Significance of LLF: Equivalence principle

Weak equivalence principle: Motion of freely falling particles are the same in a grav. field and in accelerated frame in flat spacetime, in small regions.

Einstein equivalence principle:

In small enough regions of spacetime, the non gravitational laws of physics reduce to SR. (ie. cannot detect gravitational field by local experiment)

Strong equivalence principle:

Including gravitational physics.

<u>and</u> particle w/ self gravity acts same as other particles.

Local experiments $\rightarrow$ LLF

LLF is an instantaneously freely-falling frame, or local rest frame.

So in a particle's own LLF, e.g. $U^0 = (1, 0)$

So $\vec{u} \cdot \vec{u} = -1$ in LLF

$\Rightarrow$ $\vec{u} \cdot \vec{u} = -1$ in all coordinate systems.

Commutators of vectors

vector = directional deriv.

Consider two vector fields $\vec{u}, \vec{v}$ $\qquad \vec{u} = \partial_{\vec{u}}$, $\quad \vec{v} = \partial_{\vec{v}}$

Define $\quad [\vec{u}, \vec{v}] = \partial_{\vec{u}} \partial_{\vec{v}} - \partial_{\vec{v}} \partial_{\vec{u}}$

looks tensor ish. But this is a vector!

Huh? Proof in coord basis:

$$[\vec{u}, \vec{v}] = \left(u^\alpha \frac{\partial}{\partial x^\alpha}\right)\left(v^\beta \frac{\partial}{\partial x^\beta}\right) - \left(v^\beta \frac{\partial}{\partial x^\beta}\right)\left(u^\alpha \frac{\partial}{\partial x^\alpha}\right)$$

$$= u^\alpha v^\beta \frac{\partial^2}{\partial x^\alpha \partial x^\beta} + u^\alpha \frac{\partial v^\beta}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} - v^\beta u^\alpha \frac{\partial^2}{\partial x^\alpha \partial x^\beta} - v^\beta \frac{\partial u^\alpha}{\partial x^\beta} \frac{\partial}{\partial x^\alpha}$$

$$= u^\alpha v^\beta{}_{,\alpha} \frac{\partial}{\partial x^\beta} - v^\alpha u^\beta{}_{,\alpha} \frac{\partial}{\partial x^\beta}$$

$$= \underbrace{\left(u^\alpha v^\beta{}_{,\alpha} - v^\alpha u^\beta{}_{,\alpha}\right)}_{\text{component}} \underbrace{\frac{\partial}{\partial x^\beta}}_{\text{basis vector}} \quad \Rightarrow \quad \text{a vector}$$

Note $\quad \left[\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right] = 0 \qquad$ ("holonomic")

Can be thought of as defn of coord basis: $\quad [\vec{e}_\alpha, \vec{e}_\beta] = 0$

For non-coord basis $[\vec{e}_\alpha, \vec{e}_\beta] \neq 0$
(nonholonomic)

but it is still a vector, so it can be written as $V^\gamma \vec{e}_\gamma$

Define commutation coefficients $C_{\alpha\beta}{}^{\gamma}$ by $\boxed{[\vec{e}_\alpha, \vec{e}_\beta] = C_{\alpha\beta}{}^{\gamma} \vec{e}_\gamma}$

$C_{\alpha\beta}{}^{\gamma}$ has 3 indices, but is not a rank-3 tensor.

($C_{\alpha\beta}{}^{\gamma}$ are γ_i components of a vector resulting from $[\vec{e}_\alpha, \vec{e}_\beta]$)

