Components of $\nabla \overset{\approx}{T}$ in arbitrary basis

$$\nabla_{\vec{u}} \overset{\approx}{T} = \nabla_{\vec{u}} \left[T^{\alpha}{}_{\beta}\, \vec{e}_{\alpha} \otimes \tilde{\omega}^{\beta} \right]$$

$$= \left(\nabla_{\vec{u}} T^{\alpha}{}_{\beta}\right) \vec{e}_{\alpha} \otimes \tilde{\omega}^{\beta} + \left(\nabla_{\vec{u}} \vec{e}_{\alpha} \otimes \tilde{\omega}^{\beta}\right) T^{\alpha}{}_{\beta}$$
$$+ \left(\vec{e}_{\alpha} \otimes \nabla_{\vec{u}} \tilde{\omega}^{\beta}\right) T^{\alpha}{}_{\beta}$$

– What is $\nabla_{\vec{u}} \vec{e}_{\alpha}$?

$$\nabla_{\vec{u}} \vec{e}_{\alpha} = u^{\beta} \nabla_{\beta} \vec{e}_{\alpha} = u^{\beta} \nabla_{\vec{e}_{\beta}} \vec{e}_{\alpha}$$

$\nabla_{\vec{e}_{\beta}} \vec{e}_{\alpha}$ is a vector, so can be expressed as sum of basis vectors.

<u>Define</u> connection coefficients $\boxed{\nabla_{\vec{e}_{\beta}} \vec{e}_{\alpha} = \Gamma^{\lambda}{}_{\alpha\beta}\, \vec{e}_{\lambda}}$

or $\Gamma^{\lambda}{}_{\alpha\beta} = \langle \tilde{\omega}^{\lambda}, \nabla_{e_{\beta}} \vec{e}_{\alpha} \rangle$

Convention: deriv index on $\Gamma^{\lambda}{}_{\alpha\beta}$ is last index

Can compute $\Gamma^{\lambda}{}_{\alpha\beta}$ from basis vectors (see how later...)

– OK, what is $\nabla_{\vec{u}} \tilde{\omega}^{\beta}$?

$$\nabla_{\vec{u}} \tilde{\omega}^{\beta} = u^{\alpha} \nabla_{\alpha} \tilde{\omega}^{\beta} = u^{\alpha} \nabla_{\vec{e}_{\alpha}} \tilde{\omega}^{\beta}$$

to find $\nabla_{\alpha} \tilde{\omega}^{\beta}$:

note $\langle \tilde{\omega}^{\alpha}, \vec{e}_{\beta} \rangle = \delta^{\alpha}{}_{\beta}$

$$\text{So} \quad \nabla_\gamma \langle \tilde{\omega}^\alpha, \vec{e}_\beta \rangle = 0$$

$$= \langle \nabla_\gamma \tilde{\omega}^\alpha, \vec{e}_\beta \rangle + \underbrace{\langle \tilde{\omega}^\alpha, \nabla_\gamma \vec{e}_\beta \rangle}_{\Gamma^\alpha_{\beta\gamma}}$$

From above.

$$\text{so} \quad \langle \nabla_\gamma \tilde{\omega}^\alpha, \vec{e}_\beta \rangle = -\Gamma^\alpha_{\beta\gamma}$$

$$\text{or} \quad \boxed{\nabla_\gamma \tilde{\omega}^\alpha = -\Gamma^\alpha_{\beta\gamma} \tilde{\omega}^\beta}$$

So, back to $\nabla_{\vec{u}} \tilde{\tilde{T}}$

$$\nabla_{\vec{u}} \tilde{\tilde{T}} = \underbrace{(\nabla_{\vec{u}} T^\alpha{}_\beta)}_{u^\gamma T^\alpha{}_{\beta,\gamma}} \vec{e}_\alpha \otimes \tilde{\omega}^\beta + (u^\gamma \Gamma^\lambda{}_{\alpha\gamma} \vec{e}_\lambda \otimes \tilde{\omega}^\beta) T^\alpha{}_\beta$$

$$+ (\vec{e}_\alpha \otimes u^\gamma (-\Gamma^\beta{}_{\lambda\gamma} \tilde{\omega}^\lambda)) T^\alpha{}_\beta$$

$$= (\nabla_{\vec{u}} T^\alpha{}_\beta) \vec{e}_\alpha \otimes \tilde{\omega}^\beta + (u^\gamma \Gamma^\alpha{}_{\lambda\gamma} T^\lambda{}_\beta) \vec{e}_\alpha \otimes \tilde{\omega}^\beta$$

$$- (u^\gamma \Gamma^\lambda{}_{\beta\gamma} T^\alpha{}_\lambda) \vec{e}_\alpha \otimes \tilde{\omega}^\beta$$

$$= u^\gamma (T^\alpha{}_{\beta,\gamma} + \Gamma^\alpha{}_{\lambda\gamma} T^\lambda{}_\beta - \Gamma^\lambda{}_{\beta\gamma} T^\alpha{}_\lambda) \vec{e}_\alpha \otimes \tilde{\omega}^\beta$$

