Solutions Ph 236a – Week 7

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Problem 1

Part (a)

Recall that

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R, \tag{1.1}$$

where

$$\begin{aligned} R_{\alpha\beta} &= R^{\mu}{}_{\alpha\mu\beta}, \\ R &= R^{\nu}{}_{\nu}. \end{aligned} \tag{1.2}$$

And

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\mu\gamma}\Gamma^{\mu}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\nu\delta}\Gamma^{\nu}{}_{\beta\gamma}, \tag{1.3}$$

where

$$\Gamma^{\gamma}{}_{\alpha\beta} = \frac{1}{2} g^{\gamma\mu} \left(g_{\alpha\mu,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu} \right). \tag{1.4}$$

Now we split $R_{\alpha\beta}$ into the terms involving second time derivatives of the metric and all the other terms not involving second time derivatives of the metric. We denote terms that do not contain second time derivatives of the metric with $(\cdots)$. We find

$$\begin{aligned} R_{\alpha\beta} &= R^{\mu}{}_{\alpha\mu\beta} = \Gamma^{\mu}{}_{\alpha\beta,\mu} - \Gamma^{\mu}{}_{\alpha\mu,\beta} + (\cdots) \\ &= \frac{1}{2} g^{\mu\nu} \left(g_{\alpha\nu,\beta,\mu} + g_{\nu\beta,\alpha,\mu} - g_{\alpha\beta,\nu,\mu} - g_{\alpha\nu,\mu,\beta} - g_{\nu\mu,\alpha,\beta} + g_{\alpha\mu,\nu,\beta} \right) + (\cdots) \\ &= \frac{1}{2} g^{\mu\nu} \left(g_{\alpha\mu,\nu,\beta} + g_{\beta\mu,\nu,\alpha} - g_{\alpha\beta,\mu,\nu} - g_{\mu\nu,\alpha,\beta} \right) + (\cdots), \end{aligned} \tag{1.5}$$

where we changed the order of the derivatives, relabeled dummy indices, and used that the metric is symmetric. Note that $R_{\alpha\beta} = R_{\beta\alpha}$, as required. The Ricci scalar becomes

$$\begin{aligned} R &= R^{\beta}{}_{\beta} = g^{\alpha\beta} R_{\alpha\beta} \\ &= \frac{1}{2} g^{\alpha\beta} g^{\mu\nu} \left(g_{\alpha\mu,\nu,\beta} + g_{\beta\mu,\nu,\alpha} - g_{\alpha\beta,\mu,\nu} - g_{\mu\nu,\alpha,\beta} \right) + (\cdots) \\ &= \frac{1}{2} g^{\alpha\beta} g^{\mu\nu} \left(2 g_{\alpha\mu,\nu,\beta} - 2 g_{\alpha\beta,\mu,\nu} \right) + (\cdots) \\ &= g^{\alpha 0} g^{\mu 0} g_{\alpha\mu,0,0} - g^{\alpha\beta} g^{00} g_{\alpha\beta,0,0} + (\cdots) \\ &= \left(g^{\mu 0} g^{\nu 0} - g^{00} g^{\mu\nu} \right) g_{\mu\nu,0,0} + (\cdots). \end{aligned} \tag{1.6}$$

We find

$$\begin{aligned}R^{00} &= g^{0\alpha}g^{0\beta}R_{\alpha\beta}\\ &= \frac{1}{2}\left(g^{0\alpha}g^{00}g^{\mu0}g_{\alpha\mu,0,0} + g^{00}g^{0\beta}g^{\mu0}g_{\beta\mu,0,0} - g^{0\alpha}g^{0\beta}g^{00}g_{\alpha\beta,0,0}\right.\\ &\quad\left. - g^{00}g^{00}g^{\mu\nu}g_{\mu\nu,0,0}\right) + (\cdots)\\ &= \frac{1}{2}g^{00}\left(g^{\mu0}g^{\nu0} - g^{00}g^{\mu\nu}\right)g_{\mu\nu,0,0} + (\cdots)\\ &= \frac{1}{2}g^{00}R + (\cdots),\end{aligned} \tag{1.7}$$

