Solutions Ph 236a – Week 8

Kevin Barkett, Jonas Lippuner, and Mark Scheel

December 1, 2015

Contents

Problem 1 2
Part (a) 2
Part (b) 2
Problem 2 3
Part (a) 3
Part (b) 3
Part (c) 4
Part (d) 4
Problem 3 5
Part (a) 5
Part (b) 5
Problem 4 6
Part (a) 6
Part (b) 6
Part (c) 7
Problem 5 7
Part (a) 7
Part (b) 8
Part (c) 8

Problem 1

Part (a)

We find

$$
\begin{aligned}
a_\alpha = u_{\alpha;\mu} u^\mu &= \left(\frac{\xi_{\alpha;\mu}}{|\xi_\nu \xi^\nu|^{1/2}} - \frac{1}{2} \frac{\xi_\alpha 2\xi_{\nu;\mu}\xi^\nu}{|\xi_\nu \xi^\nu|^{3/2}} \right) \frac{\xi^\mu}{|\xi_\nu \xi^\nu|^{1/2}} \\
&= \frac{\xi_{\alpha;\mu}\xi^\mu}{|\xi_\nu \xi^\nu|} - \frac{\xi_\alpha \xi_{\nu;\mu}\xi^\mu \xi^\nu}{|\xi_\nu \xi^\nu|^2}.
\end{aligned} \tag{1.1}
$$

Recall that the defining property of a Killing vector is $\xi_{\nu;\mu} = -\xi_{\mu;\nu}$. So we have

$$
\xi_{\nu;\mu}\xi^\mu \xi^\nu = -\xi_{\mu;\nu}\xi^\mu \xi^\nu = -\xi_{\nu;\mu}\xi^\nu \xi^\mu = 0. \tag{1.2}
$$

And the first term becomes

$$
a_\alpha = -\frac{\xi_{\mu;\alpha}\xi^\mu}{|\xi_\nu \xi^\nu|}. \tag{1.3}
$$

Since $u^\alpha = \xi^\alpha / |\xi_\nu \xi^\nu|^{1/2}$ is a 4-velocity, we have

$$
\begin{aligned}
-1 = u_\alpha u^\alpha &= \frac{\xi_\alpha \xi^\alpha}{|\xi_\nu \xi^\nu|} \\
\Leftrightarrow \quad |\xi_\nu \xi^\nu| &= -\xi_\alpha \xi^\alpha,
\end{aligned} \tag{1.4}
$$

and so

$$
a_\alpha = \frac{\xi_{\mu;\alpha}\xi^\mu}{\xi_\mu \xi^\mu} = \frac{1}{2} \frac{(\xi_\mu \xi^\mu)_{;\alpha}}{\xi_\mu \xi^\mu} = \frac{1}{2} \nabla_\alpha \log |\xi_\mu \xi^\mu|, \tag{1.5}
$$

which is the desired result. □

Part (b)

Since the 4-velocity $\vec{u}$ of the fluid is parallel to some timelike Killing vector, let $\vec{\xi} = \partial/\partial x^0 = \vec{e}_0$ be this timelike Killing vector and then to have $\vec{u} \cdot \vec{u} = -1$ it follows that

$$
\vec{u} = \frac{\vec{\xi}}{(-\vec{\xi} \cdot \vec{\xi})^{1/2}}, \tag{1.6}
$$

since $\vec{\xi} \cdot \vec{\xi} < 0$ because $\vec{\xi}$ is timelike. Using part (a) we know that

$$
\nabla_{\vec{u}} \vec{u} = \frac{1}{2} \nabla \log |\vec{\xi} \cdot \vec{\xi}| = \nabla \log |\vec{\xi} \cdot \vec{\xi}|^{1/2} = \nabla \log(-g_{00})^{1/2}, \tag{1.7}
$$

because

$$|\vec{\xi} \cdot \vec{\xi}| = -\vec{\xi} \cdot \vec{\xi} = -\vec{e}_0 \cdot \vec{e}_0 = -g_{00}. \tag{1.8}$$

