Geodesic Deviation

Equivalence principle: no such thing as uniform grav. Field
→ simply an accelerated frame.

Freely Falling particle $\nabla_{\vec{u}} \vec{u} = 0$

Particle sitting on earth $\nabla_{\vec{u}} \vec{u} = \vec{a}$ $a^i = g$ upward in rest frame

So is there a gravitational Field? Yes! tidal Field, tidal forces

∴ Nearby geodesics initially parallel don't stay parallel.

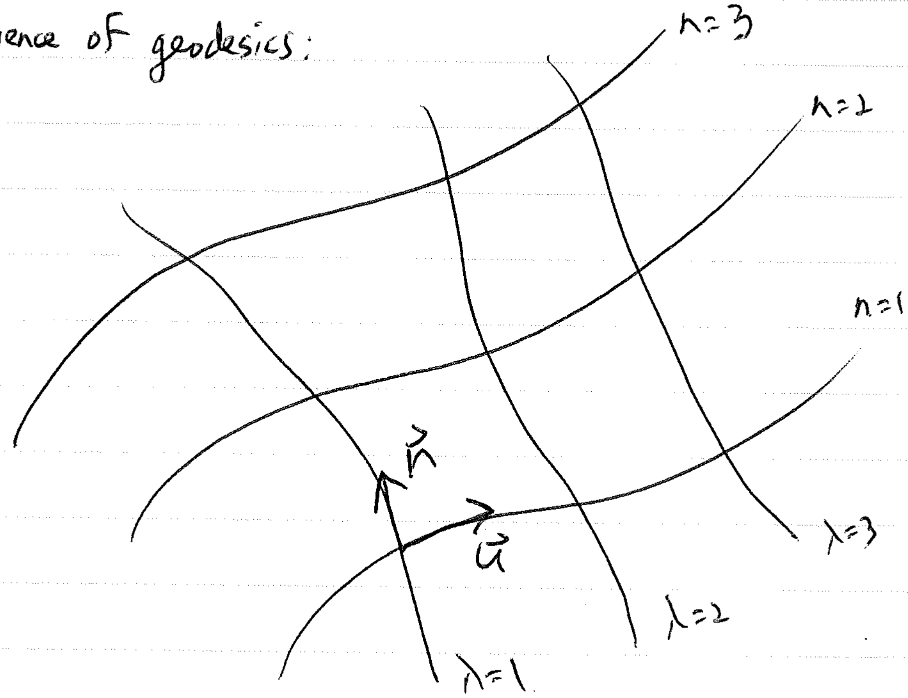
Derive geodesic deviation

Assume congruence of geodesics:

$X^\alpha = X^\alpha(\lambda, n)$

$n =$ which geodesic

$\lambda =$ affine param on each geodesic



$\vec{n} = \frac{d}{dn}$ connects points with same λ on different geodesics.

$\vec{u} = \frac{d}{d\lambda}$ tangent to geodesic.

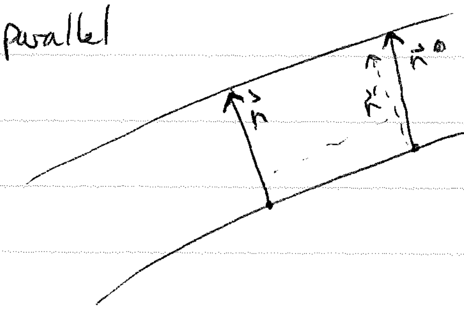
By construction, $\vec{u}, \vec{n}$ form coord basis

$$[\vec{u}, \vec{n}] = 0 = \nabla_{\vec{u}} \vec{n} - \nabla_{\vec{n}} \vec{u}$$

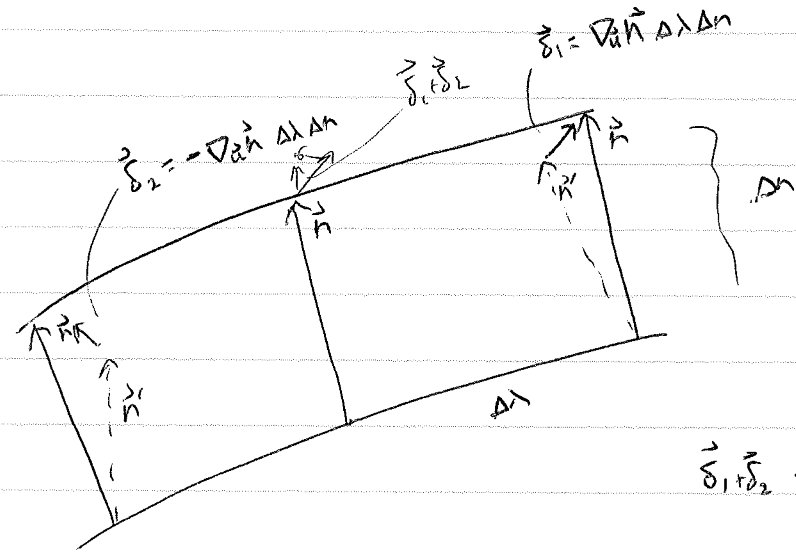
$\nabla_{\vec{u}} \vec{n}$ = how $\vec{n}$ changes along $\vec{u}$

