1-Forms

A 1-Form $\tilde{\omega}$ is an operator that takes a vector + returns a scalar.

$$\tilde{\omega}(\vec{V}) = \text{scalar}$$

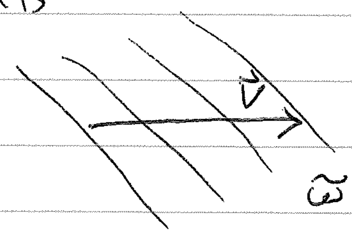
MTW writes $\tilde{\omega}(\vec{V})$ as $\langle \tilde{\omega}, \vec{V} \rangle$

- Linear: $\tilde{\omega}(a\vec{V} + b\vec{W}) = a\tilde{\omega}(\vec{V}) + b\,\tilde{\omega}(\vec{W})$

  $(a\tilde{\omega} + b\tilde{\alpha})(\vec{V}) = a\tilde{\omega}(\vec{V}) + b\,\tilde{\alpha}(\vec{V})$

  $a, b =$ scalars

picture: wavefronts



$\langle \tilde{\omega}, \vec{V} \rangle$ counts how many wavefronts $\vec{V}$ crosses

Basis

Given basis vectors $\vec{e}_\mu$ can choose basis one forms $\tilde{\omega}^\mu$

such that $\langle \tilde{\omega}^\mu, \vec{e}_\alpha \rangle = \delta^\mu_\alpha$ dual basis

Any one form can be written $\tilde{u} = u_\alpha \tilde{\omega}^\alpha$

u_α ↑ components of 1-Form

Notice $\langle \tilde{u}, \vec{V} \rangle = \langle u_\alpha \tilde{\omega}^\alpha, V^\beta \vec{e}_\beta \rangle = u_\alpha V^\beta \langle \tilde{\omega}^\alpha, \vec{e}_\beta \rangle$

$= u_\alpha V^\alpha$ scalar

Change of Frame: How do 1-Forms transform?

$$\langle \tilde{u}, \vec{V} \rangle = u_\alpha V^\alpha = u_{\bar\alpha} V^{\bar\alpha}$$ scalars independent of Frame

$$= u_{\bar\alpha} (\Lambda^{\bar\alpha}{}_\alpha V^\alpha)$$

so $$\boxed{u_\alpha = u_{\bar\alpha} \Lambda^{\bar\alpha}{}_\alpha}$$ component of 1-Form transforms like basis vector.

Basis 1-Forms: $$\tilde{u} = u_{\bar\alpha} \tilde{\omega}^{\bar\alpha} = u_\alpha \tilde{\omega}^\alpha$$

$$= (u_{\bar\alpha} \Lambda^{\bar\alpha}{}_\alpha) \tilde{\omega}^\alpha$$

$$\Rightarrow \boxed{\tilde{\omega}^{\bar\alpha} = \Lambda^{\bar\alpha}{}_\alpha \tilde{\omega}^\alpha}$$ Basis 1-Form Transforms like component of vector.

Consistent notation: $\tilde{\omega}^\alpha$ and V^α have upper indices → but are different objects
$\vec{e}_\alpha$ and u_α have lower indices ↗

Example: Gradient of scalar function $\phi(x^\alpha)$

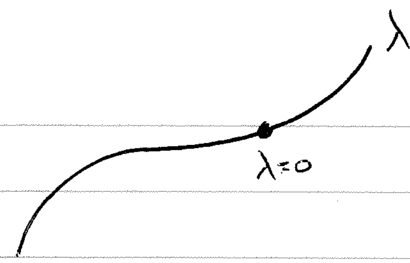
Consider $$\partial_\alpha \phi \equiv \frac{\partial \phi}{\partial x^\alpha}$$ derivative of ϕ in x^α direction

what happens in different Frame? $$\partial_{\bar\alpha} \phi = \frac{\partial \phi}{\partial x^{\bar\alpha}} = \underbrace{\frac{\partial \phi}{\partial x^\alpha}}_{\partial_\alpha \phi} \underbrace{\frac{\partial x^\alpha}{\partial x^{\bar\alpha}}}_{\Lambda^\alpha{}_{\bar\alpha}}$$

$$\Rightarrow \partial_{\bar\alpha} \phi = \Lambda^\alpha{}_{\bar\alpha} \partial_\alpha \phi$$ $\partial_{\bar\alpha}\phi$ transforms like component of 1-Form

$\partial_{\bar\alpha}\phi$ <u>is</u> component of 1-form, <u>gradient</u> $\boxed{\tilde{d}\phi = \partial_\alpha \phi \, \tilde{\omega}^\alpha}$

Directional Derivative



Curve $P(\lambda)$
Scalar field ϕ

Define <u>directional derivative</u> as derivative of ϕ along curve.