Note that $\xi^\alpha = (1, 0, 0, 0)$ and since $u^\alpha \propto \xi^\alpha$ it follows that $u^i = 0$, so

$$\nabla_{\vec{u}} p = p_{;\mu} u^\mu = p_{,0} u^0 = u^0 \frac{\partial p}{\partial t} = 0, \tag{1.9}$$

because we are in hydrodynamic equilibrium. Thus the Euler equation gives

$$\begin{aligned} &(p + \rho)\nabla_{\vec{u}}\vec{u} = -\nabla p - \vec{u}\nabla_{\vec{u}} p \\ \Leftrightarrow \quad &(p + \rho)\frac{\partial}{\partial x^\nu} \log(-g_{00})^{1/2} = -p_{,\nu} - 0, \end{aligned} \tag{1.10}$$

which is the desired result. □

Problem 2

Part (a)

The problem gives that $\vec{\xi}$ = separation between the beads and that $\ell = \vec{n} \cdot \vec{\xi}$. The stick follows a geodesic, so that

$$\vec{\nabla}_{\vec{u}}\vec{u} = \vec{a} = 0 \tag{2.1}$$

so $\vec{n}$ is parallel transported along $\vec{u} = \frac{d}{d\tau}$ which implies that

$$\vec{\nabla}_{\vec{u}}\vec{n} = 0 \tag{2.2}$$

so then

$$\begin{aligned} \frac{d\ell}{d\tau} &= \vec{\nabla}_{\vec{u}}(\vec{\xi} \cdot \vec{n}) \\ \frac{d^2\ell}{d\tau^2} &= \vec{\nabla}_{\vec{u}}\vec{\nabla}_{\vec{u}}(\vec{\xi} \cdot \vec{n}) \\ &= \vec{n} \cdot (\vec{\nabla}_{\vec{u}}\vec{\nabla}_{\vec{u}}\vec{\xi}) \end{aligned} \tag{2.3}$$

Part (b)

From part (a), $\frac{d^2\ell}{d\tau^2} = \vec{n} \cdot (\vec{\nabla}_{\vec{u}}\vec{\nabla}_{\vec{u}}\vec{\xi})$. However, the Riemann curvature of the wave (geodesic deviation) gives

$$\vec{\nabla}_{\vec{u}}\vec{\nabla}_{\vec{u}}\vec{\xi} = -\tilde{R}(\_, \vec{u}, \vec{\xi}, \vec{u}) \tag{2.4}$$

Plugging this back in gives

$$\frac{d^2\ell}{d\tau^2} = -\vec{n} \cdot \tilde{R}(\_, \vec{u}, \vec{\xi}, \vec{u}) = -\tilde{R}(\vec{n}, \vec{u}, \vec{\xi}, \vec{u}) = -R_{\alpha\beta\gamma\delta} n^\alpha u^\beta \xi^\gamma u^\delta \tag{2.5}$$

Part (c)

In vacuum, where $T_{\mu\nu} = 0$ the wave equation for $\bar{h}$ becomes

$$\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = 0 \Rightarrow \bar{h}_{\mu\nu,tt} = \bar{h}_{\mu\nu,zz} \tag{2.6}$$

The other derivative terms drop out because in a plane wave in the $z-$direction, there is no x or y dependence, also seen in the functional forms of $\bar{h}(t-z)$ given in the problem statement. Look at the results of problem 3 part (a):

$$R_{\alpha\mu\beta\nu} = \frac{1}{2}(\bar{h}_{\alpha\nu,\mu\beta} + \bar{h}_{\mu\beta,\nu\alpha} - \bar{h}_{\mu\nu,\alpha\beta} - \bar{h}_{\alpha\beta,\mu\nu}). \tag{2.7}$$

Note $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, and the partial derivative of $\eta_{\mu\nu}h$ is 0. Calculate one of the Riemann components as follows

$$R_{x0x0} = \frac{1}{2}(h_{xt,tx} + h_{tx,tx} - h_{tt,xx} - h_{xx,tt}) = -\frac{1}{2}\bar{h}_{+,tt}. \tag{2.8}$$