= "velocity of nearby geodesic relative to this geodesic"

if $\nabla_{\vec{u}} \vec{n} = 0$, geodesics are parallel



$\nabla_{\vec{u}} \nabla_{\vec{u}} \vec{n}$ = "acceleration of nearby geodesic relative to this geodesic"



$$\vec{\delta}_1 + \vec{\delta}_2 = \Delta\lambda^2 \Delta n\, \nabla_{\vec{u}} \nabla_{\vec{u}} \vec{n}$$

So deviation from parallel geodesics $\vec{\delta}_1 + \vec{\delta}_2 = \nabla_{\vec{u}} \nabla_{\vec{u}} \vec{n} \, \Delta\lambda^2 \Delta n$

$$\text{or} \quad \nabla_{\vec{u}} \nabla_{\vec{u}} \vec{n} = \frac{\vec{\delta}_1 + \vec{\delta}_2}{\Delta\lambda^2 \Delta n} = \text{relative acceleration}$$

Now $\nabla_{\vec{u}} \nabla_{\vec{u}} \vec{n} = \cancel{\nabla_{\vec{u}} \nabla_{\vec{n}} \vec{u}}$ because $\nabla_{\vec{u}} \vec{n} = \nabla_{\vec{n}} \vec{u}$

i.e. $[\vec{u}, \vec{n}] = 0$

$$+ \tilde{R}(-, \vec{u}, \vec{u}, \vec{n})$$

$$- \left(\cancel{\nabla_{\vec{u}} \nabla_{\vec{n}}} - \nabla_{\vec{n}} \nabla_{\vec{u}} - \nabla_{[\vec{u}, \vec{n}]} \right) \vec{u}$$

$$= \underbrace{\nabla_{\vec{n}} \nabla_{\vec{u}} \vec{u}}_{0 \text{ geodesic}} + \underbrace{\nabla_{[\vec{u}, \vec{n}]} \vec{u}}_{[\vec{u}, \vec{n}] = 0} + \tilde{R}(-, \vec{u}, \vec{u}, \vec{n})$$

$$\Rightarrow \boxed{\nabla_{\vec{u}} \nabla_{\vec{u}} \vec{n} = \tilde{R}(-, \vec{u}, \vec{u}, \vec{n})}$$

or if we write $\nabla_{\vec{u}} \nabla_{\vec{u}} \vec{n} = \dfrac{D^2 \vec{n}}{d\tau^2}$

$$\boxed{\frac{D^2 \vec{n}}{d\tau^2} = R^{\alpha}{}_{\beta\gamma\delta} u^{\beta} u^{\gamma} n^{\delta}}$$

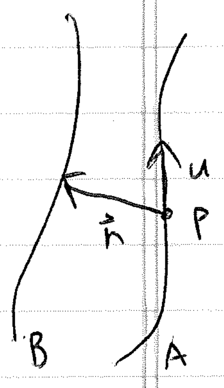
Geodesic deviation

Curvature causes nearby geodesics to converge/diverge. Tidal Force

To keep two slightly separated particles both parallel to each other, must apply force.

More on tidal Force

Assume observer A w/ 4-velocity $\vec{u}$.



Assume $\vec{n}$ is purely spatial as seen by A.

$$\vec{n} \cdot \vec{u} = 0$$

Then in rest Frame of $\vec{u}$, $\frac{D^2\vec{n}}{d\tau^2}$ = acceleration of Particle B measured by A.

$$= \frac{d^2\vec{n}}{dt^2} = \Delta\vec{a}$$

$$\frac{d^2 n^i}{dt^2} = R^i{}_{\beta\gamma\delta} u^\beta u^\gamma n^\delta$$
$$= R^i{}_{00j} n^j \quad \text{in rest Frame}$$

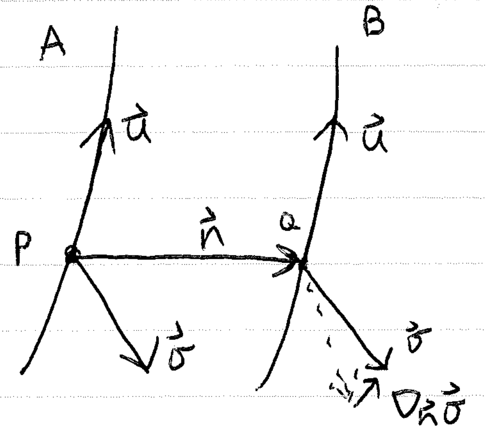
Sometimes we define $R^i{}_{0j0} = \mathcal{E}^i{}_j$

"Electric part of Riemann tensor"
"Tidal Field"

So $\Delta a^i = -\mathcal{E}^i{}_j n^j$

Another effect of curvature:

Local Frame Dragging



Observers A+B separated by vector $\vec{n}$.

