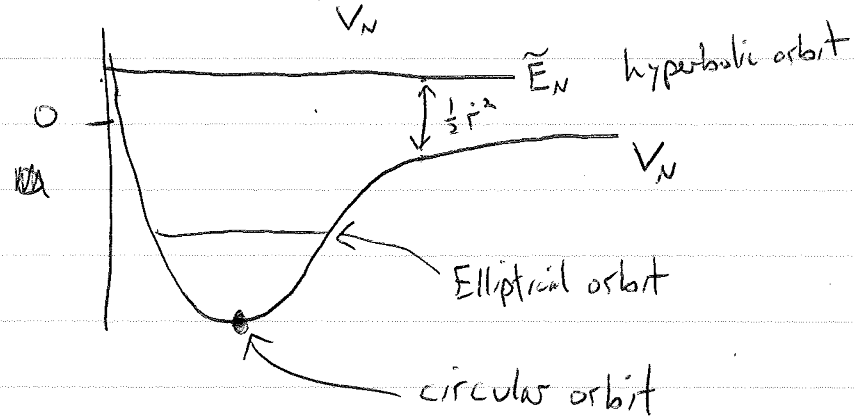
Nonradial motion

$$\left(\frac{dr}{d\tau}\right)^2 + \underbrace{\left(1-\frac{2M}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)}_{\text{call this } V_L^2(r)} = \tilde{E}^2$$

effective potential

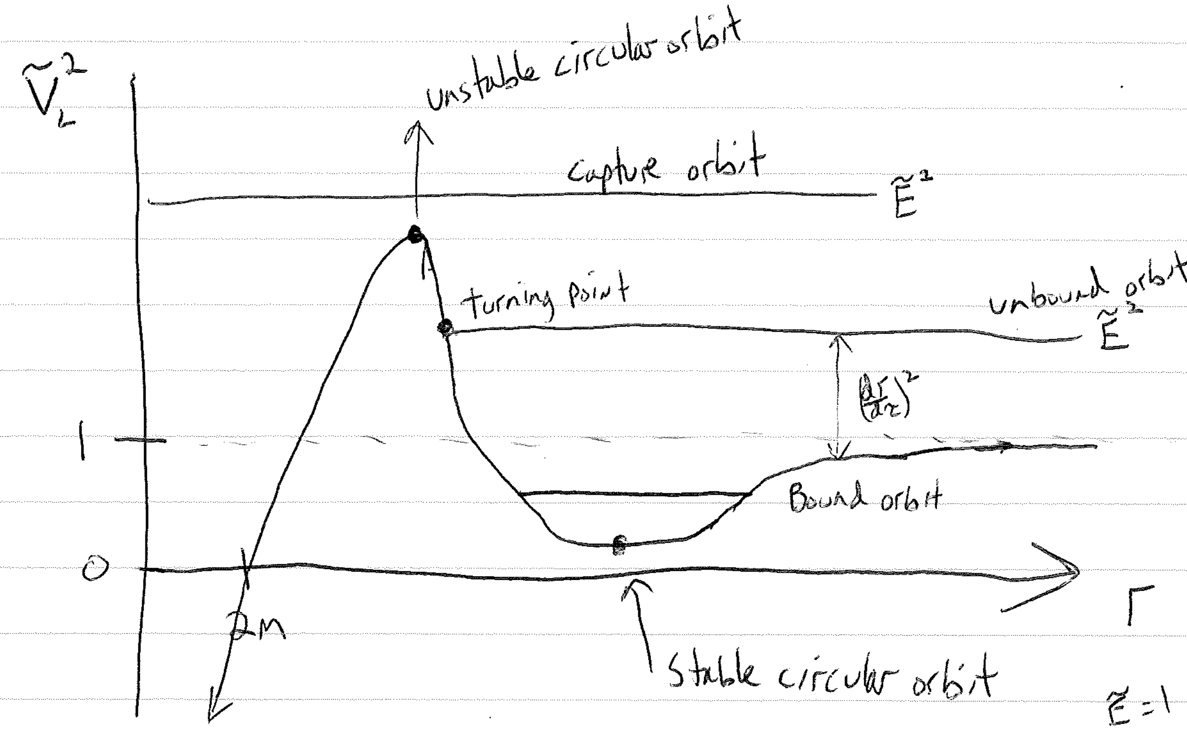
Newtonian:

$$\frac{1}{2}\dot{r}^2 \underbrace{- \frac{M}{r} + \frac{\tilde{L}^2}{2r^2}}_{V_N} = \tilde{E}_N$$



