{\alpha\beta}$ in EM

EM: can always change $A_\alpha \to A_\alpha + \phi_{,\alpha}$ w/out changing $F_{\alpha\beta}$

Maxwell's eqs. determine only some of A_α (up to gauge)

GR: can change $g_{\mu\nu} \to g_{\bar{\mu}\bar{\nu}}$ via coord xformation

w/out changing physics (comps of $G_{\alpha\beta}$ change too)

Einstein's Eqs determine $g_{\mu\nu}$ only up to coord xformations.

<u>Example</u> $T_{\alpha\beta} = 0$ Flat space

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

or

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$$

or

$$ds^2 = dv^2 - v^2 du^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$$

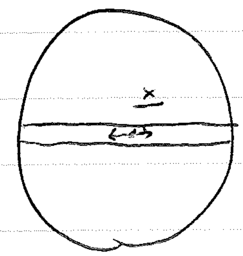
all flat space.

But unlike EM, where you can eliminate A_μ, in GR you can't eliminate $g_{\mu\nu}$

The constant K

Bore a hole in the Earth, drop a particle inside.

Let it oscillate



Geodesic Deviation in LLF at center of Earth, small oscillations:

$$\frac{d^2x^i}{dt^2} = R^i{}_{00j}x^j$$

or $$\frac{d^2x}{dt^2} = R^x{}_{00x}\,X$$

Newtonian physics:

$$\frac{d^2x}{dt^2} = \frac{-G}{X^2}\,M(\text{inside sphere of radius } x)$$

$$= \frac{-G}{x^2}\left(\frac{4}{3}\pi x^3 \rho\right)$$

$$= -\frac{4\pi}{3}\rho x \qquad (\text{let } G=1)$$

Compare: $R^x{}_{00x} = -\frac{4\pi}{3}\rho = -R^x{}_{0x0}$

$$\Rightarrow R^\alpha{}_{0\alpha 0} = \boxed{R_{00} = 4\pi\rho}$$

Compare with $G_{\alpha\beta} = KT_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$

Take trace: $R - 2R = KT = -R$

$$\text{So} \quad R_{00} = KT_{00} - \frac{1}{2} g_{00} K T$$

$$= K\left(T_{00} + \frac{1}{2}\left(T^0{}_0 + T^k{}_k\right)\right)$$

$$= \frac{1}{2} K T_{00} + \frac{1}{2} K T^k{}_k$$

Why? speed of sound $\sim \left(\frac{P}{\rho}\right)^2 \sim \frac{T_{ij}}{T_{00}} \ll 1$

$T^k{}_k \ll T^0{}_0$

$$\text{So} \quad R_{00} = \frac{1}{2} K T_{00} = \frac{1}{2} K \rho$$

$K = 8\pi$

$$\boxed{G_{\mu\nu} = 8\pi T_{\mu\nu}}$$ Einsteins equations

Cosmological Constant

Einstein considered evolution of the universe.

He assumed:
- Uniform density ρ
- Homogeneous
- Isotropic
- Closed universe
- Perfect fluid

$\Rightarrow$ Restricts metric to $-dt^2 + a^2(t)\left[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$

3-sphere

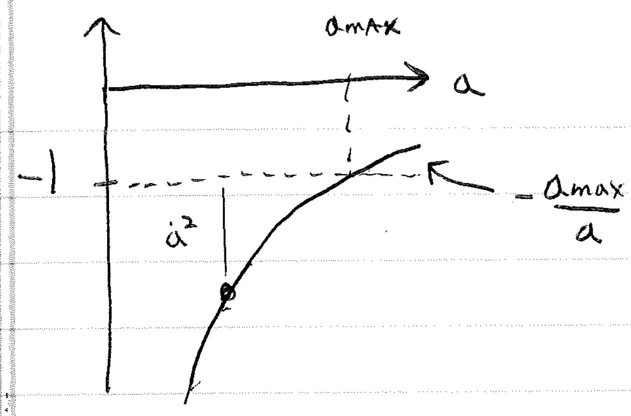
$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \quad \Rightarrow \quad \begin{cases} \left(\dfrac{da}{dt}\right)^2 + 1 = \dfrac{8}{3}\pi\rho a^2 \\ 2a\dfrac{d^2a}{dt^2} = -\left(\dfrac{da}{dt}\right)^2 - 1 - 8\pi P a^2 \end{cases}$$

Also $T^{\alpha\beta}{}_{;\beta} = 0 \quad \Rightarrow \quad \frac{d}{dt}(\rho a^3) = -P\frac{d}{dt}(a^3)$

