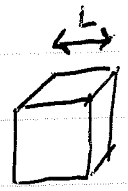
# Conservation of stress-energy tensor

Show in 2 ways.

(1)





cube of side L

x component of Force on box

$$F^x = -L^2\left[T^{xy}(+y) - T^{xy}(-y) + T^{xz}(+z) - T^{xz}(-z) + T^{xx}(+x) - T^{xx}(-x)\right]$$

$$= -L^3\left[T^{xy}{}_{,y} + T^{xz}{}_{,z} + T^{xx}{}_{,x}\right] \qquad \text{as } L \to 0$$

$$= -L^3\left[T^{xi}{}_{,i}\right]$$

but this is $\dfrac{dP^x}{dt} = \dfrac{d}{dt}(T^{x0})L^3 = T^{x0}{}_{,0}L^3 = -L^3 T^{xi}{}_{,i}$

$$\Rightarrow \quad \boxed{T^{x\alpha}{}_{,\alpha} = 0}$$

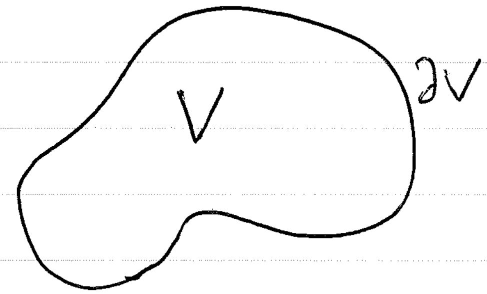
Same for y, z

What about $T^{0\alpha}$?

Rate of change of energy in box $= T^{00}{}_{,0}L^3$

Energy Flux into box $= -T^{0x}(+x)L^2 + T^{0x}(-x)L^2$
$-T^{0y}(+y)L^2 + T^{0y}(-y)L^2$
$-T^{0z}(+z)L^2 + T^{0z}(-z)L^2$
$= -T^{0i}{}_{,i}L^3 \qquad \Rightarrow \boxed{T^{0\alpha}{}_{,\alpha} = 0}$

② Other way:




V is 4d volume

$\vec{P}$ conserved, cannot pile up inside V.

$$\vec{P}_{\text{outflow}} = \sum_{\substack{\text{boundary}\\ \text{segments}}} \tilde{\tilde{T}}(-, \tilde{\Sigma}_{\text{segment}}) = 0$$

volume 3-form

$$= \sum_{s} T^{\alpha\beta} \tilde{\Sigma}^{(s)}_{\beta} \xrightarrow[\text{limit}]{} \int_{\partial V} T^{\alpha\beta} (d^3\Sigma_\beta)$$

Stokes $\rightarrow$ $$\int_V T^{\alpha\beta}{}_{,\beta}\, d^4x = 0$$ (Flat space for now)

$$\left[\text{or} \quad \int_V \left(\sqrt{-g}\, T^{\alpha\beta}\right)_{,\beta} d^4x \quad \text{in curved space}\right]$$

V is arbitrary $\Rightarrow$ $\boxed{T^{\alpha\beta}{}_{,\beta} = 0}$

# Curved Spacetime

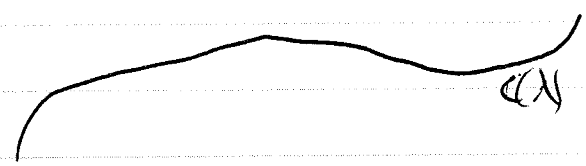
bye bye Flat space

Coordinates are now merely <u>labels</u>.

No physical significance

Most things are the same as before:

- Events - points in spacetime
- Scalars - invariant Functions [scalar Fields] or constants
- Curves - Parameterized sets of events




- Vectors - only <u>local</u> vectors

$$\vec{u} = \frac{d}{d\lambda}$$

$$= \partial_{\vec{u}}$$ directional deriv.

- Basis vectors $\vec{e}_\alpha$

Could be coord. basis vectors $\vec{e}_\alpha = \frac{d}{dx^\alpha}$

For some coords $x^\alpha$

but don't need to be.

$\boxed{\text{difference}}$ Transformations: $\vec{e}_{\bar{\alpha}} = L^\beta{}_{\bar{\alpha}} \vec{e}_\beta$

For some nonsingular matrix $L^\beta{}_{\bar{\alpha}}$

Need not be Lorentz xFormations

$$f_{,\alpha} = \langle \tilde{d}f, \vec{e}_\alpha \rangle = \partial_{\vec{e}_\alpha} f = \partial_\alpha f$$

$$\boxed{= \frac{\partial f}{\partial x^\alpha} \quad \text{in coord basis only}}$$

<u>Coord</u> <u>basis</u> 1-forms $\tilde{d}x^\alpha$ dual to $\frac{\partial}{\partial x^\alpha}$

$$\langle \tilde{d}x^\alpha, \frac{\partial}{\partial x^\beta} \rangle = \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha_\beta$$

