Solutions Ph 236a – Week 9

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Problem 1

Part (a)

Recall that T^{0y} is the y-component of the momentum density. If the sphere is rotating about the z-axis, the velocity of point (r, θ, ϕ) is

$$\underset{\sim}{v} = (-r\Omega \sin\theta \sin\phi, r\Omega \sin\theta \cos\phi, 0). \tag{1.1}$$

And so

$$T^{0y} = r\Omega\rho \sin\theta \cos\phi, \tag{1.2}$$

where ρ is the mass density. Since the total mass of the shell is M and the radius is R, we find

$$\rho = \frac{M}{4\pi R^2}\delta(r - R), \tag{1.3}$$

and thus

$$T^{0y} = \frac{Mr\Omega}{4\pi R^2} \sin\theta \cos\phi \, \delta(r - R), \tag{1.4}$$

which is the desired result. □

Part (b)

In linearized gravity with the Lorentz gauge we have

$$\Box \bar{h}_{\alpha\beta} = -16\pi T_{\alpha\beta}. \tag{1.5}$$

But note that the source term (right-hand side) is stationary (independent of time), and so the field $\bar{h}$ must also be stationary. Hence the above field equation becomes a spatial Poisson equation. Thus

$$\nabla^2 \bar{h}_{0y} = -16\pi T_{0y} = 16\pi T^{0y}. \tag{1.6}$$

Using the hint, we write

$$T^{0y} = \frac{Mr\Omega}{2\sqrt{3\pi}R^2} Y_{11}(\theta, \phi)\delta(r - R), \tag{1.7}$$

where we have used the real spherical harmonic

$$Y_{11}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \sin\theta \cos\phi. \tag{1.8}$$

Since $\nabla^2 Y_{\ell m} = -\ell(\ell+1)Y_{\ell m}/r^2$ and T^{0y} is a multiple of Y_{11}, it follows that $\bar{h}_{0y}$ is also a multiple of Y_{11}. Thus we write

$$\bar{h}_{0y} = F(r)Y_{11}(\theta,\phi). \tag{1.9}$$

Substituting this into (1.6) gives

$$\begin{aligned}
&Y_{11}(\theta,\phi)\nabla^2 F(r) - F(r)\frac{2Y_{11}(\theta,\phi)}{r^2} = \sqrt{\frac{\pi}{3}}\frac{8Mr\Omega}{R^2}Y_{11}(\theta,\phi)\delta(r-R)\\
\Leftrightarrow\ &\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dF}{dr}\right) - \frac{2F}{r^2} = \alpha r\delta(r-R)\\
\Leftrightarrow\ &F'' + \frac{2F'}{r} - \frac{2F}{r^2} = \alpha r\delta(r-R),
\end{aligned} \tag{1.10}$$

where

$$\alpha = \sqrt{\frac{\pi}{3}}\frac{8M\Omega}{R^2}. \tag{1.11}$$

Solving the above ODE with Mathematica gives

$$\begin{aligned}
F(r) &= C_1 r + \frac{C_2}{r^2} + \frac{\alpha R(r^3-R^3)H(r-R)}{3r^2}\\
&= r\left(C_1 + \frac{\alpha R}{3}H(r-R)\right) + \frac{1}{r^2}\left(C_2 - \frac{\alpha R^4}{3}H(r-R)\right),
\end{aligned} \tag{1.12}$$

where C_1 and C_2 are integration constants and $H(r-R)$ is the Heaviside step function, given by

$$H(r-R) = \begin{cases} 0 & \text{if } r < R\\ 1 & \text{if } r > R. \end{cases} \tag{1.13}$$

$F(r)$ needs to remain finite as $r \to 0$, and so $C_2 = 0$, because $H(0-R) = 0$. Similarly, $F(r)$ needs to be bounded as $r \to \infty$, thus $C_1 = -\alpha R/3$, because $H(\infty - R) = 1$. So we get

$$\begin{aligned}
F(r) &= -\frac{\alpha R}{3}\left(r(1-H(r-R)) + \frac{R^3}{r^2}H(r-R)\right)\\
&= -\frac{\alpha R^2}{3}\begin{cases} r/R & \text{if } r < R\\ (R/r)^2 & \text{if } r > R. \end{cases}
\end{aligned} \tag{1.14}$$

