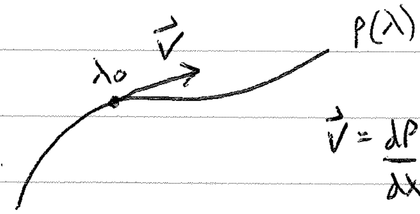
Last time:

Define vectors as tangent to a curve



$$\vec{V} = \frac{dP}{d\lambda}\Big|_{\lambda=\lambda_0}$$

Basis vectors $\vec{e}_\alpha$ $\alpha = 0,1,2,3$

Any vector can be written $\vec{V} = V^\alpha \vec{e}_\alpha$

components of vector

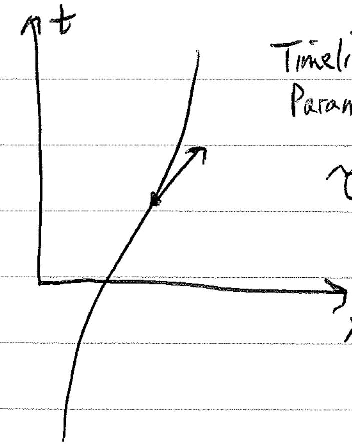
basis vectors transform like $\vec{e}_\mu = \Lambda^{\bar\nu}{}_\mu \vec{e}_{\bar\nu}$

Components transform like $V^\alpha = \Lambda^\alpha{}_{\bar\beta} V^{\bar\beta}$

vector itself is invariant $\vec{V} = \vec{e}_\alpha V^\alpha = \vec{e}_{\bar\alpha} V^{\bar\alpha}$

Strategy: express physics in terms of geometric objects, like scalars $(\Delta s)^2$ and vectors. Then equations are true in all frames.

Example Vector: 4-velocity



Timelike worldline
Parametrized by proper time τ

τ is time measured by clock moving along world line.

Coords of worldline $x^\alpha(\tau)$

4-velocity $\quad \vec{u} \equiv \frac{d\vec{x}}{d\tau}$

$$\vec{u} = u^\alpha \vec{e}_\alpha = \frac{dx^\alpha}{d\tau}\vec{e}_\alpha$$

Components in rest frame, $t = \tau$ so

$$\left.\begin{aligned} \frac{dx^0}{d\tau} &= \frac{dt}{d\tau} = 1 \\ \frac{dx^i}{d\tau} &= 0 \quad i = 1,2,3 \end{aligned}\right\} \text{rest Frame}$$

So in rest frame $\boxed{u^0 = 1,\ u^i = 0 \quad i = 1,2,3}$ "Fortran convention"

or $u^\alpha = (1, \underset{\sim}{0})$

In a boosted frame $\quad u^{\bar\alpha} = \Lambda^{\bar\alpha}{}_\alpha u^\alpha$

Assume $\Lambda^{\bar\alpha}{}_\alpha$ is pure boost, x^α is rest frame

$$\Rightarrow u^{\bar\alpha} = \Lambda^{\bar\alpha}{}_\alpha u^\alpha = \Lambda^{\bar\alpha}{}_\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow u^{\bar 0} = \gamma = \frac{1}{\sqrt{1-v^2}}$$

$$u^{\bar i} = \gamma v^{\bar i} \quad \bar i = 1,2,3$$

3-velocity of particle $\frac{dx^{\bar i}}{d\bar t}$

Sign: v in $\Lambda^{\bar\alpha}{}_\alpha$ is velocity of $x^{\bar\alpha}$ relative to x^α

= − velocity of x^α relative to $x^{\bar\alpha}$

4-momentum of particle $\vec{P} = m \vec{u}$ rest mass

Kinetic energy is $E - m$

$P^0 = \gamma m$ = energy = E

$P^i = \gamma m v^i = \gamma \cdot$ (3-momentum)

For photon, $m = 0$

Can still define 4-momentum $\vec{P} = \frac{d\vec{x}}{d\lambda}$ λ parameterizes curve

still $P^0 = E$

P^i = photon momentum.

components: $P^0 = E = \hbar \omega$ Frequency

$P^i = \hbar k^i$ wavenumber $(k^1)^2 + (k^2)^2 + (k^3)^2 = \omega^2$

Dot product and metric tensor

Define the metric tensor $\tilde{\tilde{\eta}}$ as a linear operator that takes 2 vectors and produces a scalar, the dot product.

$$\tilde{\tilde{\eta}}(\vec{A},\vec{B}) \equiv \vec{A}\cdot\vec{B} \equiv \eta_{\alpha\beta} A^\alpha B^\beta \qquad \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We already saw that $\eta_{\alpha\beta}(\Delta x)^\alpha(\Delta x)^\beta = \Delta s^2 =$ a scalar.

