Stokes Theorem: If $\tilde{\alpha}$ is an n-1 form

$$\int_{\text{volume}} \tilde{d}\tilde{\alpha} = \int_{\text{boundary}} \tilde{\alpha}$$

Exterior deriv.

n-form

Example. Given vector $\vec{V}$, can define 3-form (dual) $\tilde{\alpha} = \tilde{\varepsilon}(\vec{V}, -, -, -)$

or $\tilde{\alpha} = \frac{1}{3!} V^\alpha \varepsilon_{\alpha\beta\gamma\delta}\, \tilde{d}x^\beta \wedge \tilde{d}x^\gamma \wedge \tilde{d}x^\delta \quad = V^\alpha \tilde{\Sigma}_\alpha$ — 3-volume element.

Now compute $\tilde{d}\tilde{\alpha} = \frac{1}{3!} \partial_\varepsilon [V^\alpha \varepsilon_{\alpha\beta\gamma\delta}]\, \tilde{d}x^\varepsilon \wedge \tilde{d}x^\beta \wedge \tilde{d}x^\gamma \wedge \tilde{d}x^\delta$

But all 4-forms (in 4d) are proportional to $\tilde{\varepsilon}$, so $\partial_\varepsilon (V^\alpha \varepsilon_{\alpha\beta\gamma\delta}) = q\, \varepsilon_{\varepsilon\beta\gamma\delta}$

For some q.

Solve for q: $\partial_\varepsilon (\sqrt{-g}\, V^\alpha) \hat{\varepsilon}_{\alpha\beta\gamma\delta} = q \sqrt{-g}\, \hat{\varepsilon}_{\varepsilon\beta\gamma\delta}$

contract w/ $\varepsilon^{\varepsilon\beta\gamma\delta}$: $\frac{1}{\sqrt{-g}} \partial_\varepsilon (\sqrt{-g}\, V^\alpha) \underbrace{\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\varepsilon\beta\gamma\delta}}_{-3!\,\delta^\varepsilon_\alpha} = q \underbrace{\varepsilon_{\varepsilon\beta\gamma\delta}\varepsilon^{\varepsilon\beta\gamma\delta}}_{-4!}$

So $\frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g}\, V^\alpha)\, 3! = q\, 4! \quad \Rightarrow \quad q = \frac{1}{4} \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g}\, V^\alpha)$

Therefore $\tilde{d}\tilde{\alpha} = \frac{1}{4!} \frac{1}{\sqrt{-g}} \partial_\varepsilon (\sqrt{-g}\, V^\varepsilon)\, \varepsilon_{\alpha\beta\gamma\delta}\, \tilde{d}x^\alpha \wedge \tilde{d}x^\beta \wedge \tilde{d}x^\gamma \wedge \tilde{d}x^\delta$

$= \partial_\varepsilon (\sqrt{-g}\, V^\varepsilon)\, \tilde{d}x^0 \wedge \tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \tilde{d}x^3$

So $\int \tilde{d}\tilde{\alpha} = \int \partial_\varepsilon (\sqrt{-g}\, V^\varepsilon)\, d^4x$

So

$$\int_{\text{volume}} \partial_\mu \left(\sqrt{-g}\, V^\mu\right) d^4x = \int_{\text{boundary}} V^\alpha \Sigma_\alpha$$

3-volume element

$$= \int \frac{1}{3!} V^\alpha \varepsilon_{\alpha\beta\gamma\delta}\, dx^\beta \wedge dx^\gamma \wedge dx^\delta$$