In a general basis,

$$[\vec{u}, \vec{v}] = [u^\alpha \vec{e}_\alpha, v^\beta \vec{e}_\beta]$$

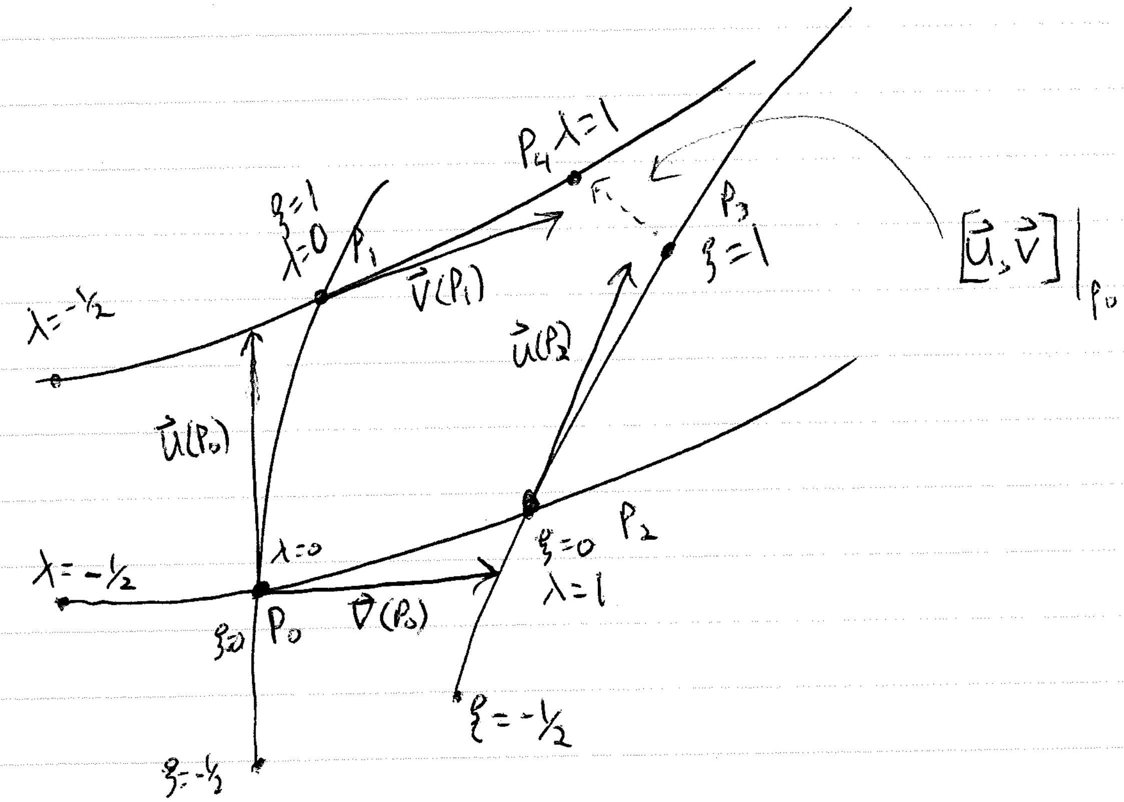
$$= u^\alpha \vec{e}_\alpha v^\beta \vec{e}_\beta - v^\beta \vec{e}_\beta u^\alpha \vec{e}_\alpha$$

$$= u^\alpha (\partial_\alpha v^\beta) \vec{e}_\beta + u^\alpha v^\beta \vec{e}_\alpha \vec{e}_\beta - v^\beta (\partial_\beta u^\alpha) \vec{e}_\alpha - v^\beta u^\alpha \vec{e}_\beta \vec{e}_\alpha$$

$$= u^\alpha v^\beta \underbrace{[\vec{e}_\alpha, \vec{e}_\beta]}_{C_{\alpha\beta}{}^{\gamma} \vec{e}_\gamma} + u^\alpha (\partial_\alpha v^\gamma) \vec{e}_\gamma - v^\beta (\partial_\beta u^\gamma) \vec{e}_\gamma$$

$$= (u^\alpha v^\beta + u^\alpha \partial_\alpha v^\gamma - v^\alpha \partial_\alpha u^\gamma) \vec{e}_\gamma$$