Key point: scalars are invariants, ie. the same value in any frame.

SR only (will generalize $\tilde{\tilde{\eta}}$ soon...)

If $\vec{V}\cdot\vec{V} > 0$ V is spacelike
$\vec{V}\cdot\vec{V} < 0$ V is timelike
$\vec{V}\cdot\vec{V} = 0$ V is null

just like Δs^2

4-velocity revisited

previously: $\vec{u} = \frac{d\vec{x}}{d\tau}$ $\quad u^0 = \gamma$, $\quad u^i = \gamma v^i$ $\quad$ in frame boosted w.r.t. rest frame

$\vec{u}\cdot\vec{u} = u^\alpha u^\beta g_{\alpha\beta}$ $\quad$ in SR $\quad \vec{u}\cdot\vec{u} = u^\alpha u^\beta \eta_{\alpha\beta}$

$$= -(u^0)^2 + (u^x)^2 + (u^y)^2 + (u^z)^2$$

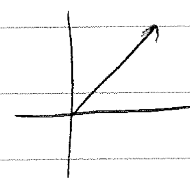
$$= \gamma^2\big[-1 + \underbrace{{v^x}^2 + {v^y}^2 + {v^z}^2}_{v^2}\big]$$

$$= -1$$

$$\boxed{\vec{u}\cdot\vec{u} = -1}$$

This eqn involves geometric objects (vectors, scalars, tensors) only. So it works in any frame.
Also works in curved spacetime, full GR.

Photon 4-momentum $\quad \vec{p} = \frac{d\vec{x}}{d\lambda}$



Photon trajectories are null, so $\boxed{\vec{p}\cdot\vec{p} = 0}$

Particle 4-momentum $\quad \vec{p} = m\vec{u}$ $\quad$ so $\boxed{\vec{p}\cdot\vec{p} = -m^2}$

Example: observer with 4-velocity $\vec{u}$ measures energy of particle/photon with 4-momentum $\vec{p}$, and gets E.
Frame-invariant statement.

How to write in geometric language? $\quad$ In rest frame of $\vec{u}$: $\quad \vec{u} = (1,0,0,0)$, $\quad \vec{p} = (E, p^x, p^y, p^z)$

$$-\vec{p}\cdot\vec{u} = -\eta_{\alpha\beta}p^\alpha u^\beta = +E$$

$\boxed{-\vec{p}\cdot\vec{u} = E}$ $\quad$ True in every frame, i.e. all observers can compute $\vec{p}, \vec{u}$ in their own frame, and get $-\vec{p}\cdot\vec{u}$ = the same E.

4-acceleration

$$\vec{a} = \frac{d\vec{u}}{d\tau} = \frac{d^2\vec{x}}{d\tau^2}$$

Note that $\vec{u}\cdot\vec{u} = -1$

$$\frac{d}{d\tau}(\vec{u}\cdot\vec{u}) = 0 = 2\vec{u}\cdot\vec{a} \qquad \Rightarrow \boxed{\vec{u}\cdot\vec{a} = 0}$$

$\left(\text{note } \frac{d}{d\tau}(\eta_{\alpha\beta}) = 0\right)$

In instantaneous comoving frame of accelerating particle $u^\alpha = (1,0,0,0)$

$a^\alpha = (0, a^x, a^y, a^z)$

a^i $(i = 1,2,3)$ is newtonian acceleration measured in instantaneous comoving frame.

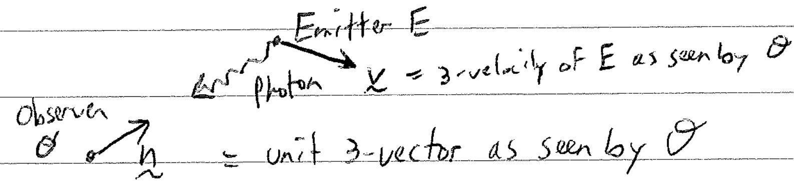
Note that magnitude of acceleration ${a^x}^2 + {a^y}^2 + {a^z}^2 = a^2$ is $a^\alpha a_\alpha$ in rest frame.

$\Rightarrow$ $a^2 = \vec{a}\cdot\vec{a}$ in all frames.