Can write as $\left.\dfrac{d}{d\lambda}\phi\right|_{\lambda=0}$

Can also write as $\partial_{\vec{V}}\phi$ where $\vec{V}$ is tangent vector to $P(\lambda)$ at $\lambda=0$

$$\vec{V} = \left.\frac{dP}{d\lambda}\right|_{\lambda=0}$$

Gradient of ϕ, $\tilde{d}\phi$, is the one-form such that

$$\left\langle \tilde{d}\phi, \left.\frac{dP}{d\lambda}\right|_{\lambda=0} \right\rangle = \left.\frac{d\phi}{d\lambda}\right|_{\lambda=0} = \partial_{\vec{V}}\phi$$

Information about function ϕ — Information about curve

$$= \langle \tilde{d}\phi, \vec{V} \rangle$$

In components

$P(\lambda)$: $x^\alpha = x^\alpha(\lambda)$

$\vec{V}$: $V^\alpha = \dfrac{dx^\alpha}{d\lambda}$

$$\partial_{\vec{V}}\phi = \frac{d}{d\lambda}\phi[x^\alpha(\lambda)] = \frac{\partial\phi}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda}$$
$$= \partial_\alpha\phi\, V^\alpha$$

Note: Directional derivative operator $\partial_{\vec{V}} = \dfrac{d}{d\lambda}$ is isomorphic to $\vec{V}$

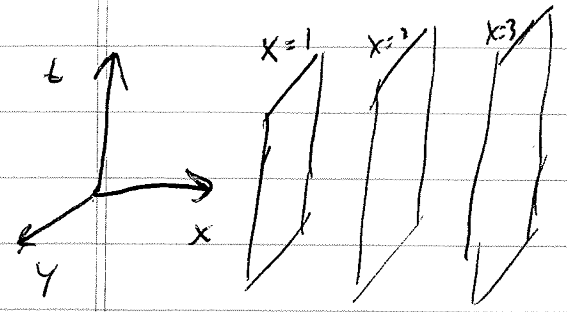
$$\partial_{\vec{V}} + \partial_{\vec{u}} = \partial_{(\vec{V}+\vec{u})}, \text{ etc.}$$

So we can consider the vector $\vec{V}$ to be the <u>same</u> as $\partial_{\vec{V}}$.

(See MTW 9.2) Note: components of $\partial_{\vec{V}}(\phi)$ are V^α if $\dfrac{\partial}{\partial x^\alpha}$ is identified with $\vec{e}_\alpha$

$$\partial_{\vec{V}}\phi = \langle \tilde{d}\phi, \vec{V}\rangle = \langle \partial_\alpha\phi\,\tilde{\omega}^\alpha, V^\beta\vec{e}_\beta \rangle$$
$$= \underbrace{\langle \tilde{\omega}^\alpha, \vec{e}_\beta \rangle}_{\delta^\alpha_\beta \text{ (dual basis)}} V^\beta \partial_\alpha\phi$$
$$= V^\alpha \partial_\alpha \phi$$

Coordinate basis vectors and 1-forms



Consider surfaces defined by

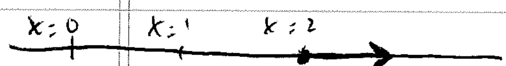
$$x^\alpha = \text{constant}$$

Treat each x^α $\alpha = 0,1,2,3$ as a scalar function (but not really a scalar)

Gradients are 1-forms $\tilde{d}x^\alpha$

Can use these as basis 1-forms $\boxed{\tilde{\omega}^\alpha = \tilde{d}x^\alpha}$ coordinate basis

Consider curve defined by $t = \text{const}$, $y = \text{const}$, $z = \text{const}$ curve parametrized by x



tangent to this curve is a vector, call it $\frac{\partial}{\partial x}$ or $\partial_{\vec{x}}$

similarly, curves (along t, y, z) can be parameterized by t, y, z. tangents are $\frac{\partial}{\partial t}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$

Can use these for basis vectors $\boxed{\vec{e}_\alpha = \frac{\partial}{\partial x^\alpha}}$ Coordinate basis

Note: arbitrary vector $\vec{V} = \partial_{\vec{V}} = \frac{d}{d\lambda} = \frac{dx^\alpha}{d\lambda}\frac{\partial}{\partial x^\alpha}$

so in a coord basis $\frac{dx^\alpha}{d\lambda}\frac{\partial}{\partial x^\alpha} = \frac{dx^\alpha}{d\lambda}\vec{e}_\alpha$

so $\frac{dx^\alpha}{d\lambda}$ are components V^α of vector, in a coord. basis.

Tensors

A tensor $\tilde{\tilde{\tilde{T}}}$ is a linear operator that takes a fixed number of vectors and/or a fixed number of 1-forms, in a particular order, and makes a scalar.

$$\tilde{\tilde{\tilde{T}}}(\vec{V}_1, \vec{V}_2, \ldots, \tilde{W}_1, \tilde{W}_2, \ldots) = \text{a scalar}$$

$$\tilde{\tilde{S}}(\vec{V}_1, \tilde{W}_1, \tilde{W}_2, \vec{V}_2) = \text{a scalar}$$

A tensor has rank $\binom{n}{m}$ if it takes n 1-forms and m vectors.