As before, let $[\vec{n},\vec{u}] = 0$

Also, let $\vec{n}$ be spatial in Frame of A: $\quad \vec{n}\cdot\vec{u} = 0$

Also, let A+B be parallel at point P: $\quad \nabla_{\vec{n}}\vec{u} = 0$ at P.

Finally, A+B have gyroscopes, pointing along $\vec{\sigma}$.

$\vec{\sigma}$ is spatial: $\quad \vec{\sigma}\cdot\vec{u} = 0$

$\vec{\sigma}$ is FW transported: $\quad \nabla_{\vec{u}}\vec{\sigma} = (\vec{a}\cdot\vec{\sigma})\vec{u} - \cancel{(\vec{u}\cdot\vec{\sigma})}^{\,0}\vec{a}$

Difference between σ at Q and P is $\quad \nabla_{\vec{n}}\vec{\sigma}$

How does $\nabla_{\vec{n}}\vec{\sigma}$ change along world line?

Time rate of change of $\vec{\sigma}$ at Q, as measured by observer at P.

i.e. we want $\quad \nabla_{\vec{u}}\nabla_{\vec{n}}\vec{\sigma}$.

$$\nabla_{\vec{u}}\nabla_{\vec{n}}\vec{\sigma} = \nabla_{\vec{n}}\nabla_{\vec{u}}\vec{\sigma} + \tilde{\tilde{R}}(\_,\vec{\sigma},\vec{u},\vec{n})$$

$$= \vec{u}\,\nabla_{\vec{n}}(\vec{a}\cdot\vec{\sigma}) + \tilde{\tilde{R}}(\_,\vec{\sigma},\vec{u},\vec{n})$$

or $\left(\nabla_{\vec{u}} \nabla_{\hat{n}} \vec{\sigma}\right)^\alpha = R^\alpha{}_{\beta\gamma\delta} \sigma^\beta u^\gamma n^\delta + u^\alpha \nabla_{\hat{n}} (\vec{a} \cdot \vec{\sigma})$

Spatial part of this is $P^\alpha{}_\lambda R^\lambda{}_{\beta\gamma\delta} \sigma^\beta u^\gamma n^\delta$

where $P^\alpha{}_\lambda = \delta^\alpha{}_\lambda + u^\alpha u_\lambda$ Projection operator

$P^\alpha{}_\lambda u_\alpha$
$= P^\alpha{}_\lambda u^\lambda = 0$

Let's call this $P^\alpha{}_\lambda \left(\nabla_{\vec{u}} \nabla_{\hat{n}} \vec{\sigma}\right)^\lambda = \mathring{\sigma}^\alpha = P^\alpha{}_\lambda R^\lambda{}_{\beta\gamma\delta} \sigma^\beta u^\gamma n^\delta$

For simplicity, go to rest frame. Note $\mathring{\sigma}^\alpha \sigma_\alpha = 0$
$\mathring{\sigma}^\alpha u_\alpha = 0$

so write $\mathring{\sigma}^k = R^k{}_{i0j} \sigma^i n^j$ in rest frame $u^0 = 1$

Let $B_{ij} = \frac{1}{2} \varepsilon_{ipq} R^{pq}{}_{j0}$ "Magnetic part of Riemann tensor"

"Frame-drag Field"

Then $\varepsilon^k{}_{ps} B^s{}_q = R^k{}_{pq0} = -R^k{}_{p0q}$

$\Rightarrow \mathring{\sigma}^k = \varepsilon^k{}_{si} (B^s{}_j n^j) \sigma^i$

compare with $\dfrac{d\underset{\sim}{V}}{dt} = \underset{\sim}{\Omega} \times \underset{\sim}{V}$ Rotation w Frequency Ω

so $\dot{\underset{\sim}{\sigma}} = \underset{\sim}{\Omega} \times \underset{\sim}{\sigma}$ where $\boxed{\Omega^i = B^i{}_j n^j}$

Gyroscope at Q precesses relative to gyroscope at P.

Bianchi Identity

$$R^{\alpha}{}_{\beta[\gamma\delta;\varepsilon]} = 0$$

Why? in LLF $\quad R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta}$

and $R^{\alpha}{}_{\beta\gamma\delta;\varepsilon} = R^{\alpha}{}_{\beta\gamma\delta,\varepsilon}$

$$\begin{aligned} \text{so} \quad R^{\alpha}{}_{\beta[\gamma\delta;\varepsilon]} = {} & \cancel{\Gamma^{\alpha}{}_{\beta\delta,\gamma\varepsilon}} - \cancel{\Gamma^{\alpha}{}_{\beta\gamma,\delta\varepsilon}} \\ & + \cancel{\Gamma^{\alpha}{}_{\beta\gamma,\varepsilon\delta}} - \cancel{\Gamma^{\alpha}{}_{\beta\delta,\varepsilon\gamma}} \\ & + \cancel{\Gamma^{\alpha}{}_{\beta\varepsilon,\delta\gamma}} - \cancel{\Gamma^{\alpha}{}_{\beta\varepsilon,\gamma\delta}} = 0 \end{aligned}$$

Will be important later...