Metric Tensor $\tilde{\tilde{g}}$ is a $\binom{0}{2}$ tensor.

- $\tilde{\tilde{g}}$ defines dot product between vectors $\tilde{\tilde{g}}(\vec{A},\vec{B}) \equiv \vec{A}\cdot\vec{B}$ = scalar
- gives spacetime interval when $\vec{A},\vec{B}$ are $\Delta\vec{x}$

Components: $g_{\alpha\beta} = \tilde{\tilde{g}}(\vec{e}_\alpha,\vec{e}_\beta) = \vec{e}_\alpha\cdot\vec{e}_\beta$

$$\tilde{\tilde{g}} = g_{\alpha\beta}\,\tilde{\omega}^\alpha\tilde{\omega}^\beta$$

so $\vec{A}\cdot\vec{B} = g_{\alpha\beta}A^\alpha B^\beta$

$\vec{A}\cdot\vec{A} < 0$ timelike

$\vec{A}\cdot\vec{A} > 0$ spacelike

$\vec{A}\cdot\vec{A} = 0$ null

- $\tilde{\tilde{g}}$ is symmetric: $\tilde{\tilde{g}}(\vec{A},\vec{B}) = \tilde{\tilde{g}}(\vec{B},\vec{A})$

Raising/Lowering indices

- Define upper index $g^{\alpha\beta}$ = matrix inverse of $g_{\alpha\beta}$, i.e. $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha{}_\gamma$

$\tilde{\tilde{g}}$ produces a correspondence between vectors + 1 forms.

- For vector $\vec{A} = A^\alpha\vec{e}_\alpha$, there is a corresponding 1-form

$$\tilde{A} = A_\beta\,\tilde{\omega}^\beta, \text{ where } A_\beta = g_{\alpha\beta}A^\alpha \quad \text{or } \tilde{A} = \tilde{\tilde{g}}(\vec{A},\_\_)$$

- Likewise, $A^\alpha = g^{\alpha\beta}A_\beta$

- In flat space, (special relativity), for usual coordinate basis,

$$g_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Can use $\tilde{\tilde{g}}$ to define corresponding tensors.

If $\tilde{\tilde{\vec{T}}}$ is a $\binom{2}{1}$ tensor $\tilde{\tilde{\vec{T}}} = T^{\alpha\beta}{}_{\gamma}\, \vec{e}_\alpha \otimes \vec{e}_\beta \otimes \tilde{\omega}^\gamma$

Then can define corresponding $\binom{3}{0}$ tensor $\tilde{\tilde{\vec{T}}}' = T^{\alpha\beta\gamma}\, \vec{e}_\alpha \otimes \vec{e}_\beta \otimes \vec{e}_\gamma$

where $T^{\alpha\beta\gamma} = g^{\gamma\sigma} T^{\alpha\beta}{}_{\sigma}$

and can define a corresponding $\binom{1}{2}$ tensor $\tilde{\tilde{T}}'' = T^{\alpha}{}_{\beta\gamma}\, \vec{e}_\alpha \otimes \tilde{\omega}^\beta \otimes \tilde{\omega}^\gamma$

where $T^{\alpha}{}_{\beta\gamma} = g_{\beta\sigma} T^{\alpha\sigma}{}_{\gamma}$

etc.

Indices are raised by $g^{\alpha\beta}$

lowered by $g_{\alpha\beta}$

Example: $\vec{V}$ has a corresponding one-form $\tilde{V} = V_\alpha \tilde{\omega}^\alpha$

$\vec{V} = V^\alpha \vec{e}_\alpha$

where $V_\alpha = g_{\alpha\beta} V^\beta$

And $V^\alpha = g^{\alpha\beta} V_\beta$

Symmetries of Tensors

In general, $\tilde{\tilde{T}}(\vec{V}_1, \vec{V}_2, \tilde{\omega}^1) \neq \tilde{T}(\vec{V}_2, \vec{V}_1, \tilde{\omega}^1)$ order matters.

similarly $T^{\alpha\beta}{}_{\gamma} \neq T^{\beta\alpha}{}_{\gamma}$ in general.

But some tensors have symmetries.