In other words $\boxed{(\nabla \tilde{\tilde{T}})^\alpha{}_{\beta\gamma} = T^\alpha{}_{\beta;\gamma} = T^\alpha{}_{\beta,\gamma} + \Gamma^\alpha{}_{\lambda\gamma} T^\lambda{}_\beta - \Gamma^\lambda{}_{\beta\gamma} T^\alpha{}_\lambda}$

Rules to reduce screwups when computing $T^{\alpha\beta\ldots}{}_{\mu\nu\ldots;\gamma}$

① One Γ term for each index on T being "corrected"
② Term is + for upper index, − for lower.
③ Deriv index is last index on Γ.
④ Corrected index moves from T to Γ, replaced on T with dummy index contracted with remaining index on Γ

Example:

$$f_{;\alpha} = f_{,\alpha} \qquad \text{scalar}$$

$$S_{\alpha\beta}{}^{\gamma}{}_{;\delta} = S_{\alpha\beta}{}^{\gamma}{}_{,\delta} - \Gamma^{\lambda}{}_{\alpha\delta} S_{\lambda\beta}{}^{\gamma} - \Gamma^{\lambda}{}_{\beta\delta} S_{\alpha\lambda}{}^{\gamma} + \Gamma^{\gamma}{}_{\lambda\delta} S_{\alpha\beta}{}^{\lambda}$$

So how do you compute $\Gamma^{\alpha}{}_{\beta\gamma}$?

commutation coefficients

① Recall $[\vec{e}_\alpha, \vec{e}_\beta] = C_{\alpha\beta}{}^{\gamma} \vec{e}_\gamma$

But from definition,
$$[\vec{e}_\alpha, \vec{e}_\beta] = \nabla_{\vec{e}_\alpha} \vec{e}_\beta - \nabla_{\vec{e}_\beta} \vec{e}_\alpha$$
$$= \left(\Gamma^{\gamma}{}_{\beta\alpha} - \Gamma^{\gamma}{}_{\alpha\beta}\right) \vec{e}_\gamma$$
$$= 2\Gamma^{\gamma}{}_{[\beta\alpha]} \vec{e}_\gamma$$

so
$$\boxed{\Gamma^{\gamma}{}_{[\alpha\beta]} = \frac{-1}{2} C_{\alpha\beta}{}^{\gamma}}$$

In coord basis, $C_{\alpha\beta}{}^{\gamma} = 0$ so $\Gamma^{\gamma}{}_{[\alpha\beta]} = 0$

Γ symmetric on last 2 indices (coord basis)

② $g_{\alpha\beta;\gamma} = 0 = g_{\alpha\beta,\gamma} - \Gamma^{\lambda}{}_{\alpha\gamma} g_{\lambda\beta} - \Gamma^{\lambda}{}_{\beta\gamma} g_{\alpha\lambda}$

Define $\Gamma_{\alpha\beta\gamma} \equiv g_{\alpha\delta} \Gamma^{\delta}{}_{\beta\gamma}$

Then $0 = g_{\alpha\beta,\gamma} - \Gamma_{\beta\alpha\gamma} - \Gamma_{\alpha\beta\gamma}$

$$\Rightarrow \boxed{\Gamma_{(\alpha\beta)\gamma} = \frac{1}{2} g_{\alpha\beta,\gamma}}$$

① + ② $\Rightarrow$ $\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(-g_{\beta\gamma,\alpha} + g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} + C_{\alpha\beta\gamma} + C_{\alpha\gamma\beta} - C_{\beta\gamma\alpha}\right)$

(Where $C_{\alpha\beta\gamma} \equiv g_{\gamma\delta} C_{\alpha\beta}{}^{\delta}$)

So
$$\boxed{\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu}\left(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu} + C_{\mu\beta\gamma} + C_{\mu\gamma\beta} - C_{\beta\gamma\mu}\right)}$$

In coord basis $C = 0$, $\Gamma^{\alpha}{}_{\beta\gamma}$ called "Christoffel symbols"

Note: $\Gamma^{\alpha}{}_{\beta\gamma}$ is not a tensor

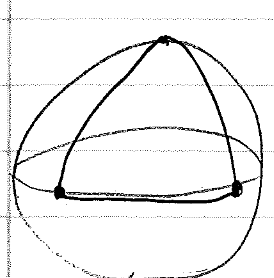
Parallel transport

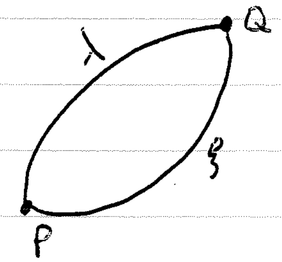
Def A tensor $\overset{\approx}{T}$ is parallel transported along a vector $\vec{u}$

iFF $\quad \nabla_{\vec{u}} \overset{\approx}{T} = 0$

Agrees with earlier definition "Components stay the same in LLF".