Thus we can write

$$\bar{h}_{0y} = F(r)Y_{11}(\theta,\phi) = F(r)\sqrt{\frac{3}{4\pi}}\sin\theta\cos\phi = f(r)\sin\theta\cos\phi, \tag{1.15}$$

where

$$
\begin{aligned}
f(r) = F(r)\sqrt{\frac{3}{4\pi}} &= -\sqrt{\frac{3}{4\pi}}\frac{\alpha R^2}{3}\begin{cases} r/R & \text{if } r < R \\ (R/r)^2 & \text{if } r > R \end{cases} \\
&= -\frac{4M\Omega}{3}\begin{cases} r/R & \text{if } r < R \\ (R/r)^2 & \text{if } r > R \end{cases} \\
&= \begin{cases} -\dfrac{4M\Omega r}{3R} & \text{if } r < R \\ -\dfrac{4M\Omega R^2}{3r^2} & \text{if } r > R, \end{cases}
\end{aligned} \tag{1.16}
$$

which is the desired result. □

Part (c)

Consider an observer at fixed r and $\theta = \pi/2$. Suppose there is a ring mirror in the plane $\theta = \pi/2$ and at radius r. The observer sends out a flash of light and some photons that hit the mirror tangentially will skim along the ring mirror and come back to the observer on the other side. There are photons traveling in the $+\phi$ and $-\phi$ directions. Since the photons are at constant r and θ, we have $dr = d\theta = 0$. Since they are photons, we have $ds^2 = 0$ and so

$$
\begin{aligned}
& 0 = ds^2 = g_{00}dt^2 + 2g_{0\phi}dt d\phi + g_{\phi\phi}d\phi^2 \\
\Leftrightarrow\ & 0 = g_{00} + 2g_{0\phi}\Omega + g_{\phi\phi}\Omega^2,
\end{aligned} \tag{1.17}
$$

where $\Omega = d\phi/dt$ is the angular velocity of the photons. Solving the above quadratic for Ω yields

$$
\Omega_\pm = \frac{-2g_{0\phi} \pm \sqrt{4g_{0\phi}^2 - 4g_{00}g_{\phi\phi}}}{2g_{\phi\phi}} = -\frac{g_{0\phi}}{g_{\phi\phi}} \pm \sqrt{\frac{g_{0\phi}^2}{g_{\phi\phi}^2} - \frac{g_{00}}{g_{\phi\phi}}}. \tag{1.18}
$$

If the observer is rotating with angular velocity ω, then the observed angular velocities of the photons will be $\Omega'_\pm = \Omega_\pm - \omega$. We want to find ω so that the observer sees no difference in the $+\phi$ and $-\phi$ directions, i.e. so that it looks to the observer as if she was not rotating. Hence we want $\Omega'_+ = -\Omega'_-$, therefore

$$
\begin{aligned}
& 0 = \Omega'_+ + \Omega'_- = -2\frac{g_{0\phi}}{g_{\phi\phi}} - 2\omega \\
\Leftrightarrow\ & \omega = -\frac{g_{0\phi}}{g_{\phi\phi}}.
\end{aligned} \tag{1.19}
$$

Thus we have shown that observers need to rotate with an angular velocity $\omega = -g_{0\phi}/g_{\phi\phi}$ in order to be considered stationary in the sense that spacetime looks symmetric in the ϕ direction for them. □

Part (d)

Note that the basis vectors $\vec{e}_y = \partial/\partial y$ and $\vec{e}_\phi = \partial/\partial\phi$ are parallel at $\phi = 0$. But $\vec{e}_y$ has length 1 while $\vec{e}_\phi$ has length $r\sin\theta$. Thus we get

$$g_{0\phi} = r\sin\theta\,(\bar{h}_{0y})\big|_{\phi=0} = \begin{cases} -\dfrac{4M\Omega r^2\sin^2\theta}{3R} & \text{if } r < R \\ -\dfrac{4M\Omega R^2\sin^2\theta}{3r} & \text{if } r > R. \end{cases} \tag{1.20}$$