For present universe, $P \ll \rho$ ("matter dominated")

so $\frac{d}{dt}(\rho a^3) = 0$ or $\rho a^3 = \text{const} = \frac{3}{8\pi}a_{max}$

So $G_{\alpha\beta} = 8\pi T_{\alpha\beta} \Rightarrow$
$$\boxed{\begin{aligned} \left(\frac{da}{dt}\right)^2 - \frac{a_{max}}{a} &= -1 \\ \frac{d^2a}{dt^2} &= \frac{-a_{max}}{2a^2} \end{aligned}}$$



Like energy diagram

$$\underbrace{\left(\frac{da}{dt}\right)^2}_{\text{Kinetic}} \underbrace{- \frac{a_{max}}{a}}_{\text{Potential}} = \underbrace{-1}_{\text{Total}}$$

$\Rightarrow$ No static solution!

Universe expands to $a = a_{max}$, Then contracts.

What did Einstein do?

Give up "no curvature in empty space"

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$$

cosmological constant.
Can get static universe!

13 years later (1929) Hubble discovered universe is expanding.

Einstein's "Greatest Blunder".

Λ has 2 effects: - Effective density $\rho = \frac{\Lambda}{8\pi}$

Adds

$\left(\text{Think } T_{ab} = \begin{pmatrix} \rho & & \\ & P & \\ & & P \end{pmatrix}\right)$ - Effective negative pressure $P = \frac{-\Lambda}{8\pi}$

Λ Thought to be zero $1929 \simeq 1998$

Since late 1990s, observations show expansion of Universe accelerating

Can be (so far) explained by $\Lambda \sim 1.3 \times 10^{-56}\ cm^{-2}$

$\sim 3 \times 10^{-46}\ M_\odot^{-2}$

"dark energy"

Λ is small

No effect on lab experiments, black holes, non cosmological astrophysics.

Only in low density large regions (cosmology) does it matter.

~70% of mass-energy in universe.

Lie Derivative

Def A Suppose we have a vector field $\vec{V}$

Suppose we choose coordinates such that $\vec{V} = \frac{\partial}{\partial x^1}$

i.e. one of the coord basis vectors is $\vec{V}$.

Lie derivative along $\vec{v}$

Then in this coord system, define $\mathcal{L}_{\vec{v}} \overset{\approx}{T} = \frac{\partial T^{\alpha\beta\ldots}{}_{\gamma\delta\ldots}}{\partial x^1} \vec{e}_\alpha \otimes \vec{e}_\beta \otimes \tilde{\omega}^\gamma \otimes \tilde{\omega}^\delta \ldots$

Tensor field

i.e. $$\left(\mathcal{L}_{\vec{v}} \overset{\approx}{T}\right)^\alpha{}_\beta = \frac{\partial}{\partial x^1}\left(T^\alpha{}_\beta\right)$$

Def B

Scalar fields: define $\mathcal{L}_{\vec{v}} f = \partial_{\vec{v}} f$ directional deriv.

Vector fields: define $\mathcal{L}_{\vec{v}} \vec{u} = [\vec{v}, \vec{u}]$

Demand Leibnitz Rule, commutation w/ contraction, linearity

Show defs A + B same for vectors:

IF $\vec{v} = \frac{\partial}{\partial x^1}$ then $V^\alpha = \begin{cases} 1 & \alpha = 1 \\ 0 & \text{otherwise} \end{cases}$

so $$[\vec{v}, \vec{u}]^\beta = \underbrace{V^\alpha}_{\delta^\alpha_1} \frac{\partial u^\beta}{\partial x^\alpha} - u^\alpha \underbrace{\cancel{\frac{\partial V^\beta}{\partial x^\alpha}}}_{0}$$

because V^α const (Coord basis)

$$= \frac{\partial u^\beta}{\partial x^1}$$

agrees with Def A.