Since $\bar{h}_{\mu\nu}$ is a function of $(t-z)$, i.e., $\bar{h}_{\mu\nu}(t-z)$, we can conclude that $\bar{h}_{\mu\nu,z} = -\bar{h}_{\mu\nu,t}$ and similarly for other terms. For example, another component is

$$R_{x0xz} = -\frac{1}{2}h_{xx,tz} = +\frac{1}{2}h_{xx,tt} = +\frac{1}{2}\bar{h}_{+,tt}. \tag{2.9}$$

From here, the rest of the Riemann tensor can be computed similarly. Up to symmetries of the Riemann tensor like $R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta][\gamma\delta]} = R_{\gamma\delta\alpha\beta}$, the nonzero terms can be written as

$$\begin{aligned} R_{x0x0} = -R_{y0y0} = -R_{x0xz} = +R_{y0yz} = +R_{xzxz} = -R_{yzyz} = -\frac{1}{2}\bar{h}_{+,tt} \\ R_{x0y0} = -R_{x0yz} = R_{xzyz} = -R_{xzy0} = -\frac{1}{2}\bar{h}_{\times,tt} \end{aligned} \tag{2.10}$$

Part (d)

In the local Lorentz frame of the stick, $\xi = \ell\vec{n}$, $\vec{u} = 1,0,0,0$. Then from parts (b) and (c),

$$\begin{aligned} \frac{d^2\ell}{d\tau^2} =& -\ell R_{\alpha 0\gamma 0}n^\alpha n^\gamma \\ =& -\ell[2R_{x0y0}n^x n^y + R_{x0x0}n^x n^x - R_{y0y0}n^y n^y] \\ =& -\ell[-\bar{h}_{\times,tt}n^x n^y - \frac{1}{2}\bar{h}_{+,tt}(n^x n^x - n^y n^y)] \\ =& \ell[\bar{h}_{\times,tt}(\sin^2\theta\cos\phi\sin\phi) + \frac{1}{2}\bar{h}_{+,tt}(\sin^2\theta\cos^2\phi - \sin^2\sin^2\phi)] \\ =& \ell[\bar{h}_{\times,tt}(\frac{1}{2}\sin^2\theta\sin 2\theta) + \frac{1}{2}\bar{h}_{+,tt}(\sin^2\theta\cos 2\phi)] \end{aligned} \tag{2.11}$$

As the perturbations of the stick's length are small because $\bar{h}$ is small, let $\ell \approx \ell_0$. Now, to lowest order here $\frac{d}{d\tau} = \frac{d}{dt}$ so the differential equation can be integrated in order to obtain $\ell(\tau)$

$$\ell = A + B\tau + \ell_0[\frac{1}{2}\bar{h}_\times \sin^2\theta \sin 2\theta + \frac{1}{2}\bar{h}_+ \sin^2\theta \cos 2\phi] \tag{2.12}$$

and applying initial constraints (i.e. initial velocity = 0 and $\bar{h} = 0 \Rightarrow \ell = \ell_0$), then we get

$$\ell = \ell_0[1 + \frac{1}{2}\bar{h}_\times \sin^2\theta \sin 2\theta + \frac{1}{2}\bar{h}_+ \sin^2\theta \cos 2\phi]. \tag{2.13}$$

Problem 3

Part (a)

In linearized gravity theory, recall from class that the Christoffel symbols are

$$\Gamma_{\alpha\mu\beta} = \frac{1}{2}(h_{\alpha\beta,\mu} + h_{\alpha\mu,\beta} - h_{\beta\mu,\alpha}) \tag{3.1}$$