Example Doppler shift

Avoid Lorentz XFormations

Much easier to use invariants



Let ω_E be Frequency of photon in E's Frame.

what is $\omega_{\mathcal{O}}$? Frequency of photon in $\mathcal{O}$'s Frame?

$$\frac{\omega_E}{\omega_{\mathcal{O}}} = \frac{\hbar\omega_E}{\hbar\omega_{\mathcal{O}}} = \frac{E_E}{E_{\mathcal{O}}} = \frac{-\vec{P}\cdot\vec{U}_E}{-\vec{P}\cdot\vec{U}_{\mathcal{O}}} = \frac{-\vec{P}\cdot\vec{U}_E}{\hbar\omega_{\mathcal{O}}}$$

Evaluate $\vec{P}\cdot\vec{U}_E$ in $\mathcal{O}$'s Frame

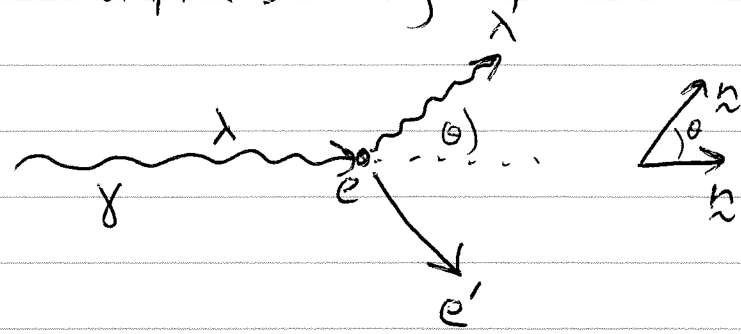
$$\mathcal{O}\text{'s Frame:} \left\{ \begin{aligned} \vec{U}_{\mathcal{O}} &= (1,0,0,0) \\ \vec{U}_E &= \gamma(1, \underset{\sim}{v}) \\ \vec{P} &= \hbar\omega_{\mathcal{O}}(1, -\underset{\sim}{n}) \end{aligned} \right.$$

$$\vec{P}\cdot\vec{U}_E = \eta_{\alpha\beta}P^\alpha U_E^\beta = -\gamma\hbar\omega_{\mathcal{O}} - \gamma\hbar\omega_{\mathcal{O}}\,\underset{\sim}{n}\cdot\underset{\sim}{v} = -\gamma\hbar\omega_{\mathcal{O}}(1+\underset{\sim}{n}\cdot\underset{\sim}{v})$$

$$\Rightarrow \frac{\omega_E}{\omega_{\mathcal{O}}} = \gamma(1+\underset{\sim}{n}\cdot\underset{\sim}{v})$$

Total 4-momentum is conserved w/o external forces.

Example: Compton scattering



show $\lambda' - \lambda = \frac{h}{m}(1-\cos\theta)$

$\vec{P}$ Conservation: $\vec{P}_e + \vec{P}_\gamma = \vec{P}_{e'} + \vec{P}_{\gamma'}$

$$\Rightarrow \vec{P}_{e'} = \vec{P}_e + \vec{P}_\gamma - \vec{P}_{\gamma'}$$

dot eqn into itself: $\underbrace{\vec{P}_{e'}\cdot\vec{P}_{e'}}_{-m^2} = \underbrace{\vec{P}_e\cdot\vec{P}_e}_{-m^2} + \underbrace{\vec{P}_\gamma\cdot\vec{P}_\gamma}_{0} + \underbrace{\vec{P}_{\gamma'}\cdot\vec{P}_{\gamma'}}_{0} + 2\vec{P}_e\cdot\vec{P}_\gamma - 2\vec{P}_e\cdot\vec{P}_{\gamma'} - 2\vec{P}_\gamma\cdot\vec{P}_{\gamma'}$

$$\Rightarrow 0 = \vec{P}_e\cdot\vec{P}_\gamma - \vec{P}_e\cdot\vec{P}_{\gamma'} - \vec{P}_\gamma\cdot\vec{P}_{\gamma'}$$

in lab frame: $\vec{P}_e = (m, 0)$

$\vec{P}_\gamma = \frac{h}{\lambda}(1, \underset{\sim}{n})$

$\vec{P}_{\gamma'} = \frac{h}{\lambda'}(1, \underset{\sim}{n}')$

so $\vec{P}_e\cdot\vec{P}_\gamma = \frac{-hm}{\lambda}$ $\quad \vec{P}_e\cdot\vec{P}_{\gamma'} = \frac{-hm}{\lambda'}$

$\vec{P}_{\gamma'}\cdot\vec{P}_\gamma = \frac{-h^2}{\lambda\lambda'}(1-\cos\theta)$

so $0 = \frac{-hm}{\lambda} + \frac{hm}{\lambda'} + \frac{h^2}{\lambda\lambda'}(1-\cos\theta)$

$$\Rightarrow \frac{(\lambda - \lambda')hm}{\lambda\lambda'} = \frac{h^2}{\lambda\lambda'}(1-\cos\theta)$$