Part (e)

Recall that $g_{\phi\phi} = r^2\sin^2\theta$ and so inside the shell we have

$$\omega = -\frac{g_{0\phi}}{g_{\phi\phi}} = \frac{4M\Omega r^2\sin^2\theta}{3R}\frac{1}{r^2\sin^2\theta} = \frac{4M\Omega}{3R}, \tag{1.21}$$

which is the desired result. □

Problem 2

Outside the neutron star we have a Schwarzschild spacetime, which is spherically symmetric and so we can choose Paul's orbit to be in the $\theta = \pi/2$ plane, so $u^\theta = 0$. Since the orbit is circular, we have $u^r = 0$. The geodesic equation

$$\frac{d^2x^\alpha}{d\lambda^2} = -\Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} \tag{2.1}$$

now gives for $\alpha = r$

$$0 = -\Gamma^r{}_{\beta\gamma}u^\beta u^\gamma. \tag{2.2}$$

Recall that the Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta\,d\phi^2, \tag{2.3}$$

and so

$$\begin{aligned}\Gamma^r{}_{\beta\gamma} &= \frac{1}{2}g^{r\mu}(g_{\mu\gamma,\beta} + g_{\beta\mu,\gamma} - g_{\beta\gamma,\mu}) \\ &= \frac{1}{2}g^{rr}(g_{r\gamma,\beta} + g_{\beta r,\gamma} - g_{\beta\gamma,r}) \qquad \text{(no summation over } r\text{)},\end{aligned} \tag{2.4}$$

since the metric is diagonal. Since $u^r = u^\theta = 0$, we are only interested for $\beta, \gamma = t, \phi$. We find

$$\begin{aligned}
\Gamma^r{}_{tt} &= -\frac{1}{2} g^{rr} g_{tt,r} = -\frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(-\frac{2M}{r^2}\right) = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right), \\
\Gamma^r{}_{t\phi} &= 0, \\
\Gamma^r{}_{\phi\phi} &= -\frac{1}{2} g^{rr} g_{\phi\phi,r} = -\frac{1}{2}\left(1 - \frac{2M}{r}\right) 2r\sin^2\theta = -\left(1 - \frac{2M}{r}\right) r,
\end{aligned} \tag{2.5}$$

since $\theta = \pi/2$. Now the geodesic equation gives

$$\begin{aligned}
0 &= -\Gamma^r{}_{tt}(u^t)^2 - \Gamma^r{}_{\phi\phi}(u^\phi)^2 \\
\Leftrightarrow \quad \left(\frac{u^\phi}{u^t}\right)^2 &= -\frac{\Gamma^r{}_{tt}}{\Gamma^r{}_{\phi\phi}} = \frac{M}{r^3} = \frac{1}{216M^2},
\end{aligned} \tag{2.6}$$

since Paul's orbit is at $r = 6M$. Note that

$$\omega^2 = \left(\frac{d\phi}{dt}\right)^2 = \left(\frac{d\phi}{d\lambda}\frac{d\lambda}{dt}\right)^2 = \left(\frac{u^\phi}{u^t}\right)^2, \tag{2.7}$$

and so

$$\omega = \frac{d\phi}{dt} = \frac{1}{6\sqrt{6}M}. \tag{2.8}$$

Between the two meetings of Paul and Peter, Paul completes 10 orbits, so $\Delta\phi = 20\pi$ and thus this takes

$$\Delta t = 20\pi \times 6\sqrt{6}M = 120\sqrt{6}\pi M \tag{2.9}$$

in coordinate time. Recall that $u^\alpha u_\alpha = -1$, so

$$\begin{aligned}
-1 &= g_{tt}(u^t)^2 + g_{\phi\phi}(u^\phi)^2 = g_{tt}(u^t)^2 + g_{\phi\phi}\omega^2(u^t)^2 \\
&= \left(-1 + \frac{2M}{r} + \frac{Mr^2\sin^2\theta}{r^3}\right)(u^t)^2 \\
&= \left(-1 + \frac{3M}{r}\right)(u^t)^2 \\
\Leftrightarrow \quad u^t &= \left(1 - \frac{3M}{r}\right)^{-1/2} = \sqrt{2}.
\end{aligned} \tag{2.10}$$

