## Let's get better coords

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$= \left(1-\frac{2m}{r}\right)\left(-dt^2 + dr^{*2}\right) + r^2 d\Omega^2$$

$$dr^* = \frac{dr}{1-\frac{2m}{r}} \quad \Rightarrow \quad r^* = r + 2M \ln\left(\frac{r}{2m} - 1\right)$$

"tortoise coordinate"

Radial light rays now have $\frac{dt}{dr^*} = \pm 1$ 45° on spacetime diagram!

$t \pm r^* = \text{const.}$

but we now have both $r^*$ and $r$ .... need to do better.

let $\tilde{u} = t - r^*$

$$\Rightarrow \boxed{ds^2 = -\left(1-\frac{2m}{r}\right)d\tilde{u}^2 - 2\, d\tilde{u}\, dr + r^2 d\Omega}$$

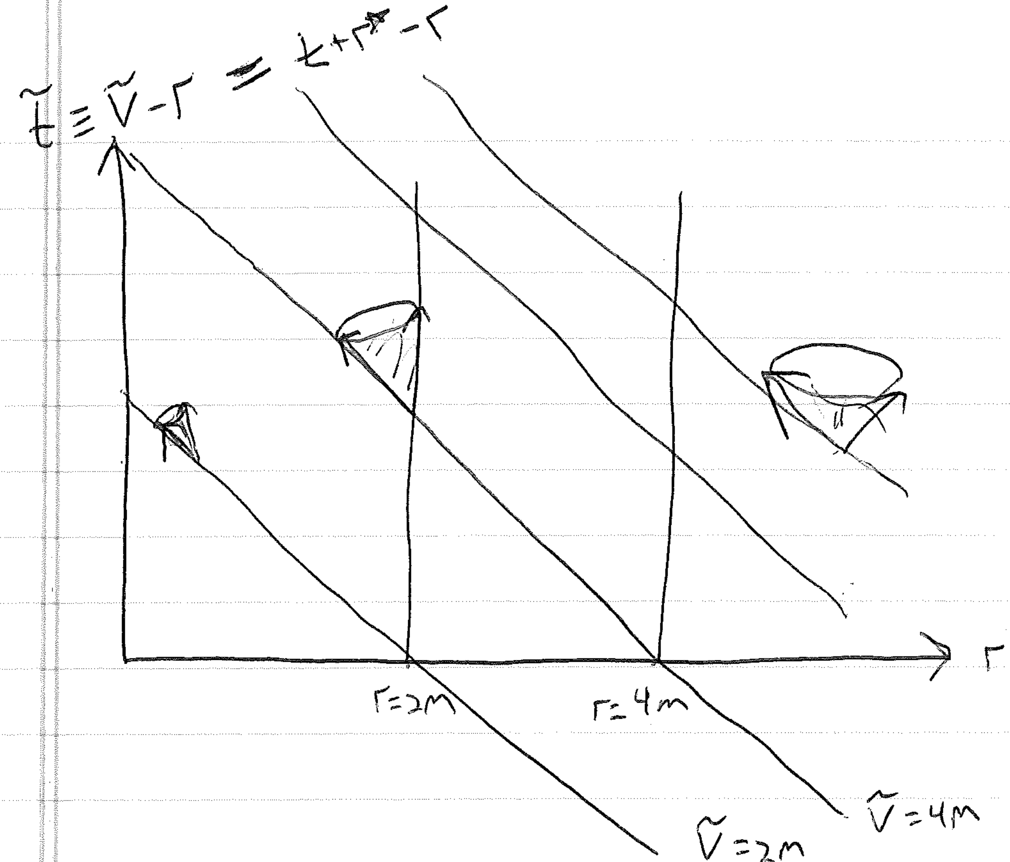
Outgoing Eddington – Finkelstein coords

$\tilde{u} = \text{const}$ is outgoing null geodesic

or let $\tilde{v} = t + r^*$

$$\Rightarrow \boxed{ds^2 = -\left(1-\frac{2m}{r}\right)d\tilde{v}^2 + 2\, d\tilde{v}\, dr + r^2 d\Omega^2}$$

Ingoing E-F coords




ingoing null rays : $\tilde{v} = \text{const}$

or $\dfrac{d\tilde{t}}{dr} = -1$

outgoing " " $\dfrac{d\tilde{v}}{dr} = \dfrac{2}{1-\frac{2M}{r}}$

or $\dfrac{d\tilde{t}}{dr} = \dfrac{1+2M/r}{1-2M/r}$ $\left( \begin{array}{l} +1 \text{ at large } r \\ \to \infty \text{ for } r=2M \end{array} \right)$

$\Rightarrow$ if inside $r=2M$, can't get out!