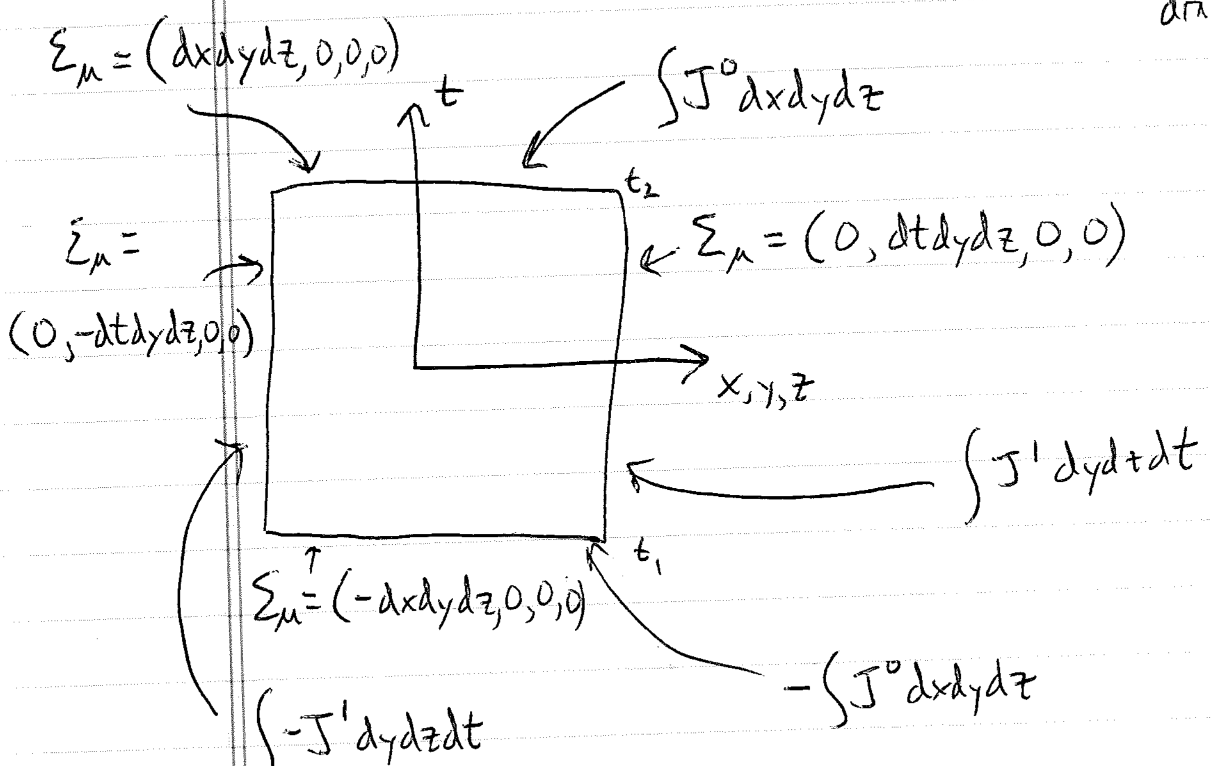
$$= \int V^\alpha n_\alpha \tilde{\tilde{\Sigma}}_{\text{surface}} \qquad n_\alpha = \text{unit normal to surface}$$

Example: Charge conservation in Flat space

$$\partial_\mu J^\mu = 0$$

$$\int \partial_\mu J^\mu d^4x = 0 = \int_{\text{boundary}} J^\alpha \Sigma_\alpha$$

draw bdry as 4-d box



$$\int_{t_1} J^0 \, dx\,dy\,dz - \int_{t_2} J^0 \, dx\,dy\,dz + \int J^k \cdot n_k \, dA = 0$$

Final charge

Initial charge

Flow of Charge out of box

Sometimes one talks about 3d volume 1-forms

$$\tilde{\Sigma}_\alpha = * \tilde{\tilde{\Sigma}}_\alpha$$

Volume 1-form

dual

3D volume element (3-form)

<u>Example</u>

So if $\tilde{\tilde{\Sigma}}_\alpha = dx \wedge dy \wedge dz$

Then $\tilde{\Sigma}_\alpha = dt$

Same as unit normal 1-form

Stress-energy tensor $\tilde{\tilde{T}}$

Consider distribution of matter & Fields

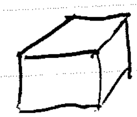
Define $\tilde{\tilde{T}}$ to be a 2nd-rank tensor, defined by:

IF $\tilde{\Sigma}$ is a 3-d volume 1-form, then $\tilde{\tilde{T}}(\_\_, \tilde{\Sigma}) = \vec{P}$

$=$ 4-momentum that crosses from $-$ side to $+$ side of $\tilde{\Sigma}$

$=$ (4-momentum in $\tilde{\Sigma}$ (if $\tilde{\Sigma}$ is timelike) or thru $\tilde{\Sigma}$ (otherwise))

<u>Components</u>:

Start with spatial box. 

Measure volume in its own rest frame $= V$

$\vec{u} =$ 4-velocity of box

Volume 1-Form $\tilde{\Sigma} = V\tilde{d}t$ $\quad \left(= \star(V\tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}\right.$

note, in rest Frame of box, $u_0 = -1$, $u_i = 0$ $\quad$ so $\tilde{u} = -\tilde{d}t$

So $\tilde{\Sigma} = -V\tilde{u}$ $\quad$ tensor equation: true in all frames

$$\vec{P}_{\text{inside box}} = \tilde{\tilde{T}}(\_\_, -V\tilde{u})$$

$$\text{or} \quad p^\alpha = T^{\alpha\beta} u_\beta (-V)$$

in rest Frame $\dfrac{p^\alpha}{V} = T^{\alpha 0}$ $\quad \Rightarrow \quad$ $T^{00} =$ energy dens