Since

$$u^t = \frac{dt}{d\tau}, \tag{2.11}$$

the elapsed proper time measured by Paul is

$$\Delta\tau_{\text{Paul}} = \frac{\Delta t}{u^t} = 120\sqrt{3}\pi M \approx 653M. \tag{2.12}$$

Let the H be the maximum height that Peter reaches. Once Peter is at $r = H$, he will fall freely back to the neutron star. Due to symmetry, the time it takes Peter to fall from H to $r = 6M$ is $\Delta t/2$. Using the equations derived in class for radial free-fall, we have

$$\Delta\tau_{\text{Peter}} = 2\sqrt{\frac{H^3}{8M}}(\eta + \sin\eta), \tag{2.13}$$

where the factor of 2 comes from the fact that Peter first travels from $r = 6M$ to $r = H$ and then he freely falls from $r = H$ back to $r = 6M$. By symmetry, both directions take the same amount of proper time and coordinate time. To find η and H, we have the following two equations

$$\begin{aligned} 6M &= r = Mh(1+\cos\eta), \\ 30\sqrt{6}\pi &= \frac{\Delta t}{2M} = \ln\left|\frac{\sqrt{h-1}+\tan\eta/2}{\sqrt{h-1}-\tan\eta/2}\right| + \sqrt{h-1}\left(\eta + \frac{h}{2}(\eta+\sin\eta)\right), \end{aligned} \tag{2.14}$$

where $h = H/(2M)$. We can find a numerical solution to the above equations with Mathematica's `FindRoot`, but we need to supply it with an initial guess. We know that Peter passes $r = 6M$ and keeps moving to a higher radius. Thus we guess $h \sim 20$ and then $\eta = \arccos(6/h - 1) \sim 2.3$. With these initial guesses, Mathematica gives

$$h \approx 27.35, \qquad \eta \approx 2.466. \tag{2.15}$$

Thus the elapsed proper time for Peter is

$$\Delta\tau_{\text{Peter}} = 2M\sqrt{h^3}(\eta + \sin\eta) \approx 884M. \tag{2.16}$$

Note that it makes sense that less proper time passes for Paul, because Paul is in a highly relativistic orbit inside a deep gravitational well, which makes his clock go much slower. Peter spends a lot of time much further away from the neutron star.

Problem 3

A rocketship is a massive object so it must move along a time-like worldline. Parameterize its trajectory by the proper time, τ, and so its 4-velocity must satisfy

$$\begin{aligned} 1 &= -\vec{u}\cdot\vec{u} = g_{\mu\nu}u^\mu u^\nu \\ &= \left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\theta}{d\tau}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 \end{aligned} \tag{3.1}$$

Inside of the horizon, ($r < 2M$), all of the terms are negative, except the $\left(\frac{dr}{d\tau}\right)^2$ term so then

$$\left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 > 0 \tag{3.2}$$

We also know that the sign of $\frac{dr}{d\tau}$ must be negative for a physical, future directed observer. From that,

$$\begin{aligned} dr &< \left(1 - \frac{2M}{r}\right)^{1/2} d\tau \\ \Rightarrow \tau_{max} &= \int_{2M}^{0} \left(1 - \frac{2M}{r}\right)^{-1/2} dr \\ \Rightarrow \tau_{max} &= \left[\sqrt{r(2M-r)} + M\cos^{-1}\left(\frac{r}{M} - 1\right)\right]\Bigg|_{2M}^{0} = \pi M \end{aligned} \tag{3.3}$$

Problem 4

Part (a)

When the particle moves in the equitorial plane, we have $L^2 = p_\phi^2$, which is a conserved quantity because p_ϕ is conserved. ($\vec{\xi} = \frac{\partial}{\partial \phi}$ is a Killing vector so $\vec{\xi} \cdot \vec{p}$ is conserved.) However, by spherical symmetry, the motion is alway in the equitorial plane for some rotated coordinate system. If p_ϕ can be written as an invarient quantity and then evaluated the invarient in the original system, that would be sufficient to prove the problem statement.