- Tensors $\tilde{\tilde{T}}(\_, \_, \_, \ldots) = \text{scalar}$ 1 forms or vectors

$$T^{\alpha\beta}{}_\gamma = L^{\bar{\gamma}}{}_\gamma L^\alpha{}_{\bar{\alpha}} L^\beta{}_{\bar{\beta}} T^{\bar{\alpha}\bar{\beta}}{}_{\bar{\gamma}}$$

- Metric $\tilde{\tilde{g}}(\vec{A}, \vec{B}) = \vec{A} \cdot \vec{B}$ defines dot product

$$g_{\alpha\beta} = \tilde{\tilde{g}}(\vec{e}_\alpha, \vec{e}_\beta)$$

$\|g_{\alpha\beta}\|$ always real, symmetric matrix, $\det g_{\alpha\beta} < 0$.

$$\tilde{\tilde{g}} = g_{\alpha\beta}\, \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta \quad \left( = g_{\alpha\beta}\, \tilde{d}x^\alpha \otimes \tilde{d}x^\beta \text{ in } \underline{\text{coord. basis}} \right)$$

<u>IF coord. basis</u> then $L^{\bar{\alpha}}{}_{\alpha} = \dfrac{\partial x^{\bar{\alpha}}}{\partial x^{\alpha}}$

i.e. $\vec{e}_{\alpha} = \dfrac{\partial}{\partial x^{\alpha}} = \underbrace{\dfrac{\partial x^{\bar{\alpha}}}{\partial x^{\alpha}}}_{L^{\bar{\alpha}}{}_{\alpha}} \underbrace{\dfrac{\partial}{\partial x^{\bar{\alpha}}}}_{\vec{e}_{\bar{\alpha}}}$

Always $\| L^{\bar{\alpha}}{}_{\alpha} \| = \| L^{\alpha}{}_{\bar{\alpha}} \|^{-1}$ i.e. $L^{\bar{\alpha}}{}_{\alpha} L^{\alpha}{}_{\bar{\beta}} = \delta^{\bar{\alpha}}_{\bar{\beta}}$

- Vector components transform like $V^{\alpha} = L^{\alpha}{}_{\bar{\alpha}} V^{\bar{\alpha}}$

$\vec{V} = V^{\alpha} \vec{e}_{\alpha}$ like before (Same as before with $\Lambda \to L$)

Other things that are the same in curved space vs flat space (with $\Lambda \to L$):

- 1 Forms $\langle \tilde{V}, \vec{u} \rangle = \text{scalar}$

- basis 1-Forms $\langle \tilde{\omega}^{\beta}, \vec{e}_{\alpha} \rangle = \delta^{\beta}{}_{\alpha}$ dual basis

- Transformations

1-form $\tilde{u} = u_{\alpha} \tilde{\omega}^{\alpha}$

$u_{\bar{\alpha}} = L^{\alpha}{}_{\bar{\alpha}} u_{\alpha}$

$\tilde{\omega}^{\bar{\alpha}} = L^{\bar{\alpha}}{}_{\alpha} \tilde{\omega}^{\alpha}$

(same with $\Lambda \to L$)

- Gradient 1-form

$\langle \tilde{d}f, \vec{u} \rangle = \partial_{\vec{u}} f = \vec{u}[f]$

## Line element

Let $\vec{\xi} = \Delta x^\alpha \vec{e}_\alpha$ = small displacement

Then $\tilde{\tilde{g}}(\vec{\xi},\vec{\xi}) = g_{\alpha\beta}\Delta x^\alpha \Delta x^\beta$

$= ds^2$

often written $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$

invariant (infinitesimal) interval

## Raising / Lowering

$\tilde{\tilde{g}}$ gives correspondence btwn. vectors + 1-forms

Like before, $\tilde{\tilde{g}}$ raises + lowers indices of tensors

1-form $\tilde{A}$ corresponds to $\vec{A}$ if $\tilde{A} = g(-,\vec{A})$

$A_\mu = g_{\mu\nu} A^\nu$

**Note:** Metric of the same spacetime looks different with different coordinates.

Ex $\quad ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \quad$ Minkowski metric

$= -dt^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

looks curved, b/c basis vectors $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}$ change from pt. to pt.

but still flat space.

# Local Lorentz Frame (LLF)

Can any curved-space metric look Flat by a coord transformation?

Lets' try. Assume metric $g_{\bar\alpha\bar\beta}$, coords $x^{\bar\alpha}$ at some point $P$, WLOG at the origin of $x^{\bar\alpha}$ coord system.

Let $x^{\alpha}$ be another coord system centered at same point $P$.

