Components of Riemann tensor

$$R^{\alpha}{}_{\beta\gamma\delta} = \tilde{\tilde{\bar{R}}}(\tilde{\omega}^{\alpha}, \vec{e}_{\beta}, \vec{e}_{\gamma}, \vec{e}_{\delta})$$

$$= \left\langle \tilde{\omega}^{\alpha}, \left(\nabla_{\gamma}\nabla_{\delta} - \nabla_{\delta}\nabla_{\gamma} - \nabla_{[\vec{e}_{\gamma},\vec{e}_{\delta}]}\right)\vec{e}_{\beta} \right\rangle$$

But $\nabla_{\alpha}\vec{e}_{\beta} = \Gamma^{\gamma}_{\beta\alpha}\vec{e}_{\gamma}$ $\qquad [\vec{e}_{\alpha}, \vec{e}_{\beta}] = C_{\alpha\beta}{}^{\gamma}\vec{e}_{\gamma}$

so $$\left(\nabla_{\gamma}\nabla_{\delta} - \nabla_{\delta}\nabla_{\gamma} - \nabla_{[\vec{e}_{\gamma},\vec{e}_{\delta}]}\right)\vec{e}_{\beta}$$

$$= \nabla_{\gamma}\left(\Gamma^{\rho}_{\beta\delta}\vec{e}_{\rho}\right) - \nabla_{\delta}\left(\Gamma^{\rho}{}_{\beta\gamma}\vec{e}_{\rho}\right) - C_{\gamma\delta}{}^{\rho}\nabla_{\rho}\vec{e}_{\beta}$$

$$= \Gamma^{\rho}{}_{\beta\delta,\gamma}\vec{e}_{\rho} + \Gamma^{\rho}{}_{\beta\delta}\Gamma^{\lambda}_{\rho\gamma}\vec{e}_{\lambda} - \Gamma^{\rho}{}_{\beta\gamma,\delta}\vec{e}_{\rho} - \Gamma^{\rho}_{\beta\gamma}\Gamma^{\lambda}_{\rho\delta}\vec{e}_{\lambda}$$
$$- C_{\gamma\delta}{}^{\rho}\Gamma^{\lambda}_{\beta\rho}\vec{e}_{\lambda}$$

So $$\boxed{R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\rho\gamma}\Gamma^{\rho}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\rho\delta}\Gamma^{\rho}{}_{\beta\gamma} - C_{\gamma\delta}{}^{\rho}\Gamma^{\alpha}{}_{\beta\rho}}$$

Notation

tensor   comps. just functions   basis

Abstract notation: $\overset{\approx}{T} = T^{\beta}{}_{\gamma}\, \vec{e}_{\beta} \otimes \tilde{\omega}^{\gamma}$ $\qquad T^{\beta}{}_{\gamma} = \overset{\approx}{T}(\tilde{\omega}^{\beta}, \vec{e}_{\gamma})$

$$\nabla_{\vec{e}_\alpha} = \nabla_\alpha$$ cov deriv operator

So $\nabla_\alpha(\overset{\approx}{T}) = \nabla_\alpha(T^{\beta}{}_{\gamma}\, \vec{e}_{\beta} \otimes \tilde{\omega}^{\gamma})$

$$= \underbrace{(\nabla_\alpha T^{\beta}{}_{\gamma})}_{T^{\beta}{}_{\gamma,\alpha}} \vec{e}_{\beta} \otimes \tilde{\omega}^{\gamma} + T^{\beta}{}_{\gamma} \underbrace{(\nabla_\alpha \vec{e}_{\beta})}_{\Gamma^{\lambda}{}_{\beta\alpha}\vec{e}_{\lambda}} \otimes \tilde{\omega}^{\gamma} + T^{\beta}{}_{\gamma}\, \vec{e}_{\beta} \otimes \underbrace{(\nabla_\alpha \tilde{\omega}^{\gamma})}_{-\Gamma^{\gamma}{}_{\lambda\alpha}\tilde{\omega}^{\lambda}}$$

$$= \left(T^{\beta}{}_{\gamma,\alpha} + \Gamma^{\beta}{}_{\lambda\alpha} T^{\lambda}{}_{\gamma} - T^{\lambda}{}_{\gamma\alpha} T^{\beta}{}_{\lambda}\right) \vec{e}_{\beta} \otimes \tilde{\omega}^{\gamma}$$