Now try both $\tilde{u}$ & $\tilde{v}$

$$ds^2 = -\left(1-\frac{2M}{r}\right) d\tilde{u}\, d\tilde{v} + r^2 d\Omega^2$$

$r$ now considered a function of $\tilde{u}, \tilde{v}$

$r^* = \frac{1}{2}(\tilde{v} - \tilde{u})$

$r = r(r^*)$

still have $1-\frac{2M}{r}$

But notice $\left(\frac{r}{2M} - 1\right) e^{r/2M} = e^{r^*/2M} = e^{\frac{\tilde{v}-\tilde{u}}{4M}}$

so let $\bar{u} = -e^{-\tilde{u}/4M}$

$\bar{v} = e^{\tilde{v}/4M}$

$$\Rightarrow \quad ds^2 = -\frac{32M^3}{r} e^{-r/2M} d\bar{u}\, d\bar{v} + r^2 d\Omega \qquad r = r(\bar{u}, \bar{v})$$

ok at $r = 2M$ !

Nice to have spacelike + timelike coords

let $\bar{u} = v - u$

$\bar{v} = v + u$

$$\Rightarrow \quad \boxed{ds^2 = \frac{+32M^3}{r}\left(-dv^2 + du^2\right) e^{-r/2M} + r^2 d\Omega^2 \qquad r = r(v,u)}$$

Kruskal–Szekeres coords

## 2 surprises

Note $u^2 - v^2 = \left(\frac{r}{2M} - 1\right) e^{r/2M}$

① Singularity at $r=0$ is $u^2 - v^2 = -1$

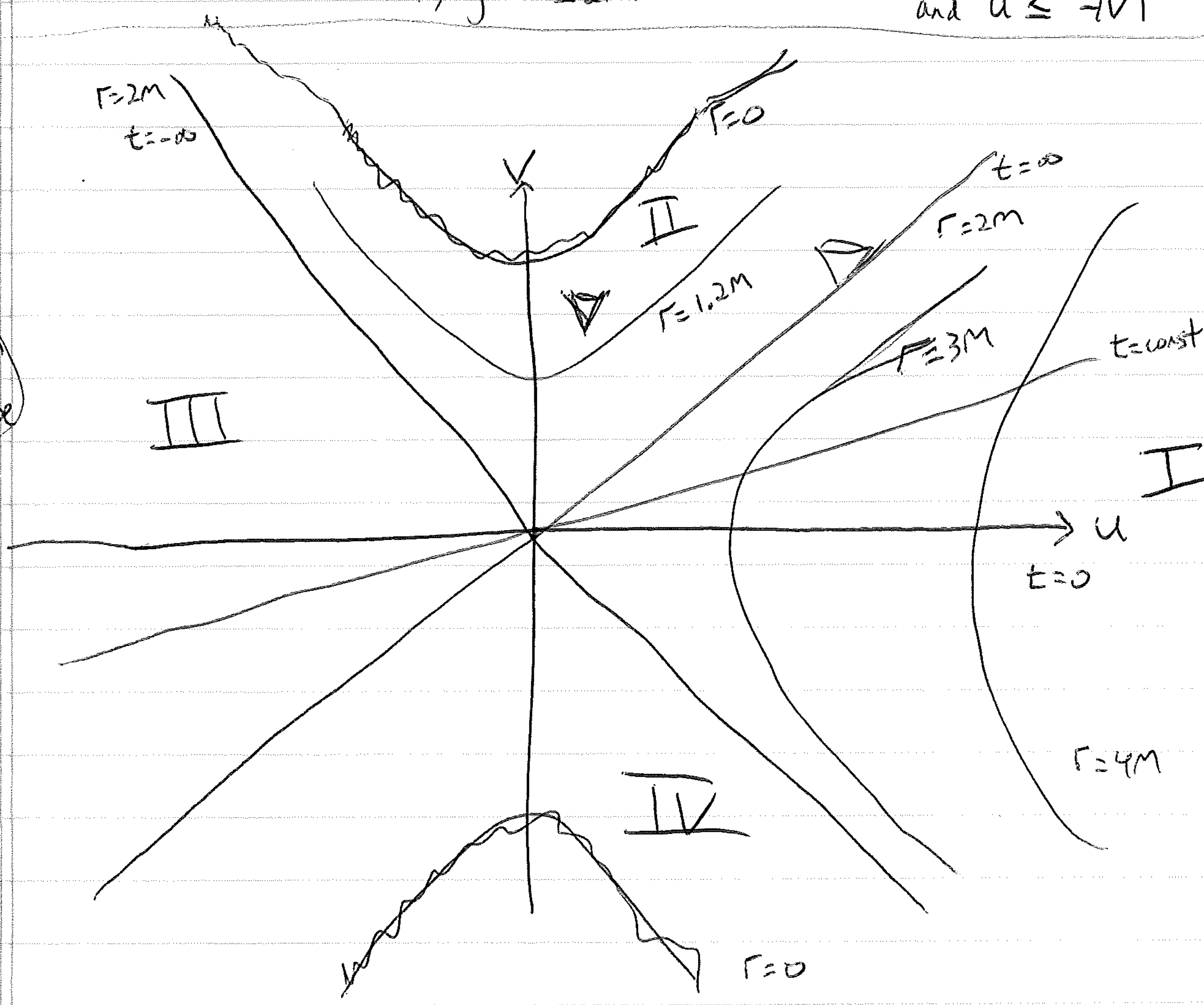
2 singularities at $v = \pm (1+u^2)^{1/2}$