Expand $x^{\bar\alpha}$ as Function of $x^{\alpha}$ (taylor series)

$$x^{\bar\alpha} = f^{\bar\alpha}(x^{\alpha}) = \cancelto{0}{f^{\bar\alpha}(0)} + f^{\bar\alpha}{}_{,\mu} x^{\mu} + \frac{1}{2} f^{\bar\alpha}{}_{,\mu\nu} x^{\nu} x^{\mu} + \frac{1}{6} f^{\bar\alpha}{}_{,\mu\nu\lambda} x^{\mu} x^{\nu} x^{\lambda}$$

write as

$$x^{\bar\alpha} = \underbrace{M^{\bar\alpha}{}_{\mu}}_{16 \text{ numbers}} x^{\mu} + \frac{1}{2} \underbrace{N^{\bar\alpha}{}_{\mu\nu}}_{40 \text{ numbers}} x^{\mu} x^{\nu} + \frac{1}{6} \underbrace{P^{\bar\alpha}{}_{\mu\nu\gamma}}_{80 \text{ numbers}} x^{\mu} x^{\nu} x^{\gamma}$$

in coord basis

$$L^{\bar\alpha}{}_{\mu} = \frac{\partial x^{\bar\alpha}}{\partial x^{\mu}} = M^{\bar\alpha}{}_{\mu} + N^{\bar\alpha}{}_{\mu\nu} x^{\nu} + \frac{1}{2} P^{\bar\alpha}{}_{\mu\nu\gamma} x^{\nu} x^{\gamma}$$

so

$$g_{\mu\nu} = L^{\bar\alpha}{}_{\mu} L^{\bar\beta}{}_{\nu}\, g_{\bar\alpha\bar\beta}$$

$$= \left(M^{\bar\alpha}{}_{\mu} + N^{\bar\alpha}{}_{\mu\lambda} x^{\lambda} + \frac{1}{2} P^{\bar\alpha}{}_{\mu\lambda\sigma} x^{\lambda} x^{\sigma}\right)\left(M^{\bar\beta}{}_{\nu} + N^{\bar\beta}{}_{\nu\lambda} x^{\lambda} + \frac{1}{2} P^{\bar\beta}{}_{\nu\lambda\sigma} x^{\lambda} x^{\sigma}\right) \times g_{\bar\alpha\bar\beta}$$

At the point P, $g_{\mu\nu} = M^{\bar{\alpha}}{}_{\mu} M^{\bar{\beta}}{}_{\nu} g_{\bar{\alpha}\bar{\beta}}$

can we make $g_{\mu\nu} = \eta_{\mu\nu}$ at point P?

<u>yes</u>! $g_{\mu\nu}$ has 10 independent components (symmetry).

$M^{\bar{\alpha}}{}_{\mu}$ has 16 degrees of Freedom

So we can choose coords s.t. $\underline{g_{\mu\nu} = \eta_{\mu\nu}}$ at P.

[Note: 6 degrees of Freedom left over: boost + rotation (corresponds to) - Lorentz transformation - ]

<u>Q</u>. Can we make $g_{\mu\nu,\lambda} = 0$ at P, by coord. choice?

Let's see:

$$g_{\mu\nu,\lambda} = M^{\bar{\alpha}}{}_{\mu} M^{\bar{\beta}}{}_{\nu} g_{\bar{\alpha}\bar{\beta},\lambda} + N^{\bar{\alpha}}{}_{\mu\lambda} M^{\bar{\beta}}{}_{\nu} g_{\bar{\alpha}\bar{\beta}} + N^{\bar{\beta}}{}_{\nu\lambda} M^{\bar{\alpha}}{}_{\mu} g_{\bar{\alpha}\bar{\beta}}$$

at P.

$g_{\mu\nu,\lambda}$ has 40 components.

$N^{\bar{\alpha}}{}_{\mu\lambda}$ has 40 degrees of Freedom.

$\Rightarrow$ <u>yes</u>!

Q. Can we choose coords. so $g_{\mu\nu,\lambda\rho} = 0$ at P?

$$g_{\mu\nu,\lambda\sigma} = \frac{1}{2}\left(P^{\bar{\alpha}}{}_{\mu\lambda\sigma} M^{\bar{\varrho}}{}_{\nu} + P^{\bar{\varrho}}{}_{\nu\lambda\sigma} M^{\bar{\alpha}}{}_{\mu}\right) g_{\bar{\alpha}\bar{\varrho}}$$

$$+ \text{ other terms without } P^{\bar{\alpha}}{}_{\mu\lambda\sigma}$$

$P^{\bar{\alpha}}{}_{\mu\lambda\sigma}$ has 80 degrees of freedom

$g_{\mu\nu,\lambda\sigma}$ has 100 components.

<u>No!</u>

20 degrees of freedom cannot be removed.
(will see later: Riemann curvature tensor has 20 comps.)

So define <u>Local Lorentz Frame</u> (LLF) at a point P to be

That frame in which $g_{\alpha\beta} \to \eta_{\alpha\beta}$ and $\partial_\lambda g_{\alpha\beta} \to 0$ at that point.