and

$$\begin{aligned}R^{0k} = R^{k0} &= g^{0\alpha}g^{k\beta}R_{\alpha\beta}\\ &= \frac{1}{2}\left(g^{0\alpha}g^{k0}g^{\mu0}g_{\alpha\mu,0,0} + g^{00}g^{k\beta}g^{\mu0}g_{\beta\mu,0,0} - g^{0\alpha}g^{k\beta}g^{00}g_{\alpha\beta,0,0}\right.\\ &\quad\left. - g^{00}g^{k0}g^{\mu\nu}g_{\mu\nu,0,0}\right) + (\cdots)\\ &= \frac{1}{2}g^{k0}\left(g^{\mu0}g^{\nu0} - g^{00}g^{\mu\nu}\right)g_{\mu\nu,0,0} + (\cdots)\\ &= \frac{1}{2}g^{k0}R + (\cdots).\end{aligned} \tag{1.8}$$

Now we get

$$\begin{aligned}G^{00} &= R^{00} - \frac{1}{2}g^{00}R = \frac{1}{2}g^{00}R - \frac{1}{2}g^{00}R + (\cdots)\\ &= 0 + (\cdots),\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}G^{0k} &= R^{0k} - \frac{1}{2}g^{0k}R = \frac{1}{2}g^{k0}R - \frac{1}{2}g^{0k}R + (\cdots)\\ &= 0 + (\cdots),\end{aligned} \tag{1.10}$$

and so we have shown that $G^{0\mu} = G^{\mu0}$ does not contain any second time derivatives of the metric. □

Part (b)

From (1.5) we get

$$\begin{aligned}R^{ij} &= g^{i\alpha}g^{j\beta}R_{\alpha\beta}\\ &= \frac{1}{2}\left(g^{i\alpha}g^{j0}g^{\mu0}g_{\alpha\mu,0,0} + g^{i0}g^{j\beta}g^{\mu0}g_{\beta\mu,0,0} - g^{i\alpha}g^{j\beta}g^{00}g_{\alpha\beta,0,0}\right.\\ &\quad\left. - g^{i0}g^{j0}g^{\mu\nu}g_{\mu\nu,0,0}\right) + (\cdots)\\ &= \frac{1}{2}\left(g^{i\mu}g^{j0}g^{0\nu} + g^{i0}g^{j\mu}g^{0\nu} - g^{i\mu}g^{j\nu}g^{00} - g^{i0}g^{j0}g^{\mu\nu}\right)g_{\mu\nu,0,0} + (\cdots).\end{aligned} \tag{1.11}$$

And so, using (1.6) we obtain

$$\begin{aligned}
G^{ij} &= R^{ij} - \frac{1}{2} g^{ij} R \\
&= \frac{1}{2}\left(g^{i\mu}g^{j0}g^{0\nu} + g^{i0}g^{j\mu}g^{0\nu} - g^{i\mu}g^{j\nu}g^{00} - g^{i0}g^{j0}g^{\mu\nu}\right. \\
&\quad \left. - g^{ij}g^{\mu 0}g^{\nu 0} + g^{ij}g^{00}g^{\mu\nu}\right) g_{\mu\nu,0,0} + (\cdots) \\
&= (0) g_{00,0,0} + \frac{1}{2}\left(g^{i0}g^{j0}g^{0k} - g^{i0}g^{jk}g^{00}\right) g_{0k,0,0} \\
&\quad + \frac{1}{2}\left(g^{i0}g^{jk}g^{00} - g^{i0}g^{j0}g^{k0}\right) g_{k0,0,0} + \frac{1}{2}\left(g^{ik}g^{j0}g^{0l} + g^{i0}g^{jk}g^{0l} - g^{ik}g^{jl}g^{00}\right. \\
&\quad \left. - g^{i0}g^{j0}g^{kl} - g^{ij}g^{k0}g^{l0} + g^{ij}g^{00}g^{kl}\right) g_{kl,0,0} + (\cdots) \\
&= \frac{1}{2}\left(g^{ik}\left(g^{j0}g^{0l} - g^{00}g^{jl}\right) + g^{i0}\left(g^{jk}g^{0l} - g^{j0}g^{kl}\right) - g^{ij}\left(g^{k0}g^{0l} - g^{00}g^{kl}\right)\right) \\
&\quad \times g_{kl,0,0} + (\cdots),
\end{aligned} \tag{1.12}$$