Lie Derivative of 1-Form?

$$\mathcal{L}_{\vec{V}} \langle \vec{U}, \tilde{\omega} \rangle = \vec{V} \langle \vec{U}, \tilde{\omega} \rangle = V^{\beta}(\nabla_{\beta} U^{\alpha})\omega_{\alpha} + V^{\beta}(\nabla_{\beta}\omega_{\alpha})U^{\alpha} \quad ①$$

But also $\mathcal{L}_{\vec{V}} \langle \vec{U}, \tilde{\omega} \rangle = \langle \mathcal{L}_{\vec{V}} \vec{U}, \tilde{\omega} \rangle + \langle \vec{U}, \mathcal{L}_{\vec{V}} \tilde{\omega} \rangle$

$$= \langle [\vec{V}, \vec{U}], \tilde{\omega} \rangle + \langle \vec{U}, \mathcal{L}_{\vec{V}} \tilde{\omega} \rangle$$

$$= (V^{\alpha} \nabla_{\alpha} U^{\beta} - U^{\alpha} \nabla_{\alpha} V^{\beta}) \omega_{\beta}$$

$$+ U^{\alpha} (\mathcal{L}_{\vec{V}} \tilde{\omega})_{\alpha} \quad ②$$

compare ① + ②

$$U^{\alpha}(\mathcal{L}_{\vec{V}} \tilde{\omega})_{\alpha} - U^{\alpha}(\nabla_{\alpha} V^{\beta})\omega_{\beta} = V^{\beta} U^{\alpha} \nabla_{\beta} \omega_{\alpha}$$

must be true for all U^{α}

$$\Rightarrow \boxed{(\mathcal{L}_{\vec{V}} \tilde{\omega})_{\alpha} = V^{\beta} \nabla_{\beta} \omega_{\alpha} + \omega_{\beta} \nabla_{\alpha} V^{\beta}}$$

In general, $\left(\mathcal{L}_{\vec{V}} \tilde{\tilde{T}}\right)^{\alpha\beta\ldots}{}_{\mu\nu\ldots} = V^{\lambda} T^{\alpha\beta\ldots}{}_{\mu\nu\ldots;\lambda}$

$$+ T^{\alpha\beta\ldots}{}_{\lambda\nu\ldots} V^{\lambda}{}_{;\mu} + \ldots$$

$$- T^{\lambda\beta\ldots}{}_{\mu\nu\ldots} V^{\alpha}{}_{;\lambda} - \ldots$$

Infinitesimal Coordinate Transformations

Let $X^{\mu}_{New}(P) = X^{\mu}_{old}(P) + \varepsilon V^{\mu}(P)$ $\qquad \vec{V}$ = vector Field

ε = small.

Then suppose you have a scalar Function f

Then $f^{New}(X^{\alpha}_{New}(P)) = f^{old}(X^{\alpha}_{old}(P))$ $\qquad$ scalars invariant at P

$$= f^{old}(X^{\alpha}_{new}(P) - \varepsilon V^{\alpha}(P))$$

$$= f^{old}(X^{\alpha}_{new}(P)) - \varepsilon V^{\alpha} f_{,\alpha}$$

$\frac{df}{d\lambda}$ if $\vec{V} = \frac{d}{d\lambda}$

How about (components of) a tensor Field?

$$T^{\alpha\ New}_{\ \ \varrho}(X^{\lambda}_{New}(P)) = \underbrace{\frac{\partial X^{\nu}_{old}}{\partial X^{\varrho}_{New}} \frac{\partial X^{\alpha}_{new}}{\partial X^{\mu}_{old}}}_{\text{Transformation law}} T^{\mu\ \ old}_{\ \nu}(X^{\lambda}_{old}(P))$$

$$= (\delta^{\nu}_{\varrho} - \varepsilon V^{\nu}_{,\varrho})(\delta^{\alpha}_{\mu} + \varepsilon V^{\alpha}_{,\mu}) T^{\mu\ \ old}_{\ \nu}(X^{\lambda}_{New}(P) - \varepsilon V^{\lambda}(P))$$

$$= T^{\alpha\ old}_{\ \ \varrho}(X^{\lambda}_{New}(P)) - \varepsilon V^{\nu}_{,\varrho} T^{\alpha\ \ old}_{\ \nu}(X^{\lambda}_{new}(P))$$

$$+ \varepsilon V^{\alpha}_{,\mu} T^{\mu\ old}_{\ \varrho}(X^{\lambda}_{new}(P))$$

$$- \varepsilon T^{\alpha}_{\ \varrho,\lambda} V^{\lambda}$$

Notice This is $T^{\alpha\ New}_{\ \ \varrho}(X^{\lambda}_{New}) = T^{\alpha\ old}_{\ \ \varrho}(X^{\lambda}_{New}) - (\mathcal{L}_{\vec{V}} \overset{=}{T})^{\alpha}_{\ \varrho}(X^{\lambda}_{New})$

- Lie derivative $\mathcal{L}_{\vec{V}}(T)$ represents change in components of T under infinitesimal coord. transform.