At some instant when the particle is at radius r, the covarient 4-velocity $\tilde{p}$ has components $(p_r, p_t, p_\theta, p_\phi)$. Consider the a "reduced" 4-velocity $\tilde{p}_\alpha^{red}$ that is constructed from $\tilde{p}_\alpha$ via a projection operator and whose construction is independent of θ and ϕ. Now when the motion is in the equitorial plane, $p_\theta = 0$ and $\theta = \pi/2$ so then

$$L^2 = g^{\alpha\beta}\tilde{p}_\alpha^{red}\tilde{p}_\beta^{red} = g^{\phi\phi} r^2 p_\phi^2 = p_\phi^2 \tag{4.1}$$

is a conserved quantity. However, in general

$$L^2 = g^{\theta\theta} r^2 p_\theta^2 + g^{\phi\phi} r^2 p_\phi^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} \tag{4.2}$$

so this must be conserved in general.

Part (b)

Using spherical symetry of the metric, we orient the coordinate axes so that the particla is at $\theta = \pi/2$ with $\dot{\theta} = 0$ at $\tau = 0$. Now consider the geodesic equation for the θ

$$\begin{aligned}\frac{d^2\theta}{d\tau^2} =& g^{\alpha\theta}\Gamma_{\theta\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} \\ =& g^{\theta\theta}\Gamma_{\theta\beta\gamma}x^\beta x^\gamma \\ =& \frac{1}{r^2}\Gamma_{\theta\beta\gamma}x^\beta x^\gamma\end{aligned} \tag{4.3}$$

using the fact that the metric is diagonal and that $g^{\theta\theta}g_{\theta\theta} = 1$. Using the equation for Christoffel symbols $\Gamma_{\theta\beta\gamma} = \frac{1}{2}(-g_{\beta\gamma,\theta} + g_{\theta\beta,\gamma} + g_{\theta\gamma,\beta})$, the only nonzero symbols are

$$\begin{aligned}\Gamma_{\theta\theta r} = \Gamma_{\theta r\theta} &= r \\ \Gamma_{\theta\phi\phi} &= r^2\sin\theta\cos\theta\end{aligned} \tag{4.4}$$

Then the geodesic equation from above becomes

$$\frac{d^2\theta}{d\tau^2} = -\frac{2}{r}\dot{r}\dot{\theta} + \sin\theta\cos\theta\dot{\phi}^2 \tag{4.5}$$

From the initial conditions, start with $\dot{\theta} = 0$ and $\cos\pi/2 = 0$ which imply that

$$\frac{d^2\theta}{d\tau^2} = 0 \tag{4.6}$$

Thus, $\dot{\theta}$ is constant and so since it starts at 0, it will remain at 0 for all τ and thus will not move out of the $\theta = \pi/2$ plane.

Part (c)

Using the constant of motion L^2 from above, start with

$$\left(\frac{d\theta}{d\lambda}\right)^2 = (g^{\theta\theta}p_\theta)^2 = \frac{1}{r^4}\left(L^2 - \frac{p_\phi^2}{\sin^2\theta}\right) \tag{4.7}$$

Let the unperturbed orbit be at $\theta = \pi/2$ with $L = p_\phi = K$ constant in which case the equation is 0. Suppose that the particle is perturbed out of the plane of the orbit, $\theta = \pi/2 + \delta\theta, L = K + \delta L, p_\phi = K + \delta p_\phi$. Taylor expand the above equation and keep terms to first order in δL and δp_ϕ and second order in $\delta\theta$, noting that the $0th$ order terms disappear because they match the equation above.

$$\left[\frac{d(\delta\theta)}{d\lambda}\right]^2 = \frac{1}{r^4}[2K(\delta L) - 2K\delta p_\phi - K^2(\delta\theta)^2] \tag{4.8}$$

where the expansion about $\delta\theta = 0$ gives $\sin^{-1}(\pi/2 + \delta\theta) = 1 + (\delta\theta)^2 + \mathcal{O}(\delta\theta^4)$. Now take the derivative of both sides by $d\lambda$ and drop the higher order terms.

