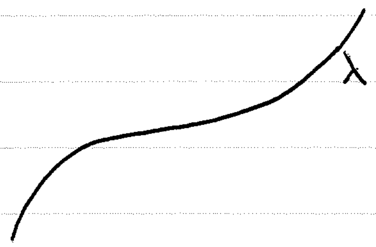
Extremality of geodesics



$$\text{Length of curve} = \int_{\text{curve}} ds$$

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$\text{so } \left(\frac{ds}{d\lambda}\right)^2 = g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}$$

Assume timelike curve, $ds^2 < 0$, so

$$\underset{\text{(proper time)}}{\text{Length}} = \int \sqrt{-ds^2} = \int \underbrace{\sqrt{-g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}}_{\text{Function of } x^\alpha,\ \dot{x}^\alpha = \frac{dx^\alpha}{d\lambda}} \, d\lambda$$

call it $L(x^\alpha, \dot{x}^\alpha)$

Extremize length $\delta[\text{Length}] = 0 \Rightarrow$ Euler-Lagrange eqs.

$$\frac{\partial}{\partial\lambda} \frac{\partial L}{\partial \dot{x}^\alpha} = \frac{\partial L}{\partial x^\alpha}$$

Now $$\frac{\partial L}{\partial \dot{x}^\alpha} = -L^{-1} g_{\alpha\beta} \dot{x}^\beta$$

$$\frac{\partial L}{\partial x^\alpha} = -\frac{1}{2} L^{-1} g_{\gamma\beta,\alpha} \dot{x}^\gamma \dot{x}^\beta$$

So $$-\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\alpha} + \frac{\partial L}{\partial x^\alpha} = 0$$

$$= \frac{d}{d\lambda}\left(L^{-1} g_{\alpha\beta}\dot{x}^\beta\right) - \frac{1}{2}L^{-1} g_{\gamma\beta,\alpha}\dot{x}^\gamma \dot{x}^\beta$$

$$= -L^{-2}\frac{dL}{d\lambda} g_{\alpha\beta}\dot{x}^\beta + L^{-1} g_{\alpha\beta}\ddot{x}^\beta + L^{-1} g_{\alpha\beta,\gamma}\dot{x}^\gamma \dot{x}^\beta$$
$$- \frac{1}{2}L^{-1} g_{\gamma\beta,\alpha}\dot{x}^\gamma \dot{x}^\beta$$

$$\Rightarrow \quad g_{\alpha\beta}\ddot{x}^\beta + \underbrace{\frac{1}{2}\left(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\gamma\beta,\alpha}\right)}_{\Gamma_{\alpha\beta\gamma}}\dot{x}^\gamma \dot{x}^\beta = L^{-1}\frac{dL}{d\lambda} g_{\alpha\beta}\dot{x}^\beta$$

Multiply by $g^{\sigma\alpha}$ $\Rightarrow$ $$\boxed{\ddot{x}^\sigma + \Gamma^\sigma_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma = L^{-1}\frac{dL}{d\lambda}\dot{x}^\sigma}$$

<u>Geodesic eq.</u>

IF λ is affine parameter,

then $\lambda = as + b$ distance along curve

so $L = \frac{ds}{d\lambda} = \text{const}$

so $\frac{dL}{d\lambda} = 0$

$$\Rightarrow \quad \boxed{\ddot{x}^\sigma + \Gamma^\sigma_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma = 0}$$

IF $\vec{u}$ is 4-velocity of a particle,

Then proper time τ is an affine parameter

$$L = \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}} = \sqrt{-g_{\alpha\beta}u^\alpha u^\beta} = \sqrt{-\vec{u}\cdot\vec{u}} = 1$$

so $\frac{dL}{d\tau} = 0$

Fermi - Walker Transport

We know how a timelike observer transports his 4-velocity:

$$\nabla_{\vec{u}} \vec{u} = \vec{a} \qquad \text{4-acceleration}$$

Q: How does an observer transport his local *spatial* basis vectors that define his instantaneous LLF?