The rank is $m+n$

A 1-form is a rank $\binom{0}{1}$ tensor. $\tilde{u}(\vec{V}) = u_\alpha V^\alpha$

A vector is a rank $\binom{1}{0}$ tensor. $\vec{V}(\tilde{u}) = u_\alpha V^\alpha$

Tensor Components

$$S^{\alpha\beta}{}_{\gamma} = \tilde{\tilde{S}}(\tilde{\omega}^\alpha, \tilde{\omega}^\beta, \vec{e}_\gamma)$$

Also $\tilde{\tilde{S}} = S^{\alpha\beta}{}_{\gamma}\, \vec{e}_\alpha \otimes \vec{e}_\beta \otimes \tilde{\omega}^\gamma$

$\otimes$ = product space. $\vec{V} \otimes \vec{W}$ is a tensor defined by $(\vec{V} \otimes \vec{W})(\tilde{u}, \tilde{q}) = \vec{V}(\tilde{u})\vec{W}(\tilde{q})$

Transformations

$$S^{\bar{\alpha}\bar{\beta}}{}_{\bar{\gamma}} = \Lambda^{\bar{\alpha}}{}_{\alpha} \Lambda^{\bar{\beta}}{}_{\beta} \Lambda^{\gamma}{}_{\bar{\gamma}} S^{\alpha\beta}{}_{\gamma}$$

why? $S^{\alpha\beta}{}_{\gamma} V^\gamma W_\alpha U_\beta = \text{scalar} = S^{\bar{\alpha}\bar{\beta}}{}_{\bar{\gamma}} V^{\bar{\gamma}} W_{\bar{\alpha}} U_{\bar{\beta}}$

$S^{\alpha\beta}{}_{\gamma} \Lambda^{\gamma}{}_{\bar{\gamma}} V^{\bar{\gamma}} \Lambda^{\bar{\alpha}}{}_{\alpha} W_{\bar{\alpha}} \Lambda^{\bar{\beta}}{}_{\beta} U_{\bar{\beta}} =$

Derivatives of tensors

We saw derivative of scalar ϕ, gradient $\tilde{d}\phi$ is a 1-form.

Components of $\tilde{d}\phi$ in coord basis are $\partial_\alpha \phi = \frac{\partial \phi}{\partial x^\alpha}$

In flat space, can make $\binom{k}{\ell}$ tensor $\rightarrow$ $\binom{k}{\ell+1}$ tensor by differentiation

$$\frac{1}{\varepsilon}\left[T(\vec{x}+\varepsilon\vec{v},\ldots)-T(\vec{x},\ldots)\right]$$

suppose $\tilde{\tilde{T}} = T^{\alpha}{}_{\beta}{}^{\gamma}\, \vec{e}_\alpha \otimes \tilde{\omega}^\beta \otimes \vec{e}_\gamma$

then $\tilde{\tilde{\nabla}} T = \partial_\sigma T^{\alpha}{}_{\beta}{}^{\gamma}\, \tilde{\omega}^\sigma \otimes \vec{e}_\alpha \otimes \tilde{\omega}^\beta \otimes \vec{e}_\gamma$

is a tensor
<u>in flat space only</u>
<u>in Minkowski coord basis</u>

<u>Will generalize this later</u>

$\partial_\sigma Q$ is sometimes written $Q_{,\sigma}$ (MTW)

Symmetries of Tensors

In general, $\tilde{\tilde{T}}(\vec{V}_1, \vec{V}_2, \tilde{\omega}^1) \neq \tilde{T}(\vec{V}_2, \vec{V}_1, \tilde{\omega}^1)$ order matters.

similarly $T^{\alpha\beta}{}_{\gamma} \neq T^{\beta\alpha}{}_{\gamma}$ in general.

But some tensors have symmetries.

<u>Example</u> $g_{\alpha\beta} = g_{\beta\alpha}$ metric is symmetric

Notation:

$$T^{(\alpha\beta)} = \frac{1}{2}\left(T^{\alpha\beta} + T^{\beta\alpha}\right) \qquad \text{symmetrization}$$

$$T^{(\alpha\beta\gamma)} = \frac{1}{6}\left(T^{\alpha\beta\gamma} + \text{all permutations}\right)$$

$$T^{[\alpha\beta]} = \frac{1}{2}\left[T^{\alpha\beta} - T^{\beta\alpha}\right] \qquad \text{antisymmetrization}$$

$$T^{[\alpha\beta\gamma]} = \frac{1}{3!}\left(T^{\alpha\beta\gamma} + (\text{all permutations of } \alpha\beta\gamma) \cdot \text{sign of permutation}\right)$$

Contraction

Can make lower-rank tensor ($\text{rank}^{Lo} = \text{rank}^{Hi} - 2n$)
from higher rank tensor.

Example: Given 4^{th} rank tensor $\tilde{\tilde{R}}(\vec{a}, \vec{b}, \tilde{u}, \tilde{v})$ can <u>contract</u> on 1st & 3rd slots

to make 2nd rank tensor $\tilde{\tilde{M}}(\vec{p}, \tilde{q}) = \tilde{\tilde{R}}(\vec{e}_\alpha, \vec{p}, \tilde{\omega}^\alpha, \tilde{q})$

Basis vectors in contracted slots

In components, $M_\alpha{}^\beta = R_{\mu\alpha}{}^{\mu\beta}$