Example $g_{\alpha\beta} = g_{\beta\alpha}$ metric is symmetric

Notation:

$$T^{(\alpha\beta)} = \frac{1}{2}\left(T^{\alpha\beta} + T^{\beta\alpha}\right) \quad \text{symmetrization}$$

$$T^{(\alpha\beta\gamma)} = \frac{1}{6}\left(T^{\alpha\beta\gamma} + \text{all permutations}\right)$$

$$T^{[\alpha\beta]} = \frac{1}{2}\left[T^{\alpha\beta} - T^{\beta\alpha}\right] \quad \text{antisymmetrization}$$

$$T^{[\alpha\beta\gamma]} = \frac{1}{3!}\left(T^{\alpha\beta\gamma} + (\text{all permutations of } \alpha\beta\gamma), \text{ sign of permutation}\right)$$

Contraction

Can make lower-rank tensor $(\text{rank}^{Lo} = \text{rank}^{Hi} - 2n)$
From higher rank tensor.

Example: Given 4^{th} rank tensor $\tilde{\tilde{R}}(\vec{a}, \vec{b}, \tilde{u}, \tilde{v})$ can contract on 1st & 3rd slots

to make 2nd rank tensor $\tilde{\tilde{M}}(\vec{p}, \tilde{q}) = \tilde{\tilde{R}}(\vec{e}_\alpha, \vec{p}, \tilde{\omega}^\alpha, \tilde{q})$

Basis vectors in contracted slots

In components, $M_\alpha{}^\beta = R_{\mu\alpha}{}^{\mu\beta}$

Levi-Civita Tensor

$\tilde{\tilde{\varepsilon}}$ = 4th rank tensor, totally antisymmetric

$$\tilde{\tilde{\varepsilon}}(\vec{V}_1, \vec{V}_2, \vec{V}_3, \vec{V}_4) = -\tilde{\tilde{\varepsilon}}(\vec{V}_2, \vec{V}_1, \vec{V}_3, \vec{V}_4)$$

same for interchanging any 2 slots
interchange brings minus sign.

Components are $\varepsilon_{\alpha\beta\gamma\delta}$

Normalization Demand $\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta\gamma\delta} = -4!$

define $\varepsilon_{\alpha\beta\gamma\delta} = \underset{\text{unknown}}{a}\,\hat{\varepsilon}_{\alpha\beta\gamma\delta}$ where

$\hat{\varepsilon}_{0123} = +1$
$\hat{\varepsilon}_{0132} = -1$
etc

$$\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta\gamma\delta} = \varepsilon_{\mu\nu\sigma\lambda}\varepsilon_{\alpha\beta\gamma\delta}\, g^{\mu\alpha} g^{\nu\beta} g^{\sigma\gamma} g^{\lambda\delta}$$

$$= a^2\,\hat{\varepsilon}_{\mu\nu\sigma\lambda}\hat{\varepsilon}_{\alpha\beta\gamma\delta}\, g^{\mu\alpha} g^{\nu\beta} g^{\sigma\gamma} g^{\lambda\delta}$$

From linear algebra, determinant of n-dim square matrix is

$$\det(A) = \frac{1}{n!}\hat{\varepsilon}_{\mu\nu\sigma\lambda\ldots}\hat{\varepsilon}_{\alpha\beta\gamma\delta\ldots} A^{\mu\alpha} A^{\nu\beta} A^{\sigma\gamma} A^{\lambda\delta}\ldots$$

So $\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta\gamma\delta} = a^2 \det(g^{\alpha\beta})(4!)$

must equal $-4!$ $\Rightarrow$ $a^2 = \frac{1}{-\det g^{\alpha\beta}} = -\det(g_{\alpha\beta}) \equiv -\det g$

so $\varepsilon_{\alpha\beta\gamma\delta} = \sqrt{-g}\,\hat{\varepsilon}_{\alpha\beta\gamma\delta}$ or $\boxed{\varepsilon_{0123} = +\sqrt{-g}}$

Notation: Wedge product

$$\vec{u} \wedge \vec{v} = \vec{u} \otimes \vec{v} - \vec{v} \otimes \vec{u}$$ antisymmetric tensor

With 1-forms, can make 2-form $\tilde{\alpha} \wedge \tilde{v}$ antisymmetric

or 3-form $\tilde{u} \wedge \tilde{v} \wedge \tilde{w} = \tilde{\alpha} \otimes \tilde{v} \otimes \tilde{w}$ + even permutations
- odd permutations

Hodge Dual tensors

In n dimensions, if F is an antisymmetric tensor of rank p,

$\star F$ is an antisymmetric tensor of rank $n-p$

Contracting by From Levi-Civita

Ex Dual of $\vec{\vec{J}}$ is $\star\vec{\vec{J}}$

components:

$$(\star \vec{\vec{J}})_{\alpha\beta\gamma} = J^{\mu} \varepsilon_{\mu\alpha\beta\gamma}$$