A: Assume the observer has basis vectors $\vec{e}_\alpha$

Assume they obey $\nabla_{\vec{u}} \vec{e}_\alpha = A_\alpha{}^\beta \vec{e}_\beta$ for unknown $A_\alpha{}^\beta$.

Solve for $A_\alpha{}^\beta$:

① let $\vec{e}_\alpha$ be orthonormal, i.e. $g_{\alpha\beta} = \eta_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta$

Metric Flat in LLF

$$0 = \nabla_{\vec{u}} \eta_{\alpha\beta} = \nabla_{\vec{u}} (\vec{e}_\alpha \cdot \vec{e}_\beta) = (\nabla_{\vec{u}} \vec{e}_\alpha) \cdot \vec{e}_\beta + \vec{e}_\alpha \cdot (\nabla_{\vec{u}} \vec{e}_\beta)$$

$$= A_\alpha{}^\gamma \cdot \vec{e}_\gamma \cdot \vec{e}_\beta + \vec{e}_\alpha \cdot A_\beta{}^\gamma \vec{e}_\gamma$$

$$= A_\alpha{}^\gamma \eta_{\gamma\beta} + A_\beta{}^\gamma \eta_{\alpha\gamma} \qquad = A_{\alpha\beta} + A_{\beta\alpha}$$

$\Rightarrow$ A is antisymmetric

② We want $\vec{e}_0 = \vec{u}$ so $\vec{e}_0$ is special.

If we define $V^\alpha = -A^{\alpha\beta} u_\beta = A^{\beta\alpha} u_\beta$

then $V^\alpha u_\alpha = 0$ because $A^{(\alpha\beta)} = 0$

also let $\omega^{\alpha\beta} = A^{\alpha\beta} - 2V^{[\alpha} u^{\beta]}$

then $\omega^{\alpha\beta} u_\beta = A^{\alpha\beta} u_\beta + V^\alpha = 0$

$u_\alpha \omega^{\alpha\beta} = u_\alpha A^{\alpha\beta} - V^\beta = 0$

So we can write $A^{\alpha\beta} = 2V^{[\alpha} u^{\beta]} + \omega^{\alpha\beta}$

Now $\nabla_{\vec{u}} \vec{u} = A_0{}^\beta \vec{e}_\beta$ but $A_0{}^\beta = -A^{0\beta}$

$= +u_\alpha A^{\alpha\beta}$

$= V^\beta$

but we know $\nabla_{\vec{u}} \vec{u} = \vec{a}$ so $\underline{\vec{V} = \vec{a}}$

$\Rightarrow A^{\alpha\beta} = 2a^{[\alpha} u^{\beta]} + \omega^{\alpha\beta}$

③ We want spatial vectors to be nonrotating

Note $\omega^{\alpha\beta}$ is spatial $(\omega^{\alpha\beta} u_\beta = u_\alpha \omega^{\alpha\beta} = 0)$ and antisymmetric, 3 components, represents spatial rotation,

$$\nabla_{\vec{u}} \vec{e}_i = A_i{}^\beta \vec{e}_\beta = \omega_i{}^j \vec{e}_j + a_i \vec{e}_0$$

If no rotation, $\omega^{\alpha\beta} = 0$

$$\boxed{\nabla_{\vec{u}} \vec{e}_\alpha = (a_\alpha u^\beta - u_\alpha a^\beta)\vec{e}_\beta}$$