$T^{i0} =$ i-momentum density

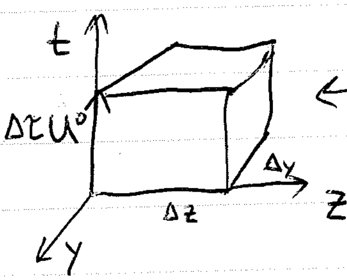
$$\underline{\text{or}} \quad E \overset{\nearrow \text{energy in box}}{} = -\vec{P}_{box} \cdot \vec{u}$$

$$= -P^\alpha u_\alpha$$

$$= V T^{\alpha\beta} u_\alpha u_\beta$$

$$= V T^{00} \text{ in rest frame}$$

Now look at different box $\Delta t = \Delta\tau u^0$



volume spanned by $\Delta y, \Delta z$ surface as it goes thru time

Volume 1-form $\tilde{\Sigma} = A \Delta t \, \tilde{d}x \qquad = {*}\left(A\Delta t \, \tilde{d}t \wedge \tilde{d}_y \wedge \tilde{d}z\right)$

area of box $\Delta y \Delta z$ in rest frame

So $\vec{P}_{box} = \overset{\approx}{T}(\_, A\Delta t \tilde{d}x)$

$$\text{or} \quad \frac{P^\alpha}{A\Delta t} = T^{\alpha\beta}(\tilde{d}x)_\beta = T^{\alpha x}$$

So $T^{0x} = \dfrac{\text{Energy}}{\text{Time}\cdot\text{Area}}$ crossing $\tilde{d}x$ = energy flux in x direction

T^{ix} = i,x component of stress = i-component of momentum flux in x direction

= i component of force per unit area in x direction

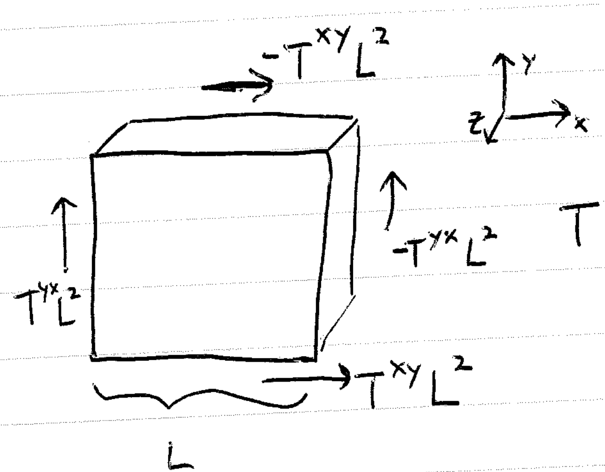
by matter at $x-\varepsilon$ acting on matter at $x+\varepsilon$

$\underset{\approx}{T}$ is symmetric

ⓐ T^{0i} = energy flux in i direction
= energy density · velocity^i
= mass density · velocity^i
= i-momentum density = T^{i0}

ⓑ

Cube of Side L



T^{yx} = y component of $\frac{\text{Force}}{\text{area}}$ on $+x$ Face

due to $x-\varepsilon$ acting on $x+\varepsilon$

T^{xy} = x component of $\frac{\text{Force}}{\text{area}}$ on $+y$ Face

So torque in z-direction about center of box

$$\tau^z = (-T^{yx}L^2)\,L/2 \quad - (-T^{xy}L^2)\,L/2$$
$$\quad + (T^{yx}L^2)(-L/2) \quad - (T^{xy}L^2)(-L/2)$$

$$\tau^z = (T^{xy} - T^{yx})L^3$$

Moment of inertia $= T^{00}L^3 \cdot L^2$

so ang. accel $= \dfrac{T^{xy} - T^{yx}}{T^{00}L^2}$

$\to \infty$ as $L \to 0$ unless $T^{xy} = T^{yx}$

Example:

Cloud of particles, mass m, all with 4-velocity $\vec{u}$.

let $n = \frac{\#\text{ of particles}}{\text{Volume}}$ in rest frame of particles   Scalar

In rest frame of particles $T^{00} = mn$
$T^{\alpha\beta} = 0 \quad \alpha \neq 0$ or $\beta \neq 0$

Can write $T^{\alpha\beta} = mn\,u^{\alpha}u^{\beta}$

Example 2:

Perfect Fluid: described by 2 quantities

ρ = energy density in rest frame
p = pressure in rest frame
scalars

Assume no heat transfer, no anisotropy, no viscosity, no shear stress.

Then in rest frame $\tilde{\tilde{T}} = \begin{pmatrix} \rho & & 0 \\ & p & \\ 0 & & p \; p \end{pmatrix}$

let $\vec{u}$ = 4-velocity of fluid

In rest frame $T^{\alpha\beta} = \rho u^{\alpha}u^{\beta} + p(\eta^{\alpha\beta} + u^{\alpha}u^{\beta})$

$\Rightarrow$ Tensor eqs. true in all frames