Physics 236

website
www.tapir.caltech.edu/~scheel/P236

Overview:

Introduce math machinery of general relativity using
- Special Relativity
- E+M
- Physics in curved spacetime

Einstein's Theory of Gravitation: GR

Special topics
- Exact solutions
- Gravitational waves
- Cosmology
- Black holes
- Numerical Relativity

Special Relativity Revisited

coordinates are just labels!

spacetime: Set of events. (t, x, y, z) labels events.

reference frame: coordinate system

inertial reference frame (Lorentz frame): Particles w/ no force move in straight line.

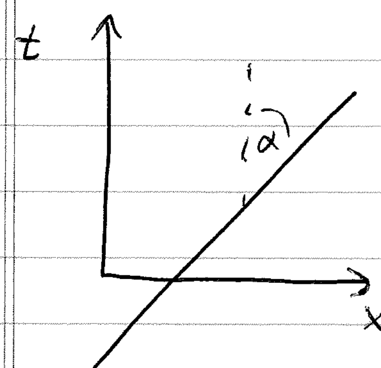
I. Relativity Principle: cannot distinguish inertial frames

II. There is a universal speed limit c

<u>Convention</u> $c \equiv 1$ measure time in cm

$\mathrm{I} + \mathrm{II} \Rightarrow$ All of SR

<u>Spacetime Diagrams</u>



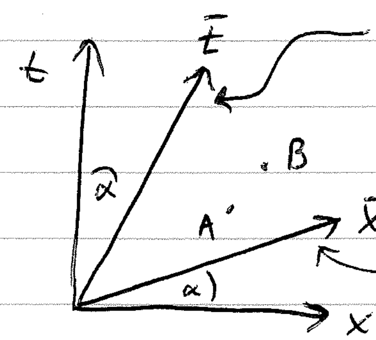
worldline of particle moving at $v = \text{const}$

$v = \tan\alpha$

$v = 1$ for $\alpha = 45°$

<u>Other observers</u>

Frame $(\bar{t}, \bar{x})$ moves with velocity $v = \tan\alpha$ w.r.t. Frame (t, x)



worldline of observer with rest Frame $(\bar{t}, \bar{x})$ $x = vt$

surface of simultaneity of observer with rest Frame $(\bar{t}, \bar{x})$

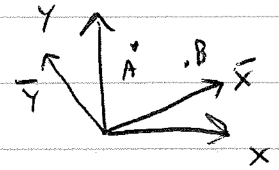
$t = vx$

Spacetime interval between 2 events A+B (arbitrary)

$$\Delta s^2 \equiv -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = -(\Delta \bar{t})^2 + (\Delta \bar{x})^2 + (\Delta \bar{y})^2 + (\Delta \bar{z})^2$$

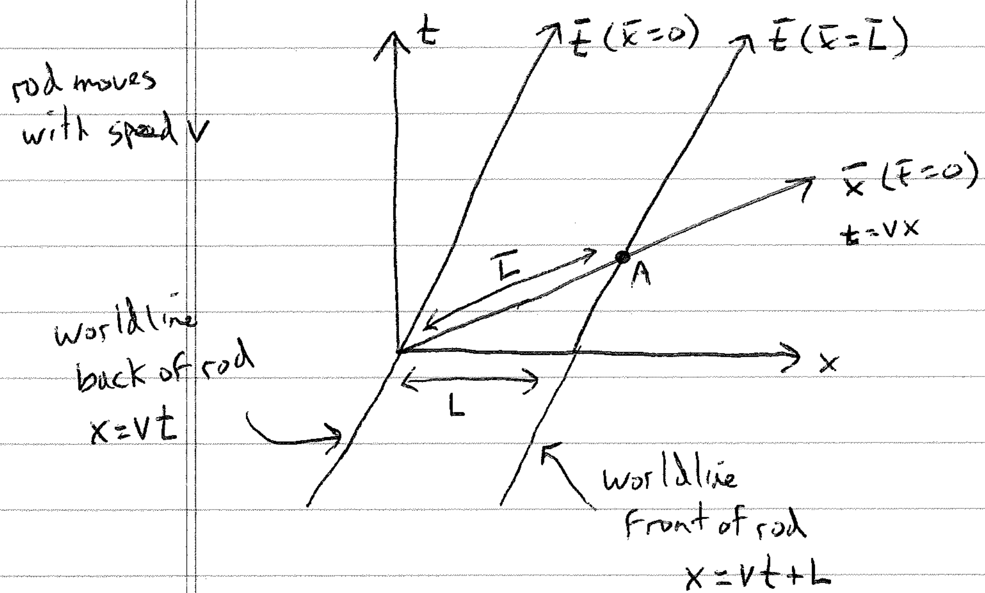
<u>independent of Frame</u> i.e. <u>invariant</u>

similar to spatial rotations in 2D:



$$(\Delta x)^2+(\Delta y)^2 = (\Delta \bar{x})^2+(\Delta \bar{y})^2 \quad \text{for all rotations.}$$