Fermi-Walker transport

Physically: observer attaches $\vec{e}_i$ to gyroscopes

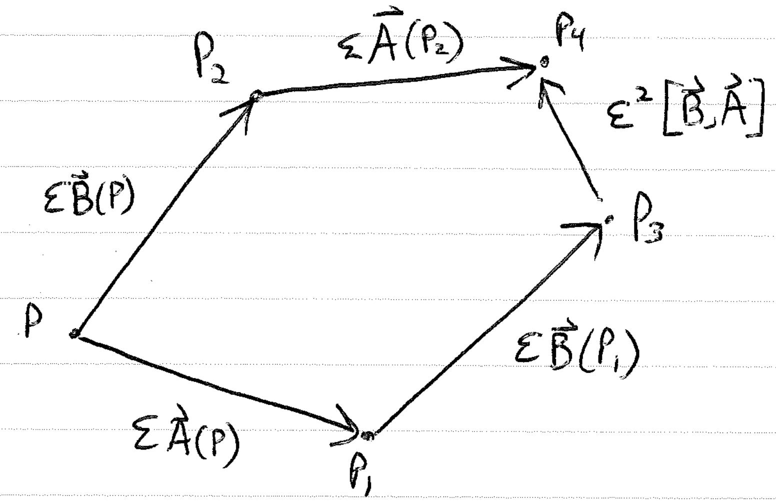
Generally $\frac{D\vec{w}}{d\tau} = (\vec{a}\otimes\vec{u} - \vec{u}\otimes\vec{a})\cdot\vec{w}$ $\vec{w}$ is FW Transported

Note: if $\vec{a} = 0$, FW transport = parallel transport

Curvature

Parallel transport a vector $\vec{V}$ around closed loop.

Specify loop by vector <u>Fields</u> $\vec{A}, \vec{B}$.



Introduce arbitrary vector field $\vec{W}$ that agrees with $\vec{V}$ at P. (Just for computation: will cancel!)

$$\delta \vec{V} = \left(\vec{V} - \vec{W}\right)_{\text{around loop}} \qquad \text{so} \quad -\delta V = \vec{W} - \vec{V}$$

$$-\delta \vec{V} \quad (P \to P_1) = \varepsilon \nabla_{\vec{A}} \vec{W} |_P$$
$$-\delta \vec{V} \quad (P_1 \to P_3) = \varepsilon \nabla_{\vec{B}} \vec{W} |_{P_1}$$
$$-\delta \vec{V} \quad (P_3 \to P_4) = \varepsilon^2 \nabla_{[\vec{B}, \vec{A}]} \vec{W} |_{P_3}$$
$$-\delta \vec{V} \quad (P_4 \to P_2) = -\varepsilon \nabla_A \vec{W} |_{P_2}$$
$$-\delta \vec{V} \quad (P_2 \to P) = -\varepsilon \nabla_B \vec{W} |_P$$

$$-\delta \vec{V}_{\text{loop}} = \varepsilon \left[\nabla_{\vec{A}} \vec{W} |_P - \nabla_{\vec{A}} \vec{W} |_{P_2} \right] + \varepsilon \left[\nabla_{\vec{B}} \vec{W} |_{P_1} - \nabla_{\vec{B}} \vec{W} |_P \right] + \varepsilon^2 \nabla_{[\vec{B}, \vec{A}]} \vec{W}$$
$$= -\varepsilon^2 \nabla_{\vec{B}} \nabla_{\vec{A}} \vec{W} + \varepsilon^2 \nabla_{\vec{A}} \nabla_{\vec{B}} \vec{W} + \varepsilon^2 \nabla_{[\vec{B}, \vec{A}]} \vec{W}$$

$$\text{So } -\delta\vec{V} = \varepsilon^2 \underbrace{\left(\nabla_{\vec{A}}\nabla_{\vec{B}} - \nabla_{\vec{B}}\nabla_{\vec{A}} - \nabla_{[\vec{A},\vec{B}]}\right)\vec{W}}_{\text{Vector}}$$

- We will show that $\tilde{\tilde{R}}(\_\_, \vec{W}, \vec{A}, \vec{B}) = \left(\nabla_{\vec{A}}\nabla_{\vec{B}} - \nabla_{\vec{B}}\nabla_{\vec{A}} - \nabla_{[\vec{A},\vec{B}]}\right)\vec{W}$

Riemann Tensor

is a tensor, and is indep. of $\vec{W}$ (as long as $\vec{W} = \vec{V}$ at P)