Singularities are spacelike

② $r \geq 2M$ is $u^2 - v^2 \geq 0$

"another universe"

2 regions of spacetime satisfying $r \geq 2M$ at $u \geq |v|$ and $u \leq -|v|$




Once you cross ~~r=2M~~ can't escape

Light cones at 45°

# Embedding Diagram

Goal: to <u>visualize</u> an n-dim curved geometry.

To do this, embed an n-dim hypersurface in an n+m-dim Euclidean space s.t. intrinsic geometry of the hypersurface is the same as the original n-dim space.

Example; Schwarzschild with $t = \text{const}$, $\theta = \pi/2$

$$2\text{-geometry is} \quad ds^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\phi^2$$

Consider Euclidean 3-geometry $\quad ds^2 = dz^2 + dr^2 + r^2 d\phi^2 \qquad$ <u>Flat</u>

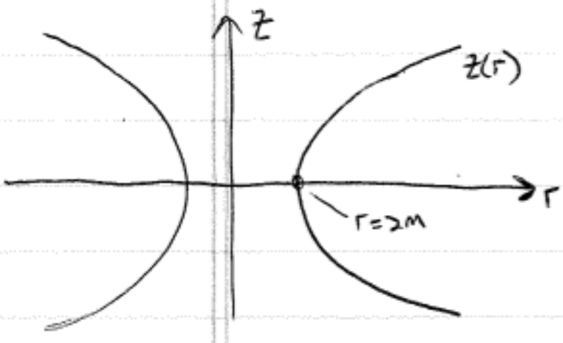
Find surface $z(r)$ (in Flat space) with same intrinsic geometry as Schwarzschild hypersurface

Assuming $z(r)$,
$$\begin{aligned} ds^2 &= dz^2 + dr^2 + r^2 d\phi^2 \\ &= \left(\frac{dz}{dr}\right)^2 dr^2 + dr^2 + r^2 d\phi^2 \\ &= \left(1 + \left(\frac{dz}{dr}\right)^2\right) dr^2 + r^2 d\phi^2 \end{aligned}$$

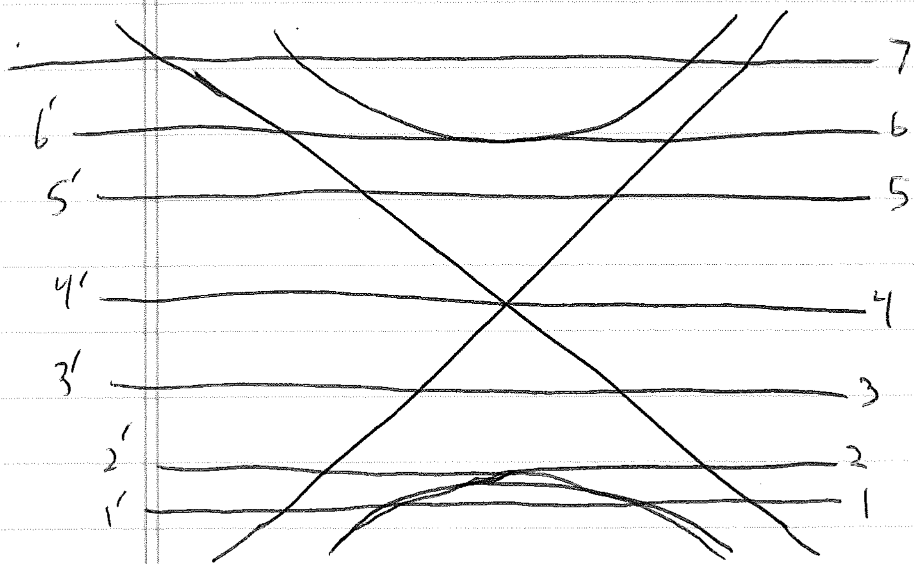
Match to schwarzschild: $\quad 1 + \left(\frac{dz}{dr}\right)^2 = \left(1 - \frac{2M}{r}\right)^{-1}$