These Christoffel symbols are of order h, so when the Riemann tensor is expanded in terms of Christoffel symbols, terms that are products of Γ's can be dropped. Then the Riemann tensor is

$$\begin{aligned} R_{\alpha\mu\beta\nu} \approx& g_{\alpha\lambda}({\Gamma^\lambda}_{\mu\nu,\beta} - {\Gamma^\lambda}_{\mu\beta,\nu}) \\ =& 2\Gamma_{\alpha\mu[\nu,\beta]} \\ =& h_{\alpha[\nu,\beta],\mu} + h_{\alpha\mu,[\beta\nu]} - h_{\mu[\nu,\beta],\alpha} \\ =& h_{\alpha[\nu,\beta],\mu} - h_{\mu[\nu,\beta],\alpha} \\ =& \frac{1}{2}(h_{\alpha\nu,\mu\beta} + h_{\mu\beta,\nu\alpha} - h_{\mu\nu,\alpha\beta} - h_{\alpha\beta,\mu\nu}) \end{aligned} \tag{3.2}$$

Part (b)

Under a gauge transformation,

$$h_{\mu\nu} \to h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \tag{3.3}$$

Plugging this into each of the expressions for $h_{\mu\nu}$ in the equation above for the Riemann tensor yields

$$R_{\alpha\mu\beta\nu} \to R_{\alpha\mu\beta\nu} - \xi_{\alpha[,\nu,\beta],\mu} + \xi_{\mu[,\nu,\beta],\alpha} - \xi_{\nu[,\alpha,\beta],\mu} + \xi_{\nu[,\mu,\beta],\alpha} \tag{3.4}$$

However, since partial derivatives commute, all of the ξ terms disappear because of the antisymmetrization in two of the partials. This leaves only the original Riemann tensor unchanged so the Riemann tensor must be gauge invarient.

Problem 4

Part (a)

Recall that the action of the Lie derivative on a general tensor is completely determined by the action of the Lie derivative on scalars and vectors. Therefore, we only need to show the identity for scalar and vectors. For some scalar function f we have

$$\begin{aligned}\mathcal{L}_{\vec{u}}\mathcal{L}_{\vec{v}}f &= \mathcal{L}_{\vec{u}}(\nabla_{\vec{v}}f) = \mathcal{L}_{\vec{u}}(f_{,\mu}v^{\mu}) = (f_{,\mu}v^{\mu})_{;\nu}u^{\nu} \\ &= f_{,\mu\nu}v^{\mu}u^{\nu} + f_{,\mu}v^{\mu}{}_{;\nu}u^{\nu},\end{aligned} \tag{4.1}$$

so

$$\begin{aligned}\mathcal{L}_{\vec{u}}\mathcal{L}_{\vec{v}}f - \mathcal{L}_{\vec{v}}\mathcal{L}_{\vec{u}}f &= f_{,\mu\nu}v^{\mu}u^{\nu} + f_{,\mu}v^{\mu}{}_{;\nu}u^{\nu} - f_{,\mu\nu}u^{\mu}v^{\nu} + f_{,\mu}u^{\mu}{}_{;\nu}v^{\nu} \\ &= f_{,\mu}(v^{\mu}{}_{;\nu}u^{\nu} - u^{\mu}{}_{;\nu}v^{\nu}) \\ &= f_{,\mu}([\vec{u},\vec{v}])^{\mu} \\ &= \nabla_{[\vec{u},\vec{v}]}f = \mathcal{L}_{[\vec{u},\vec{v}]}f,\end{aligned} \tag{4.2}$$

and so we have shown that

$$\mathcal{L}_{\vec{u}}\mathcal{L}_{\vec{v}} - \mathcal{L}_{\vec{v}}\mathcal{L}_{\vec{u}} - \mathcal{L}_{[\vec{u},\vec{v}]} = 0 \tag{4.3}$$

holds for scalars.

For vector fields, recall that $\mathcal{L}_{\vec{v}}\vec{u} = [\vec{v},\vec{u}]$, so we find

$$\begin{aligned}\mathcal{L}_{\vec{u}}\mathcal{L}_{\vec{v}}\vec{w} - \mathcal{L}_{\vec{v}}\mathcal{L}_{\vec{u}}\vec{w} - \mathcal{L}_{[\vec{u},\vec{v}]}\vec{w} &= \mathcal{L}_{\vec{u}}[\vec{v},\vec{w}] - \mathcal{L}_{\vec{v}}[\vec{u},\vec{w}] - [[\vec{u},\vec{v}],\vec{w}] \\ &= [\vec{u},[\vec{v},\vec{w}]] - [\vec{v},[\vec{u},\vec{w}]] - [[\vec{u},\vec{v}],\vec{w}] \\ &= [\vec{u},[\vec{v},\vec{w}]] + [\vec{v},[\vec{w},\vec{u}]] + [\vec{w},[\vec{u},\vec{v}]] \\ &= 0,\end{aligned} \tag{4.4}$$

where we used the Jacobi identity. Thus we have shown that (4.3) also holds for vector fields and so we are done. □

