Maxwell's Eqs

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0 = F_{[\alpha\beta,\gamma]} = \partial_{[\gamma} F_{\alpha\beta]}$$

what are possibilities for α, β, γ?

only 4:

	α	β	γ
①	1	2	3
②	0	1	2
③	0	3	1
④	0	2	3

plus permutations

① $F_{12,3} + F_{23,1} + F_{31,2} = 0$

$B_{z,z} + B_{x,x} + B_{y,y} = 0 \Rightarrow \underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0$

② $F_{01,2} + F_{20,1} + F_{12,0} = 0$

$-E_{x,y} + E_{y,x} + B_{z,t} = 0 \Rightarrow \frac{\partial B_z}{\partial t} = -(\underset{\sim}{\nabla} \times \underset{\sim}{E})_z$

③ + ④ same as ②

Note if you define 4-vector potential actually 1-form

$A_\alpha = (\phi, \underset{\sim}{A})$ and define

$F_{\alpha\beta} = \partial_{[\alpha} A_{\beta]}$

Then $\partial_{[\gamma} F_{\alpha\beta]} = 0$ automatically

$$\partial_\beta F^{\alpha\beta} = F^{\alpha\beta}{}_{,\beta} = 4\pi J^\alpha$$

$J^\alpha =$ 4-current density
$= (\rho, \underset{\sim}{J})$

ρ ← charge density

$\alpha = 0$:
$$\underbrace{F^{01}{}_{,1} + F^{02}{}_{,2} + F^{03}{}_{,3}}_{E_{x,x} + E_{y,y} + E_{z,z}} = 4\pi\rho$$

$$\boxed{\underset{\sim}{\nabla}\cdot\underset{\sim}{E} = 4\pi\rho}$$

$\alpha = 1$:
$$F^{10}{}_{,0} + F^{12}{}_{,2} + F^{13}{}_{,3} = 4\pi J^1$$
$$-E_{x,t} + B_{z,y} - B_{y,z} = 4\pi J^1$$

$$\boxed{-\frac{dE_x}{dt} + (\underset{\sim}{\nabla}\times\underset{\sim}{B})_x = 4\pi J^x}$$

<u>other maxwell</u> eqs.

<u>Charge conservation</u> (SR)

$$\partial_\alpha J^\alpha = J^\alpha{}_{,\alpha} = 0$$

Plug into maxwell: $J^\alpha{}_{,\alpha} = \frac{1}{4\pi} F^{\alpha\beta}{}_{,\beta\alpha}$ but F antisymmetric

$= 0$ consistent

Differential Forms

A p-form is a totally antisymmetric rank $\binom{0}{p}$ tensor.

$$f \rightarrow \quad \text{0-form}$$

$$\tilde{A} = A_\alpha \tilde{\omega}^\alpha \quad \text{1-form}$$

$$\tilde{\tilde{F}} = F_{\alpha\beta}\, \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta \quad \text{2-form} \ \underline{\underline{\text{if}}} \ F_{\alpha\beta} = F_{[\alpha\beta]}$$

$$\tilde{\tilde{\tilde{\Omega}}} = \Omega_{\alpha\beta\gamma}\, \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta \otimes \tilde{\omega}^\gamma \quad \text{3-form} \ \underline{\underline{\text{if}}} \ \Omega_{\alpha\beta\gamma} = \Omega_{[\alpha\beta\gamma]}$$

$$\tilde{\tilde{\tilde{\tilde{\varepsilon}}}} = \varepsilon_{\alpha\beta\gamma\delta}\, \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta \otimes \tilde{\omega}^\gamma \otimes \tilde{\omega}^\delta$$

The only 4-form in 4-d spacetime (up to a ~~const~~ scalar multiplier)

No 5-form in 4d space

Can write p-form as $\tilde{\tilde{\tilde{\Omega}}} = \frac{1}{p!} \Omega_{\alpha\beta\ldots}\, \tilde{\omega}^\alpha \wedge \tilde{\omega}^\beta \wedge \ldots$

wedge product

$$= \Omega_{[\alpha\beta\ldots]}\, \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta \otimes \ldots$$

Exterior Derivative

If $\tilde{\tilde{\Omega}}$ is a p-form, $\tilde{d}\tilde{\tilde{\Omega}}$ is a p+1 form

Define $\tilde{d}$ by:

- For 0-form f, $\tilde{d}f$ = 1-form = gradient $\langle \tilde{d}f, \vec{v} \rangle = \partial_{\vec{v}} f$
- $\tilde{d}(\Omega_1 + \Omega_2) = \tilde{d}\Omega_1 + \tilde{d}\Omega_2$ linear
- $\tilde{d}(\underset{\uparrow \text{p-form}}{\Omega_1} \wedge \underset{\uparrow \text{q-form}}{\Omega_2}) = \tilde{d}\Omega_1 \wedge \Omega_2 + (-1)^p \Omega_1 \wedge \tilde{d}\Omega_2$
- $\tilde{d}(\tilde{d}\Omega) = 0$

From this, can show

if $\Omega = \frac{1}{p!} \Omega_{\alpha\beta\ldots\gamma} \tilde{\omega}^\alpha \wedge \tilde{\omega}^\beta \ldots \wedge \tilde{\omega}^\gamma$