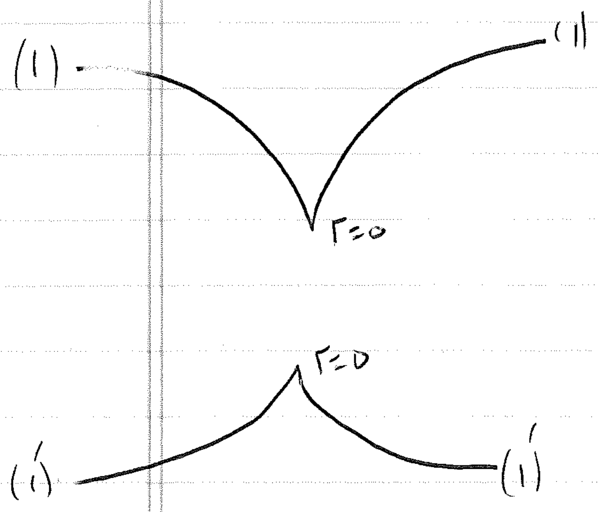
$$\left(\frac{dz}{dr}\right)^2 = \frac{1}{1 - 2M/r} - 1 = \frac{1}{r/2M - 1}$$

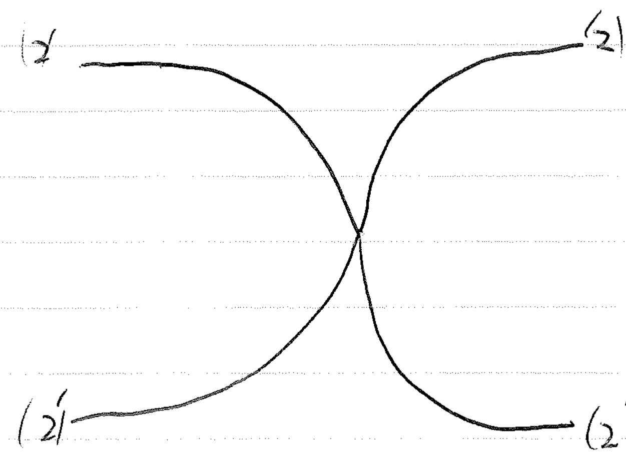
$$\frac{dz}{dr} = \frac{\pm 1}{\left(r/2M - 1\right)^{1/2}} \quad \Rightarrow \quad z = \text{const} \pm \left[8M(r-2M)\right]^{1/2}$$

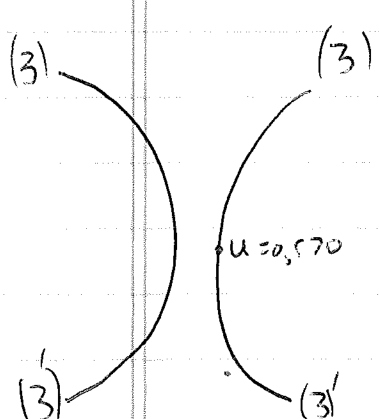


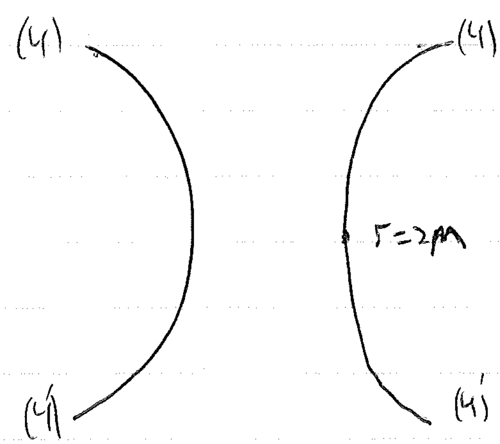
Embedding diagrams of $V = \text{const}$ slices of Schwarzschild




"Einstein – Rosen bridge", "wormhole"