Example: Length contraction



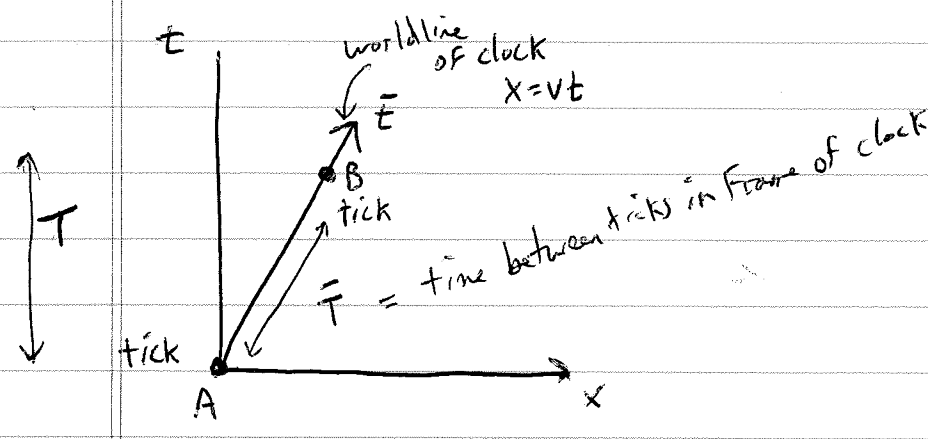
look at point A: intersection of $t = vx$ and $x = vt+L$ $\Rightarrow$ $x = \frac{L}{(1-v^2)}$

And b/w point A and the origin $\Delta s^2 = -t^2+x^2 = -\bar{t}^2+\bar{x}^2$
$= \bar{L}^2$

combine $\bar{L}^2 = -v^2x^2+x^2$
$= \frac{L^2}{1-v^2}$

$$\Rightarrow \boxed{L = \frac{\bar{L}}{\gamma} \qquad \gamma = \frac{1}{\sqrt{1-v^2}}} \qquad \gamma > 1$$

Example: time dilation



interval from A to B: $\Delta s^2 = -\bar{t}^2 + \bar{x}^2 = -t^2 + x^2$

$$= -\bar{T}^2 \qquad = -t^2 + v^2 t^2$$

$$= -T^2(1-v^2)$$

$$\Rightarrow \boxed{T = \gamma \bar{T}}$$

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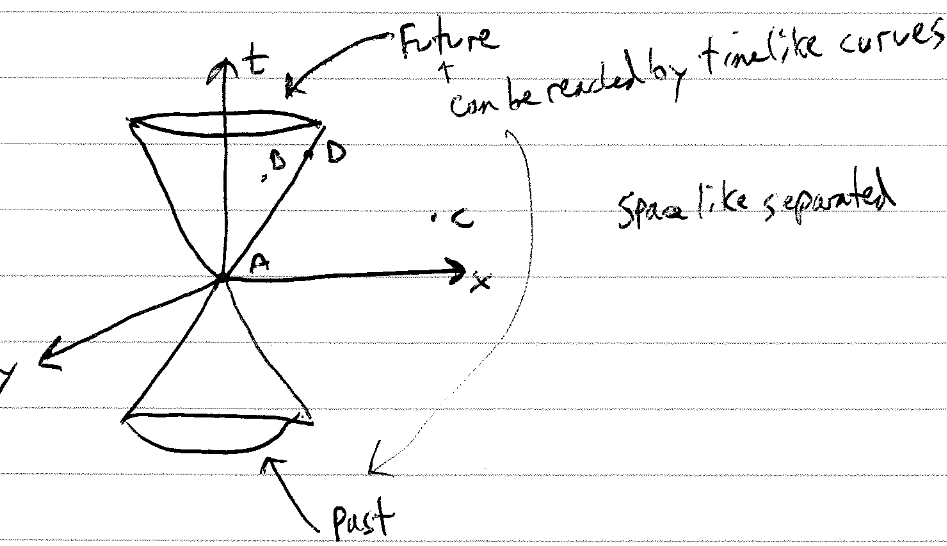
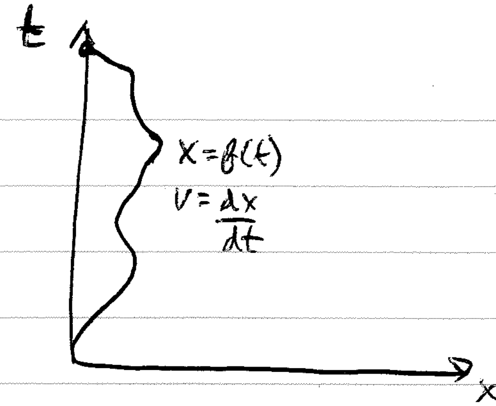
light cone