$$\begin{aligned}\frac{d}{d\lambda}\left[\frac{d(\delta\theta)}{d\lambda}\right]^2 = \frac{d(\delta\theta)}{d\lambda}\left[\frac{d^2(\delta\theta)}{d\lambda^2}\right] &= -\frac{K^2}{r^4}\delta\theta\frac{d(\delta\theta)}{d\lambda} \\ \Rightarrow \frac{d^2(\delta\theta)}{d\lambda^2} &= -\frac{K^2}{r^4}\delta\theta \end{aligned} \tag{4.9}$$

But this is simply the equation of motion for a harmonic oscillator for $\delta\theta$. Thus the perturbation $\delta\theta$ does not grow, but continues to oscillate around $\pi/2$ and so the orbit is stable.

Problem 5

Part (a)

Start by defining new coordinates $X = x, Y = y, Z(t,z)$ and $T(t,z)$. For ease of notation, define $Z_z = \frac{\partial Z}{\partial z}, T_z = \frac{\partial T}{\partial z}, ...$ so then

$$\begin{aligned} dZ^2 &= Z_z^2 dz^2 + Z_t^2 dt^2 + 2Z_z Z_t dz dt \\ dT^2 &= T_z^2 dz^2 + T_t^2 dt^2 + 2T_z T_t dz dt \end{aligned} \tag{5.1}$$

Note, that to show the Rindler metric is flat, want $-dT^2 + dZ^2 = -g^2z^2dt^2 + dz^2$ which means that

$$\begin{aligned} Z_z^2 - T_z^2 &= 1 \\ T_z T_t &= Z_z Z_t \\ T_t^2 - Z_t^2 &= g^2 z^2 \end{aligned} \tag{5.2}$$

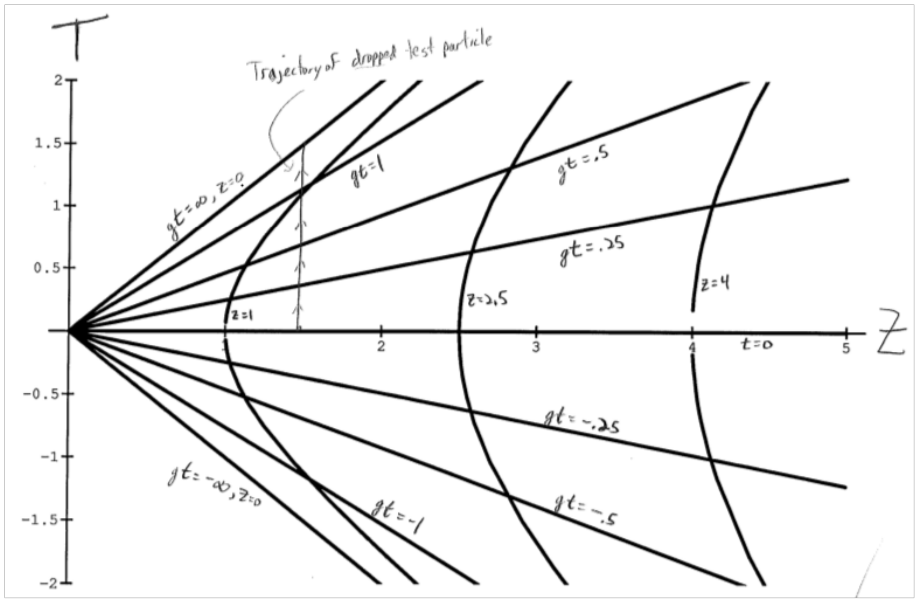
The functions which satisfy these conditions are the hyperbolic sine and cosine functions so

$$\begin{aligned} Z &= \pm z \cosh gt \\ T &= \pm z \sinh gt \end{aligned} \tag{5.3}$$

Plug those into the equation for the metric and see that $ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2$.

Part (b)

The graph below illustrates the relationship between the two coordinate systems. The curves of constant t obey $\pm \tanh gt = T/Z$ which are straight lines. Note that the slope of this family of lines is confined to between ± 1. Curves of



constant z obey $z^2 = Z^2 - T^2$ which are hyperbolae. The Rindler coordinates also break down for the region where $T > Z$.