Picture of commutator



$$\vec{U} = \frac{d}{d\xi} \qquad \vec{V} = \frac{d}{d\lambda}$$

Consider scalar function f, in coord basis

$$f(P_4) - f(P_3) = f(P_4) - f(P_1) + f(P_1) - f(P_0)$$
$$+ f(P_0) - f(P_2) + f(P_2) - f(P_3)$$

$$= \left(f_{,\alpha} V^\alpha + \frac{1}{2} f_{,\alpha\beta} V^\alpha V^\beta\right)_{P_1} + \left(f_{,\alpha} U^\alpha + \frac{1}{2} f_{,\alpha\beta} U^\alpha U^\beta\right)\Big|_{P_0}$$
$$- \left(f_{,\alpha} V^\alpha + \frac{1}{2} f_{,\alpha\beta} V^\alpha V^\beta\right)_{P_0} - \left(f_{,\alpha} U^\alpha + \frac{1}{2} f_{,\alpha\beta} U^\alpha U^\beta\right)\Big|_{P_2}$$

$$= U^\alpha \partial_\alpha \left[f_{,\epsilon} V^\epsilon + \tfrac{1}{2} f_{,\epsilon\gamma} V^\epsilon V^\gamma \right] - V^\alpha \partial_\alpha \left[f_{,\epsilon} U^\epsilon + \tfrac{1}{2} f_{,\gamma\epsilon} U^\gamma U^\epsilon \right]$$

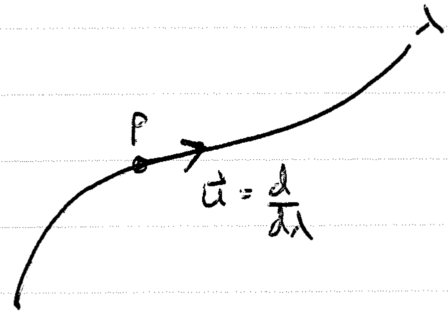
small when $u, v \to 0$

$$= U^\alpha \partial_\alpha \left[V^\epsilon \partial_\epsilon f \right] - V^\alpha \partial_\alpha \left[U^\epsilon \partial_\epsilon f \right]$$

$$= \left[U^\alpha \partial_\alpha V^\epsilon - V^\alpha \partial_\alpha U^\epsilon \right] \partial_\epsilon f$$

$$= \left([\vec{u}, \vec{v}] \right)^\epsilon \partial_\epsilon f$$

Covariant Derivative



We know how to differentiate scalar fields along a curve

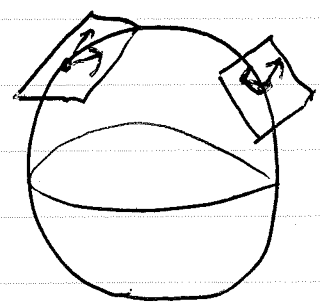
$$\frac{d}{d\lambda}\phi = \partial_{\vec{u}}\phi = u^\alpha \partial_\alpha \phi$$

$$= u^\alpha \frac{\partial \phi}{\partial x^\alpha} \quad \text{in coord basis.}$$

$$= \lim_{\varepsilon \to 0} \frac{\phi(\lambda+\varepsilon) - \phi(\lambda)}{\varepsilon}$$

compare ϕ at nearby pts along curve

But how to differentiate vectors along a curve?



Vectors at different points live in different vector spaces.

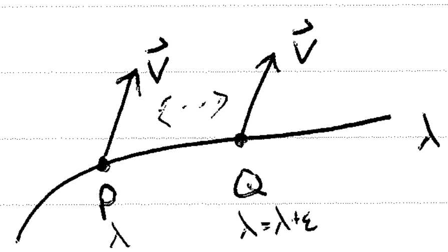
$\Rightarrow$ cannot naturally compare vectors at different points.

So we need a rule for comparing vectors at nearby points.
& tensors

Rule we will use is parallel transport:

Two derivations

① Physical.



Go to LLF at point P. Parallel transport vector at Q to vector at P by keeping components of $\vec{V}$ unchanged in LLF.