and so G^{ij} does indeed contain second time derivatives of the metric. □

Problem 2

We have

$$T^{\mu\nu} = (p+\rho) u^\mu u^\nu + p g^{\mu\nu}. \tag{2.1}$$

Recall that $T^{\mu\nu}{}_{;\mu} = 0$, so

$$0 = (p+\rho)_{,\mu} u^\mu u^\nu + (p+\rho) u^\mu{}_{;\mu} u^\nu + (p+\rho) u^\mu u^\nu{}_{;\mu} + p_{,\mu} g^{\mu\nu} + p g^{\mu\nu}{}_{;\mu}. \tag{2.2}$$

The last term is zero since $g^{\mu\nu}{}_{;\mu} = 0$. Now multiply by the projection tensor $P_{\alpha\nu} = u_\alpha u_\nu + g_{\alpha\nu}$. You get

$$\begin{aligned}
0 = P_{\alpha\nu} T^{\mu\nu}{}_{;\mu} &= P_{\alpha\nu}(p+\rho) u^\mu u^\nu{}_{;\mu} + p_{,\mu} g^{\mu\nu} T_{\alpha\nu} \\
&= (p+\rho) u^\mu u_{\alpha;\mu} + p_{,\alpha} + p_{,\mu} u^\mu u_\alpha,
\end{aligned} \tag{2.3}$$

which is the same as the relativistic Euler equation

$$(p+\rho)\nabla_{\vec{u}}\vec{u} = -\nabla p - \vec{u}\nabla_{\vec{u}} p. \tag{2.4}$$

Problem 3

Part (a)

Start by looking at the RHS of equation (3) from the problem set:

$$w_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{3}\theta P_{\alpha\beta} - a_\alpha u_\beta \tag{3.1}$$

and plugging in expressions for each of the terms, starting with $\sigma_{\alpha\beta}$ and then $w_{\alpha\beta}$ and finally $a_\alpha = u_{\alpha;\gamma}u^\gamma$.

$$\begin{aligned}
&w_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{3}\theta P_{\alpha\beta} - a_\alpha u_\beta = \\
=&w_{\alpha\beta} + \frac{1}{2}(u_{\alpha;\mu}P^\mu{}_\beta + u_{\beta;\mu}P^\mu{}_\alpha) - \frac{1}{3}\theta P_{\alpha\beta} + \frac{1}{3}\theta P_{\alpha\beta} - a_\alpha u_\beta \\
=&\frac{1}{2}(u_{\alpha;\mu}P^\mu{}_\beta - u_{\beta;\mu}P^\mu{}_\alpha) + \frac{1}{2}(u_{\alpha;\mu}P^\mu{}_\beta - u_{\beta;\mu}P^\mu{}_\alpha) - a_\alpha u_\beta \\
=&u_{\alpha;\mu}P^\mu{}_\beta - u_{\alpha;\gamma}u^\gamma u_\beta
\end{aligned} \tag{3.2}$$

Now plug in the projection operator so that

$$\begin{aligned}
u_{\alpha;\mu}P^\mu{}_\beta - u_{\alpha;\gamma}u^\gamma u_\beta =&u_{\alpha;\mu}(g^\mu{}_\beta + u^\mu u_\beta) - u_{\alpha;\gamma}u^\gamma u_\beta \\
=&u_{\alpha;\beta}
\end{aligned} \tag{3.3}$$

Part (b)

From the definition of $\theta = \nabla \cdot \vec{u}$, write

$$\begin{aligned}
\frac{d\theta}{d\tau} &= u^\beta(u^\alpha{}_{;\alpha})_{;\beta} = u^\beta u^\alpha{}_{;\alpha\beta} = u^\beta(u^\alpha{}_{;\beta\alpha} - R^\alpha{}_{\beta\alpha\gamma}u^\gamma) \\
&= u^\beta u^\alpha{}_{;\beta\alpha} - R_{\beta\gamma}u^\beta u^\gamma
\end{aligned} \tag{3.4}$$