$\tilde{E}_N = 0$ is parabolic orbit

GR



$\tilde{E} = 1$ parabolic orbit

Where are max + min of $V_L^2(r)$?

$$V_L^2 = 1 + \frac{\tilde{L}^2}{r^2} - \frac{2M\tilde{L}^2}{r^3} - \frac{2M}{r}$$

$$\frac{\partial}{\partial r}(V_L^2) = -\frac{2\tilde{L}^2}{r^3} + \frac{6M\tilde{L}^2}{r^4} + \frac{2M}{r^2}$$

Extremum when $Mr^2 - \tilde{L}^2 r + 3M\tilde{L}^2 = 0$

$$\boxed{r = \frac{\tilde{L}^2 \pm \sqrt{\tilde{L}^4 - 12M^2\tilde{L}^2}}{2M}}$$

(No max or min if $\tilde{L} < 2\sqrt{3}M$

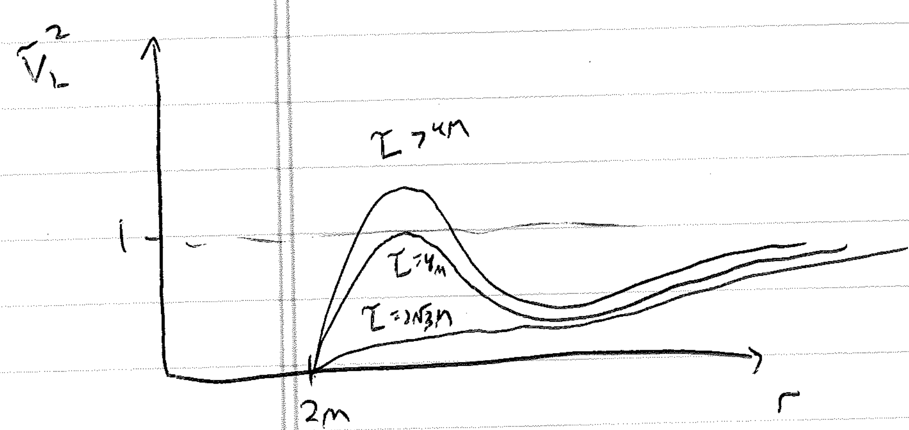
all orbits are capture orbits)

$V_L^2 = 1$ at maximum when

$$\frac{\tilde{L}^2}{r^2} - \frac{2M\tilde{L}^2}{r^3} - \frac{2M}{r} = 0 \qquad \text{and} \qquad Mr^2 - \tilde{L}^2 r + 3M\tilde{L}^2 = 0$$

$$\Rightarrow \quad r = 4M, \quad \tilde{L} = 4M$$

$\boxed{\text{so } \tilde{L} < 4M \text{ Particles from infinity captured}}$



Circular orbits

$$\frac{\partial r}{\partial \tau} = 0 \quad (\tilde{V}_L^2 = \tilde{E}^2) \qquad \underline{\text{and}} \qquad \frac{\partial \tilde{V}_L^2}{\partial r} = 0$$

$$\tilde{L}^2 = \frac{Mr^2}{r-3M} \quad \Rightarrow \quad \tilde{E}^2 = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{M}{r-3M}\right)$$

$$\Rightarrow \quad \tilde{E}^2 = \frac{(r-2M)^2}{r(r-3M)}$$

Circular orbits possible for $r \geq 3M$

but for $r = 3M$ $\quad \tilde{E}^2 = \infty$

$\tilde{L}^2 = \infty$

particle is a photon

(see better later)