Also $(\nabla \overset{\approx}{T})$ is a tensor $\nabla_{\vec{e}_\alpha} \overset{\approx}{T} = \nabla_\alpha \overset{\approx}{T} = (\nabla \overset{\approx}{T})(-,-,\vec{e}_\alpha)$

and $(\nabla \overset{\approx}{T})(\tilde{\omega}^{\beta}, \vec{e}_{\gamma}, \vec{e}_{\alpha}) = T^{\beta}{}_{\gamma;\alpha}$

Component notation: $T^{\beta}{}_{\gamma;\alpha} = T^{\beta}{}_{\gamma,\alpha} + \Gamma^{\beta}{}_{\lambda\alpha} T^{\lambda}{}_{\gamma} - \Gamma^{\lambda}{}_{\gamma\alpha} T^{\beta}{}_{\lambda}$

Covariant Derivatives don't commute

Let $\vec{v}$ be a vector. Then $\nabla\vec{v}$ is a rank 2 tensor

$\nabla\nabla\vec{v}$ is a rank 3 tensor.

$$\nabla\vec{v}(\tilde{\omega}^\alpha, \vec{e}_\beta) = V^\alpha{}_{;\beta} \qquad \nabla\nabla\vec{v}(\tilde{\omega}^\alpha, \vec{e}_\beta, \vec{e}_\gamma) = V^\alpha{}_{;\beta\gamma}$$

This is not $\nabla_\gamma \nabla_\beta \vec{V} = \nabla_{\vec{e}_\gamma} \nabla_{\vec{e}_\beta} \vec{V}$

i.e. $(\nabla_\gamma \nabla_\beta \vec{V})^\alpha \neq V^\alpha{}_{;\beta\gamma}$

Why?

$$\nabla_\gamma \nabla_\beta \vec{V} = \nabla_\gamma \left[\langle \vec{e}_\beta, \nabla\vec{V} \rangle \right]$$

$$= \underbrace{\langle \nabla_\gamma \vec{e}_\beta, \nabla\vec{V} \rangle} + \langle \vec{e}_\beta, \nabla_\gamma (\nabla\vec{V}) \rangle$$

$$= \nabla_{\nabla_\gamma \vec{e}_\beta} \vec{V} \qquad \langle \vec{e}_\beta, (\nabla\nabla\vec{V})(-,-,\vec{e}_\gamma) \rangle$$

(also, since $\nabla_\gamma \vec{e}_\beta = \Gamma^\lambda{}_{\beta\gamma} \vec{e}_\lambda$, $\nabla_{\nabla_\gamma \vec{e}_\beta} \vec{V}$ is $\Gamma^\alpha{}_{\beta\gamma} \nabla_\alpha \vec{V}$)

$$= \nabla_{\nabla_\gamma \vec{e}_\beta} \vec{V} + \nabla\nabla\vec{V}(-, \vec{e}_\beta, \vec{e}_\gamma)$$

So $\nabla_\gamma \nabla_\beta \vec{V} = \nabla_{\nabla_\gamma \vec{e}_\beta} \vec{V} + \nabla\nabla\vec{V}(-, \vec{e}_\beta, \vec{e}_\gamma)$