For a point particle dropped at $t = 0, z = z_0$, its trajectory is a straight line in the $+T-$direction in the T, Z coordinates, because it is moving in geodesic in Minkowski spacetime. However, the particle will not appear to move in a straight line in the Rindler coordinates, but it still moves along geodesics so it obeys the geodesic equation of motion

$$\frac{d^2x^\alpha}{d\lambda^2} = -\Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma \tag{5.4}$$

and since in a coordinate basis, $\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha})$. However, the only derivative of the metric that is not zero is

$$\begin{aligned} & & g_{00,z} &= -2zg^2 \\ \Rightarrow \Gamma_{00z} = \Gamma_{0z0} &= -g^2z, & \Gamma_{z00} &= g^2z \\ \Rightarrow \Gamma^0{}_{0z} = \Gamma^0{}_{z0} &= \frac{1}{z}, & \Gamma^z{}_{00} &= g^2z \end{aligned} \tag{5.5}$$

Plugging this into the equation of motion for each of the components yields

$$\begin{aligned} \ddot{x} &= \ddot{y} = 0 \\ \ddot{t} &= -\frac{2}{z}\dot{t}\dot{z} \\ \ddot{z} &= -g^2z\dot{t}^2 \end{aligned} \tag{5.6}$$

where the each dot reperesents $\frac{d}{d\lambda}$. To reduce them to Rindler coordinates,

$$\begin{aligned}\frac{dz}{dt} =& \frac{\dot{z}}{\dot{t}} \\ \Rightarrow \frac{d^2z}{dt^2} =& \frac{d}{dt}\frac{\dot{z}}{\dot{t}} \\ =& \frac{\frac{d}{d\lambda}(\dot{z}/\dot{t})}{\dot{t}} \\ =& \frac{\dot{t}\ddot{z} - \ddot{t}\dot{z}}{\dot{t}^3} \\ =& \frac{-\dot{t}^3 g^2 z + \frac{2}{z}\dot{t}\dot{z}^2}{\dot{t}^3} \\ \frac{d^2z}{dt^2} =& -g^2 z + \frac{2}{z}\left(\frac{dz}{dt}\right)^2 \end{aligned} \tag{5.7}$$

For large z, this looks like a harmonic oscillator potential pulling the particle towards $z = 0$, but when z gets small, the second term takes over and slows the particle down. The particle will reach the $z = 0$ plane at $t = \infty$.

Part (c)

The Schwarzchild metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{5.8}$$

Let $\tilde{u} = \left(1 - \frac{2M}{r}\right)^{1/2}$ for $r \geq 2M$. From there,

$$\begin{aligned} d\tilde{u} &= \frac{1}{2}\frac{1}{\tilde{u}}\frac{2M}{r^2}dr \\ \Rightarrow dr &= \frac{\tilde{u}r^2}{M}d\tilde{u} \end{aligned} \tag{5.9}$$

With this, the equation equation above can be rewritten with this coordinate as

$$ds^2 = -\tilde{u}^2 dt^2 + \frac{r^4}{M^2}d\tilde{u}^2 + r^2 d\Omega^2 \tag{5.10}$$

where now $r = r(\tilde{u})$. Near the event horizon, $r \approx 2M, \tilde{u}$ is small so then

$$ds^2 = -\tilde{u}^2 dt^2 + \frac{16M^4}{M^2}d\tilde{u}^2 + 4M^2 d\Omega^2 \tag{5.11}$$

Make one last substitution for $u = 4M\tilde{u}$ and then

$$\begin{aligned} ds^2 =& -\frac{u^2}{16M^2}dt^2 + du^2 + 4M^2 d\Omega^2 \\ =& -(gu)^2 dt^2 + du^2 + 4M^2 d\Omega^2 \end{aligned} \tag{5.12}$$

where $g = \frac{1}{4M}$. This is the Rindler spacetime for radial infall. This transformation can help gain insight of what happens to a particle passing through the event horizon. As it approaches $r = 2M$, a distant observer (whose proper time is just t) sees the particle slow down and take an infinite time to reach $u = 0$, just like above in part (b). But since Rindler is really just flat spacetime, an observer with the particle sees nothing special as it pass through $r = 2M$.