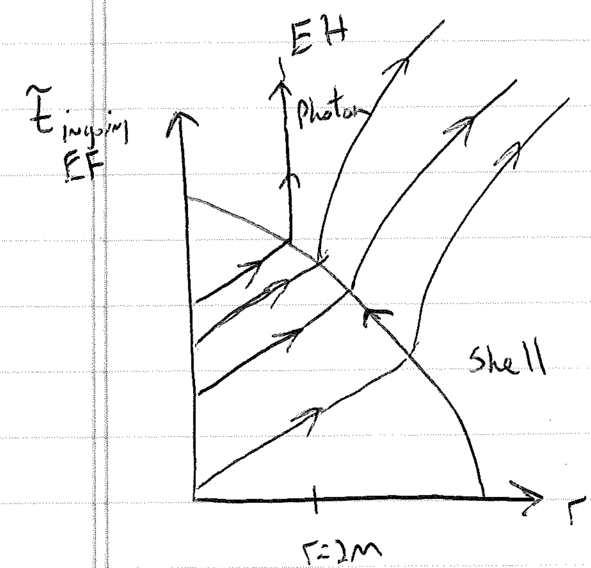
Spacetime not static to those observers!

## Shell of dust collapses to form BH

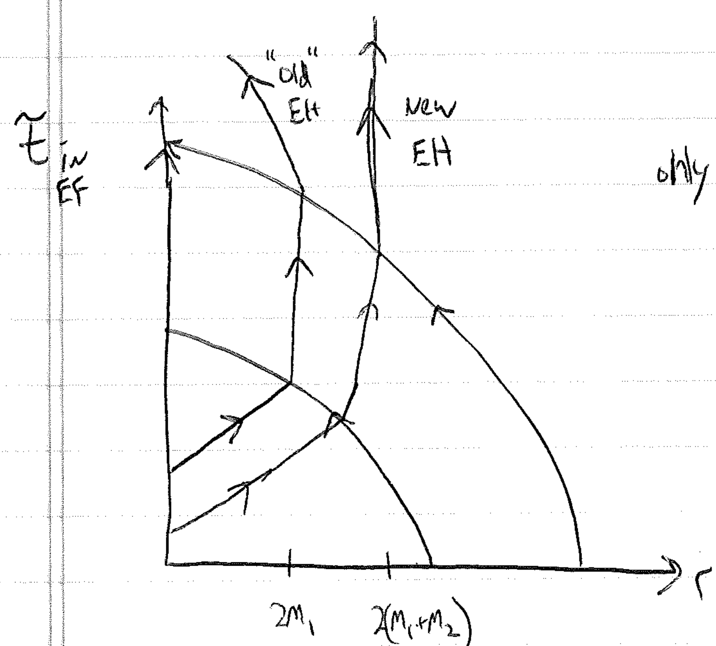
Birkhoff thm:

Exterior: Schwarzschild

Interior: Flat




2 shells:




only 1 EH, the new one

Don't know EH until full future evolution is known.

Oppenheimer - Snyder collapse:

Zero-pressure Fluid ("dust ball").

Assume uniform density, isotropic, homogeneous.

Solution for interior:

$$ds^2 = -d\tau^2 + a^2(t)\left[d\chi^2 + \sin^2\chi \, d\Omega^2\right]$$ Comoving coords

(Can use $\tau$ as coord, since proper time same for all particles, $g_{00} = -1$)

(Same as closed universe) For $0 < \chi < \chi_0$

$a \sin\chi_0 = R$ radius of star

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \Rightarrow \left(\frac{da}{d\tau}\right)^2 - \frac{a_{max}}{a} = -1$$

where $a_{max} = \text{const} = \frac{8\pi a^3}{3}\rho$

Solution

$$a = \frac{1}{2} a_{max}(1+\cos\eta)$$

$$\tau = \frac{1}{2} a_{max}(\eta + \sin\eta)$$

$\eta = 0$ at $t = 0$, $a = a_{max}$

Solution for exterior: Schwarzschild

Match interior + exterior:

Notice that particle at surface falls along geodesic (No pressure)

Recall for Schwarzschild, radial freefalling particle has

$$R = \frac{R_{init}}{2}(1+\cos\eta)$$

$$\tau = \left(\frac{R_{init}^3}{8M}\right)^{1/2}(\eta+\sin\eta)$$

So proper circumference $C = 2\pi R$
$= \pi R_{init}(1+\cos\eta)$

But interior:

$$\text{Proper circumference} = 2\pi a \sin\chi_0$$
$$= 2\pi \sin\chi_0 \frac{1}{2} a_{max}(1+\cos\eta)$$

$$\text{and } \tau = \frac{1}{2}a_{max}(\eta+\sin\eta)$$

Match: $R_{init} = a_{max}\sin\chi_0$

$$\frac{R_{init}^3}{8M} = \frac{1}{4}a_{max}^2 \quad \Rightarrow M = \frac{1}{2}\frac{R_{init}^3}{a_{max}^2}$$
$$= \frac{1}{2}a_{max}\sin^3\chi_0$$

Turns out (harder to show) entire metric matches at surface.