Part (b)

Recall that $\vec{\xi}$ is a Killing vector field if and only if $\mathcal{L}_{\vec{\xi}}\boldsymbol{g} = 0$, where $\boldsymbol{g}$ is the metric tensor. Let $\vec{\xi}$ and $\vec{\chi}$ be Killing vector fields, then using the result from part (a) we find

$$\mathcal{L}_{[\vec{\xi},\vec{\chi}]}\boldsymbol{g} = \mathcal{L}_{\vec{\xi}}\mathcal{L}_{\vec{\chi}}\boldsymbol{g} - \mathcal{L}_{\vec{\chi}}\mathcal{L}_{\vec{\xi}}\boldsymbol{g} = \mathcal{L}_{\vec{\xi}}(0) - \mathcal{L}_{\vec{\chi}}(0) = 0, \tag{4.5}$$

and so $[\vec{\xi},\vec{\chi}]$ is a Killing vector field. □

Part (c)

Let $\vec{\xi}$ and $\vec{\chi}$ be Killing vector fields, and let a and b be constant scalars. Using linearity of the Lie derivative gives

$$\mathcal{L}_{a\vec{\xi}+b\vec{\chi}}\boldsymbol{g} = a\mathcal{L}_{\vec{\xi}}\boldsymbol{g} + b\mathcal{L}_{\vec{\chi}}\boldsymbol{g} = a(0) + b(0) = 0, \tag{4.6}$$

since $\mathcal{L}_{\vec{\xi}}\boldsymbol{g} = \mathcal{L}_{\vec{\chi}}\boldsymbol{g} = 0$, and so $a\vec{\xi} + b\vec{\chi}$ is a Killing vector field. □

Problem 5

In this problem we will assume that we are always in a coordinate basis. The time basis vector is already specified as $\partial/\partial t$ and we have the freedom to pick the spatial basis vectors to be coordinate basis vectors as well.

Part (a)

Since we have a stationary spacetime, we have a timelike Killing vector field $\vec{\xi}$ and we choose the time coordinate such that $\vec{\xi} = \partial/\partial t$. Since $\vec{\xi}$ is a Killing vector field, we have

$$\begin{aligned}
0 = \mathcal{L}_{\vec{\xi}}\boldsymbol{g} &= g_{\alpha\beta;\mu}\xi^\mu + g_{\mu\beta}\xi^\mu{}_{;\alpha} + g_{\alpha\mu}\xi^\mu{}_{;\beta} \\
&= g_{\alpha\beta,\mu}\xi^\mu - \Gamma^\nu{}_{\alpha\mu}g_{\nu\beta}\xi^\mu - \Gamma^\nu{}_{\beta\mu}g_{\alpha\nu}\xi^\mu + g_{\mu\beta}\xi^\mu{}_{,\alpha} + g_{\mu\beta}\Gamma^\mu{}_{\nu\alpha}\xi^\nu \\
&\quad + g_{\alpha\mu}\xi^\mu{}_{,\beta} + g_{\alpha\mu}\Gamma^\mu{}_{\nu\beta}\xi^\nu \\
&= g_{\alpha\beta,\mu}\xi^\mu + g_{\mu\beta}\xi^\mu{}_{,\alpha} + g_{\alpha\mu}\xi^\mu{}_{,\beta} \\
&= g_{\alpha\beta,0} + 0 + 0,
\end{aligned} \tag{5.1}$$

where all the connection coefficients canceled, and we used that $\xi^\mu = (1, 0, 0, 0)$ is a constant vector field and so the last two terms above vanish. We thus have