Stability $\quad \dfrac{\partial^2 \tilde{V}_L^2}{\partial r^2} > 0 \qquad \dfrac{\partial^2 \tilde{V}_L^2}{\partial r^2} = \dfrac{6\tilde{L}^2}{r^4} - \dfrac{24 M\tilde{L}^2}{r^5} - \dfrac{4M}{r^3}$

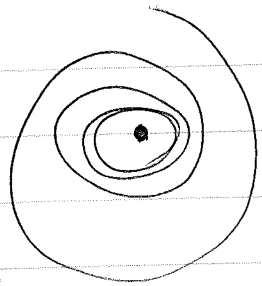
Plug in $\tilde{L}^2 = \dfrac{Mr^2}{r-3M}$

$$\frac{\partial^2 \tilde{V}_L^2}{\partial r^2} > 0 \quad \Rightarrow \quad \frac{M}{r^3}\left(\frac{r-6M}{r-3M}\right) > 0$$

circular orbits stable for $r > 6M$

unstable for $r < 6M$

Imagine particle in accretion disk



Loses energy by Friction, viscosity, etc. $\Rightarrow$ emitted as light, neutrinos, etc.

$\tilde{E} = 1$ at $r = \infty$

$\tilde{E} = \sqrt{8/9}$ at $r = 6M$

Must lose $\dfrac{\Delta E}{\text{rest mass}} = 1 - \sqrt{8/9}$

$\approx 5.7\%$

5.7% of rest mass radiated away

(Fusion $< 0.7\%$ of rest mass)

Enormous power source

Back to eqs for circular orbits:

$$\frac{d\phi}{d\tau} = \frac{\tilde{L}}{r^2} = \left(\frac{M}{r-3M}\right)^{1/2} \frac{1}{r}$$

$$\frac{dt}{d\tau} = \frac{\tilde{E}}{1-\frac{2M}{r}} = \frac{r-2M}{\sqrt{r(r-3M)}} \frac{1}{1-2M/r} = \sqrt{\frac{r}{r-3M}}$$

So $\boxed{\frac{d\phi}{dt} = \left(\frac{M}{r^3}\right)^{1/2}}$ Kepler's 3rd law!

$\frac{d\phi}{dt} = \Omega$ is angular frequency of orbit as viewed by observer at ∞.

Photon orbits

$$\frac{dt}{d\lambda} = \frac{E}{1-2M/r}$$

$$\frac{d\phi}{d\lambda} = L/r^2$$

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - L^2/r^2\left(1-\frac{2M}{r}\right)$$