Then

$$d\Omega = \frac{1}{p!} \partial_\epsilon \Omega_{\alpha\beta\ldots\gamma} \tilde{\omega}^\epsilon \wedge \tilde{\omega}^\alpha \wedge \tilde{\omega}^\beta \ldots \wedge \tilde{\omega}^\gamma$$

Maxwell's eqs in terms of p-forms

$\tilde{\tilde{F}}$ is a p-form $\qquad \tilde{\tilde{F}} = \frac{1}{2} F_{\mu\nu} \tilde{d}x^{\mu} \wedge \tilde{d}x^{\nu}$

Maxwell: $\boxed{\tilde{d}\tilde{\tilde{F}} = 0} \qquad = \frac{1}{2} \partial_{[\lambda} F_{\mu\nu]} dx^{\lambda} \wedge dx^{\mu} \wedge dx^{\nu}$

$\uparrow \; \partial_{[\lambda} F_{\mu\nu]} = 0$

Other Maxwell eqs: $\boxed{\tilde{d}({}^{*}\tilde{\tilde{F}}) = 4\pi \, {}^{*}\tilde{J}}$

Integration in Minkowski spacetime

$$\int \underbrace{d^4x}^{\text{is this invariant?}}$$

$$\int d^4x\, f(x^\alpha) = \int d^4\bar{x} \underbrace{\frac{\partial(x^0, x^1, x^2, x^3)}{\partial(x^{\bar{0}}, x^{\bar{1}}, x^{\bar{2}}, x^{\bar{3}})}}_{\det\left(\frac{\partial x^\alpha}{\partial x^{\bar{\beta}}}\right)} f(x^{\bar{\alpha}})$$

Note: $$g_{\bar{\alpha}\bar{\beta}} = \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \frac{\partial x^\beta}{\partial x^{\bar{\beta}}} g_{\alpha\beta}$$

$$\Rightarrow \det \bar{g} = \det\left(\frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}}\right) \det\left(\frac{\partial x^\beta}{\partial x^{\bar{\beta}}}\right) \det g$$

$$\Rightarrow \quad \det\left(\frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}}\right) = \frac{\sqrt{-\det \bar{g}}}{\sqrt{-\det g}}$$

$$\Rightarrow \quad d^4x\, f(x^\alpha) = \frac{\sqrt{-\det \bar{g}}}{\sqrt{-\det g}} d^4\bar{x}\, f(x^{\bar{\alpha}})$$

$\Rightarrow$ $\sqrt{-\det g}\, d^4x$ is invariant.

In SR, $\int d^4x\, f(x)$ will be invariant

Integration of p-Forms

Area / volume elements are p-forms!

Note that $\sqrt{-\det g}\, d^4x$ is natural invariant volume element.

Also note Levi-civita can be written $\tilde{\tilde{\varepsilon}} = \frac{1}{4!} \varepsilon_{\alpha\beta\gamma\delta}\, \tilde{d}x^\alpha \wedge \tilde{d}x^\beta \wedge \tilde{d}x^\gamma \wedge \tilde{d}x^\delta$

$$= \sqrt{-\det g}\; \tilde{d}t \wedge \tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z$$

Define, in n-dimensions, integral of n-form $\tilde{\tilde{\alpha}} = \alpha(x_0, \ldots x_{n-1})\, \tilde{d}x^0 \wedge \tilde{d}x^1 \wedge \ldots \wedge \tilde{d}x^{n-1}$

↑ coord basis

as $$\int_V \tilde{\tilde{\alpha}} \equiv \int_V \alpha\, dx^0 dx^1 \ldots dx^{n-1}$$

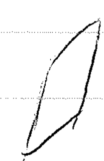
over n-volume

usual defn of integral.

So $$\int_V \tilde{\tilde{\varepsilon}} = \int_V \sqrt{-\det g}\; dt\, dx\, dy\, dz \qquad \text{in 4-dims.}$$

usual defn of integral

Now suppose you have a surface of constant x^α



Then $$\tilde{\tilde{\tilde{\Sigma}}}_\alpha \equiv \frac{1}{3!} \varepsilon_{\alpha\beta\gamma\delta}\, \tilde{d}x^\beta \wedge \tilde{d}x^\gamma \wedge \tilde{d}x^\delta$$

is a 3-form – 3-volume element + a orientation

labels which 3-form

Example $\tilde{\tilde{\tilde{\Sigma}}}_0$ is 3-volume element $\varepsilon_{0123}\, \tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z$

So can express integral as $\displaystyle\int_{\partial V\; x^0 = \text{const}} \tilde{\tilde{\tilde{\Sigma}}}_0$

Surface of constant x^α and x^β 2-surface

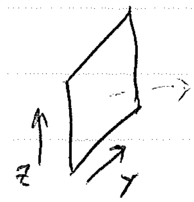
$$\tilde{\tilde{\Sigma}}_{\alpha\beta} = \frac{1}{2!}\,\varepsilon_{\alpha\beta\gamma\delta}\,\tilde{d}x^\gamma \wedge \tilde{d}x^\delta$$

2 form

area element

labels which 2-form

i.e. $\tilde{\tilde{\Sigma}}_{01} = \varepsilon_{0123}\,\tilde{d}y \wedge \tilde{d}z$



surface area element

scalar

So integrating f over the surface is written $\displaystyle\int_{\partial V\ x_0 x_1 = \text{const}} f\,\tilde{\tilde{\Sigma}}_{01}$