Other curvature tensors

Ricci $\boxed{R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta}}$ 10 indep. components

symmetric

$$R_{\alpha\beta} = g^{\gamma\delta} R_{\gamma\alpha\delta\beta}$$
$$= g^{\gamma\delta} R_{\delta\beta\gamma\alpha}$$
$$= R_{\beta\alpha}$$

Scalar Curvature

$$\boxed{R = R^{\alpha}{}_{\alpha}}$$

Einstein Curvature

$$\boxed{G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R}$$

symmetric

Note: $G_{\alpha}{}^{\beta}{}_{;\beta} = 0$ Contracted Bianchi identity

why?

Bianchi: $R^{\alpha}{}_{\beta\gamma\delta;\varepsilon} + R^{\alpha}{}_{\beta\delta\varepsilon;\gamma} + R^{\alpha}{}_{\beta\varepsilon\gamma;\delta} = 0$

contraction $\alpha\gamma$ $\Rightarrow$ $R_{\beta\delta;\varepsilon} + R^{\alpha}{}_{\beta\delta\varepsilon;\alpha} - R_{\beta\varepsilon;\delta} = 0$

contraction $\beta\delta$ $\Rightarrow$ $R_{;\varepsilon} - R_{\varepsilon}{}^{\alpha}{}_{;\alpha} - R^{\beta}{}_{\varepsilon;\beta} = 0$

$$\Rightarrow \quad R_{;\varepsilon} - 2R_{\varepsilon}{}^{\alpha}{}_{;\alpha} = 0$$

But $G_{\alpha}{}^{\beta}{}_{;\beta} = R_{\alpha}{}^{\beta}{}_{;\beta} - \frac{1}{2}(g_{\alpha}{}^{\beta}R)_{;\beta}$

$$= R_{\alpha}{}^{\beta}{}_{;\beta} - \frac{1}{2}R_{;\beta} = 0$$

Weyl Curvature

$$C^{\alpha\beta}{}_{\gamma\delta} = R^{\alpha\beta}{}_{\gamma\delta} - 2\delta^{[\alpha}_{[\gamma}R^{\beta]}_{\delta]} + \frac{1}{3}R\,\delta^{[\alpha}_{[\gamma}\delta^{\beta]}_{\delta]}$$

Same symmetries as $R_{\alpha\beta\gamma\delta}$

but contract any 2 indices $\rightarrow 0$.

10 indep components

Riemann can be decomposed into Ricci + Weyl

"Trace" Part — "Trace Free" Part

- Weyl vanishes in $\dim < 4$

- IF $C^{\alpha\beta}{}_{\gamma\delta} = 0 \iff$ "conformally flat"

$\iff$ Can find coords such that

$$ds^2 = e^{2\phi}\left(\eta_{\mu\nu}\,dx^{\mu}dx^{\nu}\right)$$

$$\phi = \phi(x^{\alpha})$$

Flat Space → Curved Space

Einstein Equivalence principle: in every LLF, all nongravitational physics is that of SR.

How to implement: ("comma → semicolon rule")

If you write a physical law in geometric form in SR,
The same law applies in GR, except $\partial_\mu \to \nabla_\mu$
or $A_{,\beta} \to A_{;\beta}$

Example: in SR, $\nabla \cdot \tilde{\tilde{T}} = 0$ meaning $T^{\alpha\beta}{}_{,\beta} = 0$

$\tilde{\tilde{T}}$ = stress-energy tensor.

- so in LLF, at origin $T^{\hat{\alpha}\hat{\beta}}{}_{,\hat{\beta}} = 0$ (E.P.)

- but in LLF at origin $T^{\hat{\alpha}\hat{\beta}}{}_{,\hat{\beta}} = T^{\hat{\alpha}\hat{\beta}}{}_{;\hat{\beta}} = 0$

- Geometric law, true in LLF ⇒ True in all frames

$$\Rightarrow \quad \underline{T^{\alpha\beta}{}_{;\beta} = \nabla \cdot \tilde{\tilde{T}} = 0}$$

Similarly, Maxwell: $F^{\alpha\beta}{}_{,\beta} = 4\pi J^\alpha \Rightarrow F^{\alpha\beta}{}_{;\beta} = 4\pi J^\alpha$

$F_{[\alpha\beta,\gamma]} = 0 \Rightarrow F_{[\alpha\beta;\gamma]} = 0$

$m a^\alpha = F^{\alpha\beta} q u_\beta \Rightarrow m a^\alpha = F^{\alpha\beta} q u_\beta$