$$0 = g_{\alpha\beta,0} = g_{\alpha\beta,t}. \tag{5.2}$$

Note that $g_{t\alpha} = \vec{\xi} \cdot \vec{e}_\alpha$. Now if $\vec{\xi} = \vec{e}_0 \to -\vec{\xi}$, the spatial basis vectors remain the same and we know the that the metric is invariant, hence

$$g_{ti} = \vec{\xi} \cdot \vec{e}_i = -\vec{\xi} \cdot \vec{e}_i = 0. \tag{5.3}$$

Thus we have shown that the first definition of static implies $g_{\alpha\beta,t} = g_{ti} = 0$.

To prove the converse, suppose that $g_{\alpha\beta,t} = g_{ti} = 0$. This means that a time coordinate is already defined. Let $\vec{\chi} = \partial/\partial t$. We need to show that $\vec{\chi}$ is a

Killing vector. We find

$$\begin{aligned}
\mathcal{L}_{\vec{\chi}}\boldsymbol{g} &= g_{\alpha\beta;\mu}\xi^\mu + g_{\mu\beta}\xi^\mu{}_{;\alpha} + g_{\alpha\mu}\xi^\mu{}_{;\beta} \\
&= g_{\alpha\beta,\mu}\xi^\mu + g_{\mu\beta}\xi^\mu{}_{,\alpha} + g_{\alpha\mu}\xi^\mu{}_{,\beta} \\
&= g_{\alpha\beta,t} = 0,
\end{aligned} \tag{5.4}$$

since $\vec{\chi} = (1, 0, 0, 0)$, and so $\vec{\chi}$ is indeed a Killing vector. We also need to show that the metric is invariant under the transformation $\vec{\chi} \to -\vec{\chi}$. We find

$$\begin{aligned}
g_{tt} &= \vec{\chi}\cdot\vec{\chi} \to (-\vec{\chi})\cdot(-\vec{\chi}) = \vec{\chi}\cdot\vec{\chi} = g_{tt}, \\
g_{ti} &= 0 \to 0, \\
g_{ij} &= \vec{e}_i \cdot \vec{e}_j \to \vec{e}_i \cdot \vec{e}_j = g_{ij},
\end{aligned} \tag{5.5}$$

and so the metric is indeed invariant under $\partial/\partial t \to -\partial/\partial t$. Thus we have shown that $g_{\alpha\beta,t} = g_{ti} = 0$ implies the first definition of static and so $g_{\alpha\beta,t} = g_{ti} = 0$ is equivalent to the first definition of static spacetime.

Part (b)

Note that

$$\xi_\alpha = g_{\alpha\beta}\xi^\beta = g_{\alpha t}, \tag{5.6}$$

since $\vec{\xi} = \partial/\partial t = (1, 0, 0, 0)$. So if $g_{ti} = 0$, then $\xi_i = 0$ and $\xi_t = g_{tt}$. So we can write

$$\xi_\alpha = g_{tt}\partial_\alpha t, \tag{5.7}$$

which is of the form $hf_{,\alpha}$, where $h = g_{tt}$ is a scalar and $f = t$ is also a scalar. Thus $\vec{\xi}$ is hypersurface orthogonal. □

Part (c)

Suppose that the Killing vector $\vec{\xi} = \partial/\partial t$ is hypersurface orthogonal. Then

$$\begin{aligned}
0 = \xi_{[\mu;\nu}\xi_{\lambda]} &= (\xi_{\mu;\nu} - \xi_{\nu;\mu})\xi_\lambda + (\xi_{\lambda;\mu} - \xi_{\mu;\lambda})\xi_\nu + (\xi_{\nu;\lambda} - \xi_{\lambda;\nu})\xi_\mu \\
&= (\xi_{\mu;\nu} - \xi_{\nu;\mu})\xi_\lambda + 2\xi_{\lambda;\mu}\xi_\nu - 2\xi_{\lambda;\nu}\xi_\mu,
\end{aligned} \tag{5.8}$$