Note: parallel transport is curve-dependent



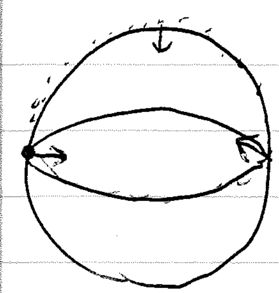


Start w/ vector at P. Parallel transport to Q

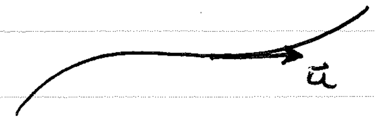
along $\vec{u} = \frac{d}{d\lambda}$ $\quad \nabla_{\vec{u}} \vec{V} = 0$

along $\vec{w} = \frac{d}{d\xi}$ $\quad \nabla_{\vec{w}} \vec{V} = 0$

get different vector at Q.



Parallel transport revisited

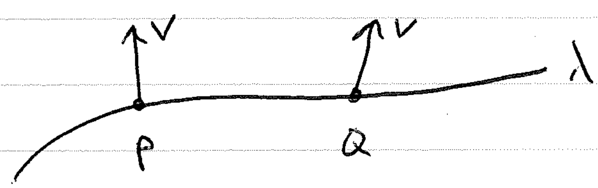


Given curve,

Tensor field $\tilde{\tilde{T}}$ is parallel transported along curve

if $\nabla_{\vec{u}} \tilde{\tilde{T}} = 0$

To Define covariant derivative of vector field $\vec{V}$ along $\vec{u}$:



Define new vector field $\vec{W}$ that is

- Parallel transported ($\nabla_{\vec{u}} \vec{W} = 0$)
- $\vec{W} = \vec{V}$ at Q.

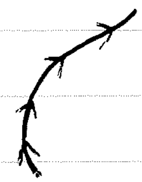
Then
$$\boxed{\nabla_{\vec{u}} \vec{V} = \lim_{Q \to P} \left(\vec{W}(P) - \vec{V}(P) \right)}$$

Geodesics: "straight line in curved space"

A curve is a geodesic if its tangent vector is parallel transported along itself. $\nabla_{\vec{u}} \vec{u} = 0$

or $u^\alpha \nabla_\alpha u^\beta = 0$

or $u^\alpha u^\beta{}_{;\alpha} = 0$



arbitrary function

- Note: Weaker condition, $\nabla_{\vec{u}} \vec{u} = f(\lambda)\vec{u}$ for $\vec{u} = \frac{d}{d\lambda}$

gives same curve as $\nabla_{\vec{u}} \vec{u} = 0$,

but parameterized differently.

However, given $P(\lambda)$ with $\nabla_{\vec{u}}\vec{u} = f(\lambda)\vec{u}$,

always possible to choose $\lambda' = g(\lambda)$ so if

$\vec{V} = \frac{d}{d\lambda'}$ Then $\nabla_{\vec{V}} \vec{V} = 0$.

Parameterization that gives $\nabla_{\vec{V}} \vec{V} = 0$ called "affine parameterization".

We will assume our curves are affinely parameterized.

Affine parameters are unique up to $\lambda' = a\lambda + b$

Affine parameter is linearly related to distance along curve. (or proper time) for non-null curves.

Geodesic eq. in components

$$\nabla_{\vec{u}} \vec{u} = 0 = u^\alpha \nabla_\alpha u^\beta = u^\alpha u^\beta{}_{;\alpha}$$

$\vec{u} = \frac{d}{d\lambda}$

$$= u^\alpha \left[u^\beta{}_{,\alpha} + \Gamma^\beta{}_{\gamma\alpha} u^\gamma \right]$$

$\left(u^\alpha u^\beta{}_{,\alpha} = \frac{d}{d\lambda} u^\beta \right)$

$$= \boxed{\frac{du^\beta}{d\lambda} + \Gamma^\beta{}_{\gamma\alpha} u^\gamma u^\alpha = 0}$$

In coord basis: $u^\alpha = \frac{dx^\alpha}{d\lambda}$

So Geodesic eq $\Rightarrow$ $$\boxed{\frac{d^2 x^\beta}{d\lambda^2} + \Gamma^\beta{}_{\gamma\alpha} \frac{dx^\gamma}{d\lambda} \frac{dx^\alpha}{d\lambda} = 0}$$

2nd order differential equation: Given $x^\gamma(\lambda_0)$

$\frac{dx^\gamma}{d\lambda}(\lambda_0)$

$\longrightarrow x^\gamma(\lambda)$ determined uniquely

4-acceleration revisited

In SR $\quad \vec{a} = \frac{d\vec{u}}{d\tau}$

D means use cov. derivative

In GR, $\quad \vec{a} = \nabla_{\vec{u}} \vec{u} = \frac{D\vec{u}}{d\tau}$

Acceleration is failure to remain on a geodesic.

Newton: person standing in room is at rest.

Einstein: " " is accelerating (not on geodesic)