Ambiguities [curvature coupling]

Example
In SR, consider Maxwell: $F_{[\alpha\beta,\gamma]} = 0 \qquad F^{\alpha\beta}{}_{,\beta} = 4\pi J^\alpha$

define 4-potential A^α so $F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}$

$\Rightarrow F_{[\alpha\beta,\gamma]} = 0$ automatically

also $F^{\alpha\beta}{}_{,\beta} = 4\pi J^\alpha$

$$\Rightarrow A^\beta{}_{,\alpha\beta} - A_{\alpha,}{}^\beta{}_\beta = 4\pi J_\alpha \qquad ①$$

In SR This is same as $A^\beta{}_{,\beta\alpha} - A_{\alpha,}{}^\beta{}_\beta = 4\pi J_\alpha$ ②

SR ⇒ GR

either $$A^\beta{}_{;\alpha\beta} - A_{\alpha;}{}^\beta{}_\beta = 4\pi J_\alpha \qquad ①'$$

or $$A^\beta{}_{;\beta\alpha} - A_{\alpha;}{}^\beta{}_\beta = 4\pi J_\alpha \qquad ②'$$

①' is same as $A^\beta{}_{;\beta\alpha} - A_{\alpha;}{}^\beta{}_\beta + R_{\alpha\beta}A^\beta = 4\pi J_\alpha$

Which is right? ①' or ②' ?

Here go back to basic law $F_{[\alpha\beta,\gamma]} = 0 \Rightarrow F_{[\alpha\beta;\gamma]} = 0$

$F^{\alpha\beta}{}_{,\beta} = 4\pi J^\alpha \Rightarrow F^{\alpha\beta}{}_{;\beta} = 4\pi J^\alpha$

so define A_α s.t. $F_{\alpha\beta} = A_{\beta;\alpha} - A_{\alpha;\beta}$

Then ①' is correct.

Curvature coupling occurs when:

1. There are 2nd derivs + order matters
2. Can't treat systems as localized in LLF
   i.e. extended objects
   $\Rightarrow$ need physical reason   example: HW problem
   precession of equinoxes.

GR in 2 sentences:

① – Spacetime curvature tells matter how to move.
② – Matter tells spacetime how to curve.

We have seen ①:

all determined by metric $g_{\mu\nu}$

Freely falling particles move on geodesics $\nabla_{\vec{u}} \vec{u} = 0$

Geodesic deviation governed by curvature $\frac{D^2 \vec{n}}{d\tau^2} = \tilde{\tilde{R}}(-, \vec{u}, \vec{u}, \vec{n})$

Now for ②.

We want an equation curvature = mass-energy.

Want a tensor equation.

Tensor expression for mass-energy is $\tilde{\tilde{T}}$, stress-energy tensor.

So we want $\tilde{\tilde{A}} = K \tilde{\tilde{T}}$

Something to do with curvature

constant

or $A_{\alpha\beta} = K T_{\alpha\beta}$

$\tilde{A}$ should satisfy

1. Rank 2 + symmetric (because $\tilde{\tilde{T}}$ is)
2. Built from Riemann + metric
3. Linear in Riemann (it's a curvature!)
4. Vanishes in flat space
5. $A^{\alpha\beta}{}_{;\beta} = 0$

(Because $T^{\alpha\beta}{}_{;\beta} = 0$)
Conservation of mass-energy

Only 1 tensor satisfies the above:

Einstein curvature tensor $\tilde{\tilde{G}}$ $\Rightarrow$ $\boxed{G_{\mu\nu} = K T_{\mu\nu}}$

PDEs that determine metric $g_{\alpha\beta}$

Important Note on point 5.

$G^{\alpha\beta}{}_{;\beta} = 0$ automatically
(Bianchi identity)

$G^{\alpha\beta}{}_{;\beta}$ is not a consequence of $T^{\alpha\beta}{}_{;\beta}$

If it were, then the 10 components of $G_{\mu\nu} = K T_{\mu\nu}$ would determine all 10 components of $g_{\mu\nu}$.

But actually, $G^{\alpha\beta}{}_{;\beta} = 0$ and $T^{\alpha\beta}{}_{;\beta} = 0$ independently

4 equations

So $G_{\alpha\beta} = K T_{\alpha\beta}$ has only 6 degrees of freedom $\begin{pmatrix} 10 \text{ eqs} \\ -4 \text{ constraints} \end{pmatrix}$

$\Rightarrow$ only 6 components of $g_{\alpha\beta}$ determined.

Other 4 components represent freedom in choosing coordinates.