where we used the fact that $\xi_{\mu;\nu} = -\xi_{\mu;\nu}$. Dotting the above with ξ^λ gives

$$\begin{aligned}
0 &= (\xi_{\mu;\nu} - \xi_{\nu;\mu})\xi_\lambda\xi^\lambda + (\xi_\lambda\xi^\lambda)_{;\mu}\xi_\nu - (\xi_\lambda\xi^\lambda)_{;\nu}\xi_\mu \\
&= (\xi_{\mu;\nu} - \xi_{\nu;\mu})\xi^2 + (\xi^2)_{;\mu}\xi_\nu - (\xi^2)_{;\nu}\xi_\mu \\
&= (\xi_{\mu;\nu} - \xi_{\nu;\mu})(\xi^2)^{-1} + (\xi^2)^{-2}(\xi^2)_{;\mu}\xi_\nu - (\xi^2)^{-2}(\xi^2)_{;\nu}\xi_\mu \\
&= ((\xi^2)^{-1}\xi_\mu)_{;\nu} - ((\xi^2)^{-1}\xi_\nu)_{;\mu},
\end{aligned} \tag{5.9}$$

where $\xi^2 = \xi_\lambda \xi^\lambda$. So for the vector $v_\mu = (\xi^2)^{-1}\xi_\mu$ we have

$$\begin{aligned} & 0 = v_{\mu;\nu} - v_{\nu;\mu} \\ \Leftrightarrow\ & v_{\mu,\nu} - \Gamma^\lambda{}_{\mu\nu} v_\lambda - v_{\nu,\mu} + \Gamma^\lambda{}_{\nu\mu} v_\lambda \\ \Leftrightarrow\ & v_{\mu,\nu} = v_{\nu,\mu}, \end{aligned} \tag{5.10}$$

since the Connection coefficients are symmetric in the last two indices, provided we are in a coordinate basis. We already have $\vec{e}_0 = \partial/\partial t$ and we can choose spatial coordinate basis vectors. Since $v_{\mu,\nu} = v_{\nu,\mu}$, it follows that $\vec{v}$ is a gradient, hence

$$v_\mu = h_{,\mu} = (\xi^2)^{-1}\xi_\mu, \tag{5.11}$$

and so

$$\xi_\mu = \xi^2 h_{,\mu} = \vec{\xi}\cdot\vec{\xi}\, h_{,\mu} = g_{tt} h_{,\mu}. \tag{5.12}$$

Recall that $\xi_\alpha = g_{\alpha t}$, so the above gives

$$g_{\alpha t} = g_{tt} h_{,\alpha}. \tag{5.13}$$

Setting $\alpha = t$, this implies that $h_{,t} = 1$ and so $h = t + f(x^i)$. Choosing a new time coordinate $t' = t + f(x^i)$, we find

$$g_{it'} = g_{tt} h_{,i} = g_{tt}(t')_{,i} = 0, \tag{5.14}$$

and

$$\frac{\partial}{\partial t'} = \vec{e}_{\alpha'} = \xi^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\beta}\xi^\beta = (1,0,0,0), \tag{5.15}$$

and so the metric is still independent of t', because $g_{\alpha'\beta'} = \vec{e}_{\alpha'}\cdot\vec{e}_{\beta'}$ and we did not change the spatial basis vectors. To complete the proof, we just need to show that the change in time coordinate did not change the Killing vector $\vec{\xi} = \partial/\partial t$. We have

$$\frac{\partial}{\partial t} = \frac{\partial x^{\alpha'}}{\partial t}\frac{\partial}{\partial x^{\alpha'}} = \frac{\partial}{\partial t'}, \tag{5.16}$$

because $x^{\alpha'} = (t + f(x^i), x^i)$. So the timelike Killing vectors in both coordinates are the same and are hypersurface orthogonal.

This means, given a stationary spacetime, we have a Killing vector $\vec{\xi}$ that is hypersurface orthogonal, and there exists a choice of time coordinate t' such that $\vec{\xi} = \partial/\partial t'$ and $g_{it'} = 0$. □