A vs. C	$\Delta s^2 > 0$	: spacelike	Δs = "proper separation"
A vs. B	$\Delta s^2 < 0$	: timelike	$\sqrt{-\Delta s^2}$ = "proper time" $\equiv \Delta\tau$
A vs. D	$\Delta s^2 = 0$	: null	

Arbitrary timelike curve



$$\text{Proper time} = \int \sqrt{-ds^2}$$

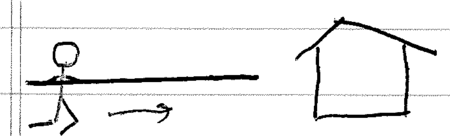
$$= \int \sqrt{-dx^2 + dt^2}$$

$$= \int \sqrt{1 - v^2}\, dt \qquad \underline{\text{less than}}\ \Delta t$$

$\Rightarrow$ timelike straight lines are <u>longest</u>

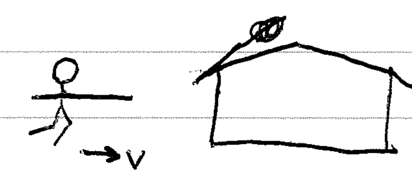
Problem: <u>Pole-vaulter thru barn "paradox"</u>

vaulter Frame:



barn looks Lorentz contracted

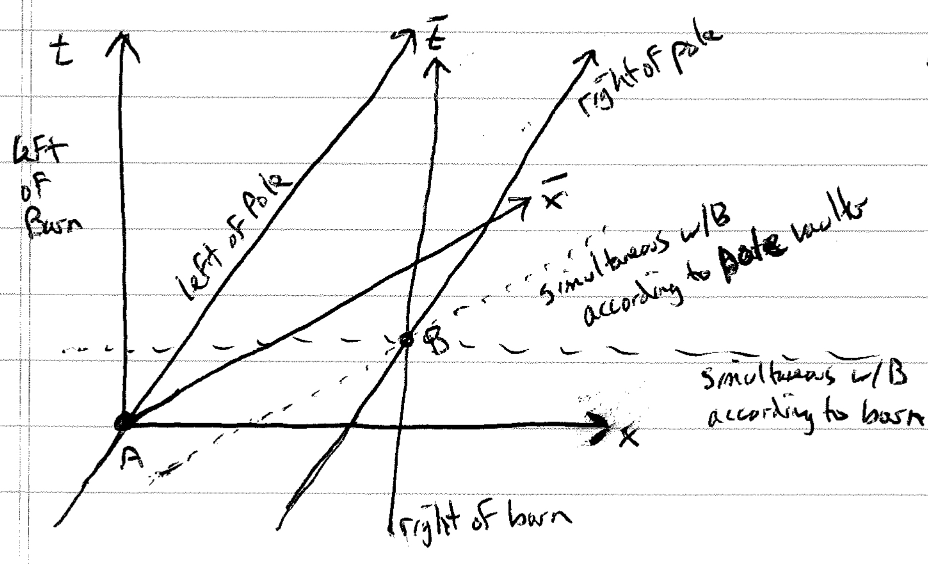
barn Frame:



Pole looks Lorentz contracted

Can Farmer close The barn door on pole?

draw in barn Frame (t, x)



A = left of pole enters barn

B = right of pole exits barn

– no paradox –

~~[illegible]~~

<u>simultaneity is relative</u>

Notation

Label coordinates of some event by x^μ $\quad \mu = 0,1,2,3$

$$x^0 = t$$
$$x^1 = x$$
$$x^2 = y$$
$$x^3 = z$$

of same event

Label coordinates in different frame by $x^{\bar\mu}$ $\quad \bar\mu = \bar0,\bar1,\bar2,\bar3$

$$x^{\bar0} = \bar t$$
$$x^{\bar1} = \bar x$$
$$x^{\bar2} = \bar y$$
$$x^{\bar3} = \bar z$$

indices are <u>up</u>

Then write interval between x^μ and origin as

$$\Delta s^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \eta_{\mu\nu} x^\mu x^\nu \qquad = \eta_{\mu\nu} x^\mu x^\nu$$

$$\eta_{\mu\nu} = \text{"minkowski metric"} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Einstein summation convention: repeated up/down indices are <u>summed</u>