Another way to write the same thing:

$$\nabla_\gamma \nabla_\epsilon \vec{V} = \nabla_\gamma \left[\nabla_{\vec{e}_\epsilon} \vec{V}\right] = \nabla_\gamma \left[\nabla \vec{V}(-, \vec{e}_\epsilon)\right] = \nabla_\gamma \left[(\nabla \vec{V})^\alpha{}_\epsilon \, \vec{e}_\alpha\right]$$

$$= \nabla_\gamma \left((\nabla \vec{V})^\alpha{}_\epsilon\right) \vec{e}_\alpha + (\nabla \vec{V})^\alpha{}_\epsilon \nabla_\gamma \vec{e}_\alpha$$

$$= (\nabla \vec{V})^\alpha{}_{\epsilon,\gamma} \vec{e}_\alpha + (\nabla \vec{V})^\alpha{}_\epsilon \Gamma^\lambda{}_{\alpha\gamma} \vec{e}_\lambda$$

$$= \left((\nabla \vec{V})^\alpha{}_{\epsilon;\gamma} \cancel{- \Gamma^\alpha{}_{\lambda\gamma} (\nabla \vec{V})^\lambda{}_\epsilon} + \Gamma^\lambda{}_{\epsilon\gamma} (\nabla \vec{V})^\alpha{}_\lambda\right) \vec{e}_\alpha$$

$$\cancel{+ (\nabla \vec{V})^\lambda{}_\epsilon \Gamma^\alpha{}_{\lambda\gamma} \vec{e}_\alpha}$$

$$= \left((\nabla \vec{V})^\alpha{}_{\epsilon;\gamma} + \Gamma^\lambda{}_{\epsilon\gamma} (\nabla \vec{V})^\alpha{}_\lambda\right) \vec{e}_\alpha$$

$$= \left((\nabla \nabla \vec{V})^\alpha{}_{\epsilon\gamma} + (\nabla_\gamma \vec{e}_\epsilon)^\lambda (\nabla \vec{V})^\alpha{}_\lambda\right) \vec{e}_\alpha$$

$$= \nabla \nabla \vec{V}(-, \vec{e}_\epsilon, \vec{e}_\gamma) + \nabla_{(\nabla_\gamma \vec{e}_\epsilon)} \vec{V}$$

So $\nabla\nabla\vec{V}(-,\vec{e}_\beta,\vec{e}_\gamma) - \nabla\nabla\vec{V}(-,\vec{e}_\gamma,\vec{e}_\beta)$

$$= \nabla_\gamma\nabla_\beta\vec{V} - \nabla_\beta\nabla_\gamma\vec{V} \underbrace{\underbrace{- \nabla_{\nabla_\gamma\vec{e}_\beta}\vec{V} + \nabla_{\nabla_\beta\vec{e}_\gamma}\vec{V}}_{-\nabla_{(\nabla_\gamma\vec{e}_\beta - \nabla_\beta\vec{e}_\gamma)}\vec{V}}}_{-\nabla_{[\vec{e}_\gamma,\vec{e}_\beta]}\vec{V}}$$

So $\nabla\nabla\vec{V}(-,\vec{e}_\beta,\vec{e}_\gamma) - \nabla\nabla\vec{V}(-,\vec{e}_\gamma,\vec{e}_\beta) = \nabla_\gamma\nabla_\beta\vec{V} - \nabla_\beta\nabla_\gamma\vec{V} - \nabla_{[\vec{e}_\gamma,\vec{e}_\beta]}\vec{V}$

$$= R(-,\vec{V},\vec{e}_\gamma,\vec{e}_\beta)$$

or $\boxed{V^\alpha{}_{;\beta\gamma} - V^\alpha{}_{;\gamma\beta} = R^\alpha{}_{\delta\gamma\beta}V^\delta}$

In components, $\boxed{V^\alpha{}_{;\beta\gamma} - V^\alpha{}_{;\gamma\beta} = R^\alpha{}_{\delta\gamma\beta} V^\delta}$

What about 1-forms?