$$\begin{pmatrix} E = -P_0 \\ L = P_\phi \end{pmatrix}$$

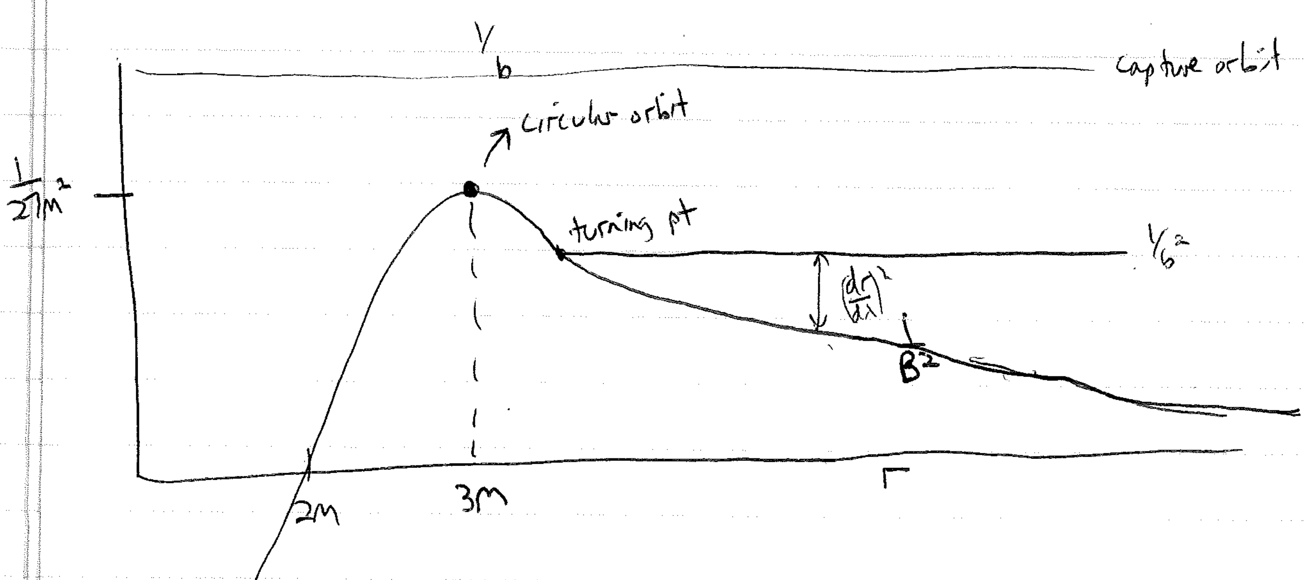
Let $\lambda_{new} = L\lambda_{old}$

let $b = L/E$

$$\Rightarrow \quad \frac{dt}{d\lambda} = \frac{1}{b\left(1-\frac{2M}{r}\right)}$$

$$\frac{d\phi}{d\lambda} = \frac{1}{r^2}$$

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} - \underbrace{\frac{1}{r^2}\left(1-\frac{2M}{r}\right)}_{=\frac{1}{B^2}}$$



unstable circular orbit at $r = 3M$

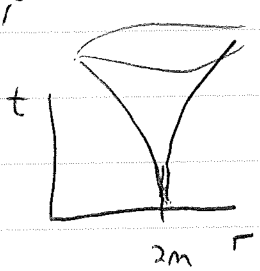
What's going on at $r = 2M$?

1. light cones
2. geodesics
3. acceleration
4. tidal forces
5. Coord transformations

① Radial light rays

$$ds^2 = 0 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2$$

$$\Rightarrow \frac{dt}{dr} = \pm\left(1-\frac{2M}{r}\right)^{-1}$$



pathological coords

② Geodesics — No problem with τ at $r = 2M$ t is pathological

③ $\vec{a} = \nabla_{\vec{u}}\vec{u}$

assume stationary observer $\vec{u}$ $r = \text{const}$, $\theta = \text{const}$, $\phi = \text{const}$

$$a^r = u^\alpha u^r{}_{;\alpha}$$
$$= u^\alpha u^r{}_{,\alpha} + \Gamma^r_{\beta\alpha}u^\beta u^\alpha$$
$$= u^0 \cancel{u^r{}_{,0}} + u^r \cancel{u^0{}_{,r}}$$
$$+ \Gamma^r_{00}u^0u^0$$

~~Compute~~ Look up $\Gamma^r_{00} = \frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1}$

Note $\vec{u}\cdot\vec{u} = -1 = u^0u^0 g_{00}$ $\Rightarrow u^0u^0 = -g_{00}^{-1}$

so $a^r = \frac{M}{r^2}$

Local acceleration magnitude $\vec{a}\cdot\vec{a}^2 = a^r a^r g_{rr} = \frac{M^2}{r^4}\left(1-\frac{2M}{r}\right)^{-1}$

(accel measured by observer $\vec{u}$)

$\Rightarrow$ Need ∞ acceleration to stay at $r=2M$

④ tidal forces

Can compute $R_{\hat{t}\hat{r}\hat{t}\hat{r}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{-2M}{r^3}$

$R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = \frac{M}{r^3}$ $= R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}$
$= -R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}}$

in orthonormal frame at const r, θ, φ

<u>No problem at $r=2M$</u>

also $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{48M^2}{r^6}$ (in all frames)

ok at $r=2M$
(not at $r=0$)