Now transform between coordinates:

$$x^{\bar\nu} = \underbrace{\Lambda^{\bar\nu}{}_{\mu}}\, x^\mu \qquad \text{(summation over } \mu \text{ implied)}$$

$$\Lambda^{\bar\nu}{}_{\mu} = \frac{\partial x^{\bar\nu}}{\partial x^\mu}$$

Note some indices are up, others are down. <u>Important</u>

For SR, consider only subset of transformations that keep Δs^2 invariant

$$\Delta s^2 = \eta_{\mu\nu} x^\mu x^\nu$$

$$= \eta_{\bar{\mu}\bar{\nu}} x^{\bar{\mu}} x^{\bar{\nu}} = \eta_{\bar{\mu}\bar{\nu}} \Lambda^{\bar{\mu}}{}_{\nu} x^\nu \Lambda^{\bar{\nu}}{}_{\mu} x^\mu$$

$$= \underbrace{\left(\eta_{\bar{\mu}\bar{\nu}} \Lambda^{\bar{\mu}}{}_{\nu} \Lambda^{\bar{\nu}}{}_{\mu}\right)}_{\text{must equal } \eta_{\mu\nu}} x^\mu x^\nu$$

$\Lambda^{\bar{\mu}}{}_{\nu}$ that satisfy are Lorentz transformations

- Rotations
- Boosts $\Lambda^{\bar{\mu}}{}_{\nu \text{ (boost in x direction)}} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$x^{\bar{\mu}}$ frame moving w.r.t. x^μ frame with velocity $+v$

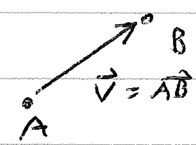
- [Also time reversals, space inversions, but usually we demand Rt. handed coords and +ive time]

- Poincaré Transformations add translations

Vectors

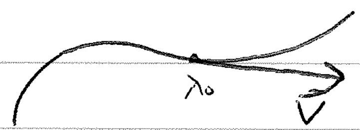
2 definitions

(a) Displacement between two events



(b) Tangent to a curve

curve $C(\lambda)$



$$\vec{V} = \left.\frac{dC}{d\lambda}\right|_{\lambda=\lambda_0}$$

In Flat space, (a) + (b) make sense
In curved space, only (b) makes sense.

(b) Vector does not need to lie in the same space as original manifold.

(b) $\Rightarrow$ defined locally (i.e. at one point)

Basis Vectors

$\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3$ example $\vec{e}_0 = \hat{t}$
$\vec{e}_1 = \hat{x}$
$\vec{e}_2 = \hat{y}$
$\vec{e}_3 = \hat{z}$

Notation: write basis vectors as $\vec{e}_\alpha$ where $\alpha = 0,1,2,3$

For $\vec{e}_\alpha$ to be a basis, any vector $\vec{V}$ can be written

$$\vec{V} = V^0\vec{e}_0 + V^1\vec{e}_1 + V^2\vec{e}_2 + V^3\vec{e}_3 \equiv \sum_\alpha V^\alpha \vec{e}_\alpha \equiv \overbrace{V^\alpha \vec{e}_\alpha}^{\text{Einstein Summation Convention}}$$

Note: V^α $(\alpha = 0,1,2,3)$ are just numbers
= components of vectors

up, down indices are significant,

Note: "dummy" indices: $V^\alpha \vec{e}_\alpha$ and $V^\beta \vec{e}_\beta$ are the same thing!
and $V^\mu \vec{e}_\mu$ (they are $\vec{V}$)

Note: we usually use $\vec{e}_0 = \hat{t}$, $\vec{e}_1 = \hat{x}$ etc. in SR.
But in general, basis vectors don't need to be orthogonal
or unit.

- Vectors can be added or multiplied by a scalar:

$$\vec{V} = a\vec{u} + b\vec{w}$$

components are $V^\alpha = a\,u^\alpha + b\,w^\alpha$

- Transformations of vectors into different frames

$\vec{\Delta x}$ displacement vector

$$= (\Delta x)^\mu \vec{e}_\mu = (\Delta x)^{\bar{\nu}} \vec{e}_{\bar{\nu}}$$

(only makes sense in flat space)

$$= (\Delta x)^\mu \Lambda^{\bar{\nu}}{}_\mu \vec{e}_{\bar{\nu}}$$

$$\Rightarrow \quad \vec{e}_\mu = \Lambda^{\bar{\nu}}{}_\mu \vec{e}_{\bar{\nu}}$$

Components: $\vec{V} = V^\mu \vec{e}_\mu = V^{\bar{\mu}} \vec{e}_{\bar{\mu}} = V^{\bar{\mu}} \Lambda^\mu{}_{\bar{\mu}} \vec{e}_\mu$

$$\Rightarrow V^\mu = \Lambda^\mu{}_{\bar{\mu}} V^{\bar{\mu}}$$