Similarly, $\nabla\nabla\tilde{\omega}(-,\vec{e}_\beta,\vec{e}_\gamma) - \nabla\nabla\tilde{\omega}(-,\vec{e}_\gamma,\vec{e}_\beta)$

$$= \left(\nabla_\gamma\nabla_\beta - \nabla_\beta\nabla_\alpha - \nabla_{[\vec{e}_\gamma,\vec{e}_\beta]}\right)\tilde{\omega}$$

↑ how to evaluate?

$$\left(\nabla_\gamma\nabla_\beta - \nabla_\beta\nabla_\gamma - \nabla_{[\vec{e}_\gamma,\vec{e}_\beta]}\right)\langle\tilde{\omega},\vec{V}\rangle = 0$$

↖ scalar

$$= \underbrace{\left\langle\left(\nabla_\gamma\nabla_\beta - \nabla_\beta\nabla_\gamma - \nabla_{[\vec{e}_\gamma,\vec{e}_\beta]}\right)\tilde{\omega},\vec{V}\right\rangle}_{V^\delta\left(\left(\nabla_\gamma\nabla_\beta - \nabla_\beta\nabla_\gamma - \nabla_{[\vec{e}_\gamma,\vec{e}_\beta]}\right)\tilde{\omega}\right)_\delta} + \underbrace{\left\langle\tilde{\omega},\left(\nabla_\gamma\nabla_\beta - \nabla_\beta\nabla_\gamma - \nabla_{[\vec{e}_\gamma,\vec{e}_\beta]}\right)\vec{V}\right\rangle}_{R^\alpha{}_{\delta\gamma\beta}V^\delta\omega_\alpha}$$

$$\text{So } \underbrace{\left[\left(\nabla_\gamma\nabla_\beta - \nabla_\beta\nabla_\gamma - \nabla_{[\vec{e}_\gamma,\vec{e}_\beta]}\right)\tilde{\omega}\right]_\delta}_{\omega_{\delta;\beta\gamma} - \omega_{\delta;\gamma\beta}} = -R^\alpha{}_{\delta\gamma\beta}\omega_\alpha$$

In general $T^{\alpha\dots}{}_{\delta\dots;\beta\gamma} - T^{\alpha\dots}{}_{\delta\dots;\gamma\beta} = R^\alpha{}_{\lambda\gamma\beta}T^{\lambda\dots}{}_{\delta\dots}$

$$- R^\lambda{}_{\delta\gamma\beta}T^{\alpha\dots}{}_{\lambda\dots}$$

$$+ \dots$$ one R for each index

Symmetries of Riemann tensor

① $R^{\alpha}{}_{\beta\gamma\delta} = -R^{\alpha}{}_{\beta\delta\gamma}$ From definition

② $R^{\alpha}{}_{[\beta\gamma\delta]} = 0$

Why? In LLF, $R^{\alpha}{}_{\beta\gamma\delta} = -\Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\beta\delta,\gamma}$

$$\text{so } R^{\alpha}{}_{[\beta\gamma\delta]} = \cancel{\Gamma^{\alpha}{}_{\beta\delta,\gamma}} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\gamma\beta,\delta} - \cancel{\Gamma^{\alpha}{}_{\gamma\delta,\beta}} + \cancel{\Gamma^{\alpha}{}_{\delta\gamma,\beta}} - \cancel{\Gamma^{\alpha}{}_{\delta\beta,\gamma}} = 0$$

③ $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$

Why? $g_{\alpha\beta;\gamma\delta} - g_{\beta\alpha;\delta\gamma} = 0 = -R^{\lambda}{}_{\alpha\delta\gamma}\, g_{\lambda\beta} - R^{\lambda}{}_{\beta\delta\gamma}\, g_{\alpha\lambda}$

$$= -R_{\beta\alpha\delta\gamma} - R_{\alpha\beta\delta\gamma}$$

④ From ①②③ $\Rightarrow$ $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$

⑤ From ①②③ $\Rightarrow$ $R_{[\alpha\beta\gamma\delta]} = 0$

Symmetries give 20 independent components for Riemann.

(Recall: 20 degrees of freedom left over when we failed to set $g_{\alpha\beta,\gamma\delta} = 0$ in LLF)