Dual of $\tilde{\tilde{F}}$ is $\star\tilde{\tilde{F}}$

$$(\star \tilde{\tilde{F}})_{\alpha\beta} = \frac{1}{2!} F^{\mu\nu} \varepsilon_{\mu\nu\alpha\beta}$$

Dual of rank $\binom{3}{0}$ $\tilde{\tilde{\tilde{T}}}$ is $\star\tilde{\tilde{\tilde{T}}}$

$$(\star \tilde{\tilde{\tilde{T}}})_{\alpha} = \frac{1}{3!} T^{\mu\nu\lambda} \varepsilon_{\mu\nu\lambda\alpha}$$

3D cross product
$\underset{\sim}{A} \times \underset{\sim}{B} = \star(\underset{\sim}{A} \otimes \underset{\sim}{B})$

$$\star\star F = -(-1)^{r(n-r)} F$$ r is rank of F
n is dim of spacetime

Lorentz Force

Force on particle with charge q:

$$\frac{d\underset{\sim}{p}}{dt} = q\left(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}\right)$$

3-momentum

What about energy?

For particle, $-m^2 = \vec{p}\cdot\vec{p}$

$= -(p^0)^2 + \underset{\sim}{p}\cdot\underset{\sim}{p}$

$\Rightarrow (p^0)^2 = m^2 + \underset{\sim}{p}\cdot\underset{\sim}{p}$

$$\frac{dp^0}{dt} = \frac{1}{2p^0}\frac{d}{dt}\left((p^0)^2\right)$$

$$= \frac{1}{2p^0}\frac{d}{dt}\left(\underset{\sim}{p}\cdot\underset{\sim}{p} + m^2\right)$$

note $\dfrac{\underset{\sim}{p}}{p^0} = \dfrac{m\gamma \underset{\sim}{v}}{m\gamma} = \underset{\sim}{v}$

$$= \frac{\underset{\sim}{p}}{p^0}\cdot\frac{d\underset{\sim}{p}}{dt}$$

$$= \underset{\sim}{v}\cdot\frac{d\underset{\sim}{p}}{dt} = q\,\underset{\sim}{v}\cdot\underset{\sim}{E}$$

Now
$$\frac{d\underset{\sim}{p}}{d\tau} = \frac{d\underset{\sim}{p}}{dt}\frac{dt}{d\tau}$$

$\rightarrow U^0 = \gamma$

$$\frac{dp^0}{d\tau} = \frac{dp^0}{dt}\frac{dt}{d\tau}$$

$\underset{\sim}{u} \equiv \gamma \underset{\sim}{v}$

$$\Rightarrow \frac{d\underset{\sim}{p}}{d\tau} = q\,U^0\underset{\sim}{E} + q\,\underset{\sim}{u}\times\underset{\sim}{B}$$

$$\frac{dp^0}{d\tau} = q\,\underset{\sim}{u}\cdot\underset{\sim}{E}$$

proportional to U^α

So can we write $\dfrac{d\vec{P}}{d\tau} = q\cdot(\text{linear operator acting on } U)$?

Faraday tensor

$$\tilde{\vec{F}} = F^{\alpha}{}_{\beta}\, \vec{e}_{\alpha} \otimes \tilde{\omega}^{\beta} \qquad\qquad \tilde{\vec{F}}(\tilde{\omega}, \vec{V}) = \text{scalar}$$

F takes a vector and produces another vector or $\tilde{\vec{F}}(-, \vec{V}) = \text{vector}$

Force on particle with charge q:

$$\frac{d\vec{P}}{d\tau} = q\,\tilde{\vec{F}}(-, \vec{u})$$

4-velocity of particle.

or
$$\frac{dP^{\alpha}}{d\tau} = q\, F^{\alpha}{}_{\beta}\, u^{\beta}$$

Components in Minkowski coords:

$$F^{\alpha}{}_{\beta} = \begin{array}{cc} & \xrightarrow{\ \beta\ } \\ \alpha\big\downarrow & \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \end{array}$$

in Minkowski coords

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

$F_{\alpha\beta}$ is antisymmetric: $F_{\alpha\beta} = -F_{\beta\alpha}$ or $F_{\alpha\beta} = F_{[\alpha\beta]}$