Then $\frac{d}{d\lambda}\vec{V}\Big|_{\lambda_0} = \lim_{\varepsilon \to 0} \dfrac{\vec{V}(\lambda_0+\varepsilon)_{\substack{\text{Parallel}\\ \text{Transported}\\ \text{to } \lambda_0}} - \vec{V}(\lambda_0)}{\varepsilon}$

Write This as $\nabla_{\vec{u}} \vec{V}$ where $\vec{u} = \frac{d}{d\lambda}$

covariant derivative along $\vec{u}$

Similarly for tensors

let $\tilde{\vec{T}} = T^{\alpha}{}_{\beta}\, \vec{e}_{\alpha} \times \tilde{\omega}^{\beta}$ in LLF

$$\nabla_{\vec{u}} \tilde{\vec{T}} = \lim_{\varepsilon \to 0} \frac{T^{\alpha}{}_{\beta}\, \vec{e}_{\alpha} \otimes \tilde{\omega}^{\beta} \Big|_{\substack{\parallel \text{ transport}\\ \text{from}\\ \varepsilon \text{ to } 0}} - T^{\alpha}{}_{\beta}\, \vec{e}_{\alpha} \otimes \tilde{\omega}^{\beta} \Big|_{\varepsilon=0}}{\varepsilon}$$

but basis vectors constant in LLF

So $\nabla_{\vec{u}} \tilde{\vec{T}} = U^{\gamma} T^{\alpha}{}_{\beta,\gamma}\, \vec{e}_{\alpha} \otimes \tilde{\omega}^{\beta}$ In LLF

② More abstract defn of cov. derivative

Define $\nabla \overset{\approx}{T}$ as a tensor — covariant derivative of $\overset{\approx}{T}$

such that $\nabla \overset{\approx}{T}(\ldots, \vec{u}) = \nabla_{\vec{u}} \overset{\approx}{T}$

slots of $\overset{\approx}{T}$

(If $\overset{\approx}{T}$ has components $T^{\alpha}{}_{\beta}$, $\nabla \overset{\approx}{T}$ has components $\nabla_{\gamma} T^{\alpha}{}_{\beta}$

$\nabla_{\vec{u}} \overset{\approx}{T}$ has components $U^{\gamma} \nabla_{\gamma} T^{\alpha}{}_{\beta}$

$\nabla_{\gamma} T^{\alpha}{}_{\beta}$ also written $T^{\alpha}{}_{\beta;\gamma}$)

and such that $\nabla \equiv$ covariant derivative operator has following properties

ⓐ Linear: $\nabla(\alpha \overset{\approx}{T} + \beta \overset{\approx}{S}) = \alpha \nabla \overset{\approx}{T} + \beta \nabla \overset{\approx}{S}$

α, β constants

ⓑ Leibnitz: $\nabla_{\vec{u}}(f \overset{\approx}{A} \otimes \overset{\approx}{B}) = (\nabla_{\vec{u}} f)\, \overset{\approx}{A} \otimes \overset{\approx}{B} + f\, \nabla_{\vec{u}} \overset{\approx}{A} \otimes \overset{\approx}{B} + f\, \overset{\approx}{A} \otimes \nabla_{\vec{u}} \overset{\approx}{B}$

ⓒ Commutes w/ contraction: if $\overset{\approx}{S}$ is rank n,

$\overset{\approx}{S}(\vec{e}_{\alpha}, \tilde{\omega}^{\alpha}, \ldots)$ is rank $n-2$

$$\nabla\left[\overset{\approx}{S}(\vec{e}_{\alpha}, \tilde{\omega}^{\alpha}, \ldots)\right] = \underbrace{\nabla \overset{\approx}{S}(\vec{e}_{\alpha}, \tilde{\omega}^{\alpha}, \ldots)}_{\text{rank } n+1}$$

Or $\quad S^{\alpha}{}_{\alpha\beta;\gamma}$ doesn't depend on if slots contracted before/after deriv

(d) For scalars,

$$\partial_{\vec{u}}\phi = \nabla_{\vec{u}}\phi = u^{\alpha}\nabla_{\alpha}\phi = u^{\alpha}\phi_{;\alpha}$$
$$= u^{\alpha}\partial_{\alpha}\phi = u^{\alpha}\phi_{,\alpha}$$

(e) $\nabla_{\vec{u}}\vec{v} - \nabla_{\vec{v}}\vec{u} = [\vec{u},\vec{v}]$

(equivalent to $(\nabla\nabla f)(\vec{u},\vec{v}) = (\nabla\nabla f)(\vec{v},\vec{u})$)

torsion free

(f) $\nabla_{\vec{u}}\tilde{\tilde{g}} = 0$ compatability of covariant derivative w/ metric