① show $\tilde{\tilde{R}}(-, f\vec{W}, \vec{A}, \vec{B}) = f\tilde{\tilde{R}}(-, \vec{W}, \vec{A}, \vec{B})$

$$\nabla_{\vec{A}}\nabla_{\vec{B}}(f\vec{W}) = \nabla_{\vec{A}}\left(\vec{W}\nabla_{\vec{B}}f + f\nabla_{\vec{B}}\vec{W}\right)$$

$$= \nabla_{\vec{A}}\vec{W}\,\nabla_{\vec{B}}f + \nabla_{\vec{A}}f\,\nabla_{\vec{B}}\vec{W} + \vec{W}\,\nabla_{\vec{A}}\nabla_{\vec{B}}f + f\,\nabla_{\vec{A}}\nabla_{\vec{B}}\vec{W}$$

$$\text{so } \left(\nabla_{\vec{A}}\nabla_{\vec{B}} - \nabla_{\vec{B}}\nabla_{\vec{A}}\right)(f\vec{W}) = f\left(\nabla_{\vec{A}}\nabla_{\vec{B}} - \nabla_{\vec{B}}\nabla_{\vec{A}}\right)W + \vec{W}\left(\nabla_{\vec{A}}\nabla_{\vec{B}} - \nabla_{\vec{B}}\nabla_{\vec{A}}\right)f \quad ①$$

$$\text{but } \nabla_{[\vec{A},\vec{B}]}(f\vec{W}) = f\nabla_{[\vec{A},\vec{B}]}\vec{W} + \vec{W}\,\nabla_{[\vec{A},\vec{B}]}f \quad ②$$

$$①-② \quad \tilde{\tilde{R}}(-, f\vec{W}, \vec{A}, \vec{B}) - f\tilde{\tilde{R}}(-, \vec{W}, \vec{A}, \vec{B})$$

$$= \vec{W}\underbrace{\left(\nabla_{\vec{A}}\nabla_{\vec{B}} - \nabla_{\vec{B}}\nabla_{\vec{A}} - \nabla_{[\vec{A},\vec{B}]}f\right)}_{0}$$

$$\left(\begin{aligned} \text{why? } \nabla_{\vec{A}}\nabla_{\vec{B}}f &= \vec{A}[\vec{B}(f)] \\ \nabla_{[\vec{A},\vec{B}]}f &= A[B(f)] - B(A(f)) \end{aligned}\right)$$

② Similarly $\tilde{R}(-,\vec{w}, f\vec{A},\vec{B}) = f\tilde{\tilde{R}}(-,w,\vec{A},\vec{B})$

same for last slot.

③ $\tilde{\tilde{R}}(-,\vec{w}+\vec{u},\vec{A},\vec{B}) = \tilde{\tilde{R}}(-,\vec{w},\vec{A},\vec{B}) + \tilde{\tilde{R}}(-,\vec{u},\vec{A},\vec{B})$

④ $\tilde{\tilde{R}}(-,\vec{w},\vec{A}+\vec{C},\vec{B}) = \tilde{\tilde{R}}(-,\vec{w},\vec{A},\vec{B}) + \tilde{\tilde{R}}(-,\vec{w},\vec{C},\vec{B})$

same for last slot.

Key property: Assume $\vec{w}' = \vec{w} + \underset{\sim}{c}^{\alpha}\vec{e}_{\alpha}$

$c^{\alpha} = 0$ at P

otherwise arbitrary

$$\Rightarrow \tilde{\tilde{R}}(-,\vec{w}',\vec{A},\vec{B}) = \tilde{\tilde{R}}(-,\vec{w},\vec{A},\vec{B}) + c^{\alpha}\tilde{\tilde{R}}(-,\quad,\vec{A},\vec{B})$$

$$= \tilde{\tilde{R}}(-,\vec{w},\vec{A},\vec{B}) \text{ at } P.$$

So $\tilde{\tilde{R}}(-,\vec{w},\vec{A},\vec{B})$ at P

depends on $\vec{w}$ <u>only</u> at point P

Can show
Same for $\vec{A},\vec{B}$

$\Rightarrow \tilde{\tilde{R}}(-,\vec{w},\vec{A},\vec{B})$ is a tensor.

So $\delta\vec{V} + \varepsilon^2\tilde{\tilde{R}}(-,\vec{V},\vec{A},\vec{B}) = 0$