Now consider the term $u^\beta u^\alpha{}_{;\beta\alpha}$ and use the results from part (a) as well as the ability to raise/lower indices that are being contracted,

$$\begin{aligned}
u^\beta u^\alpha{}_{;\beta\alpha} =&(u^\alpha{}_{;\beta}u^\beta)_{;\alpha} - u^\alpha{}_{;\beta}u^\beta{}_{;\alpha} \\
=&a^\alpha{}_{;\alpha} - u_{\alpha;\beta}u^{\beta;\alpha} \\
=&a^\alpha{}_{;\alpha} - (w_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{3}\theta P_{\alpha\beta} - a_\alpha u_\beta)(w^{\beta\alpha} + \sigma^{\beta\alpha} + \frac{1}{3}\theta P^{\beta\alpha} - a^\beta u^\alpha)
\end{aligned} \tag{3.5}$$

Now note that $w_{\alpha\beta}$ is antisymmetric and that both $\sigma_{\alpha\beta}$ and $P_{\alpha\beta}$ are symmetric. So when considering the cross terms, terms that are contractions between symmetric and antisymmetric are 0 meaning that terms like $w_{\alpha\beta}\sigma^{\alpha\beta} = w_{\alpha\beta}P^{\alpha\beta} = 0$. Also, the projection operator is orthogonal to $u^\alpha \Rightarrow P^{\alpha\beta}u_\alpha = 0$. Next look at terms like

$$u_{\alpha;\mu}P^\mu{}_\beta a^\beta u^\alpha = \frac{1}{2}(u^\alpha u_\alpha)_{;\mu}a^\beta P^\mu{}_\beta = (0) \tag{3.6}$$

This shows, along with projection operator orthoganality, that terms like

$$w^{\alpha\beta}a_\alpha u_\beta = \sigma^{\alpha\beta}a_\alpha u_\beta = \frac{1}{3}\theta P^{\beta\alpha}a_\alpha u_\beta = 0 \tag{3.7}$$

Also, $a^\alpha u_\alpha = 0$ as the 4-velocity and 4-acceleration are orthogonal to each other. After all these simplification, all that remains is

$$u^\beta u^\alpha{}_{;\beta\alpha} = a^\alpha{}_{;\alpha} + w_{\alpha\beta}w^{\alpha\beta} - (\sigma_{\alpha\beta} + \frac{1}{3}\theta P_{\alpha\beta})(\sigma^{\alpha\beta} + \frac{1}{3}\theta P^{\alpha\beta}) \tag{3.8}$$

Now consider contracting the projection operator with itself (keeping in mind u^α is orthogonal to it)

$$\begin{aligned} P_{\alpha\beta}P^{\alpha\beta} = (g^{\alpha\beta} + u^\alpha u^\beta)g_{\alpha\beta} = g^2 - u^2 = 3 \\ \Rightarrow \frac{1}{3}\theta P_{\alpha\beta} \times \frac{1}{3}\theta P^{\alpha\beta} = \frac{1}{3}\theta^2 \end{aligned} \tag{3.9}$$

The last term to consider is the contraction between $\sigma_{\alpha\beta}P^{\alpha\beta}$. Note that the projection operator acting on itself gives a projection operator, or simply put $P^{\mu\gamma}P_{\gamma\nu} = P^\mu{}_\nu$. Expanding σ, the term becomes

$$\begin{aligned} \sigma_{\alpha\beta}P^{\alpha\beta} =& \left(\frac{1}{2}(u_{\alpha;\mu}P^\mu{}_\beta + u_{\beta;\mu}P^\mu{}_\alpha) - \frac{1}{3}\theta P_{\alpha\beta}\right)P^{\alpha\beta} \\ =& \frac{1}{2}(u_{\alpha;\mu}P^{\mu\alpha} + u_{\beta;\mu}P^{\mu\beta}) - \theta \\ =& u_{\alpha;\mu}P^{\mu\alpha} - u^\alpha{}_{;\alpha} \\ =& u_{\alpha;\mu}g^{\alpha\mu} + u_{\alpha;\mu}u^\alpha u^\mu - u_\alpha{}^{;\alpha} \\ =& a_\alpha u^\alpha \\ =& 0 \end{aligned} \tag{3.10}$$

Using these results in equation (3.8) and plugging everything back into (3.4) yields the desired result:

$$\frac{d\theta}{d\tau} = a^\alpha{}_{;\alpha} + w_{\alpha\beta}w^{\alpha\beta} - \sigma_{\alpha\beta}\sigma^{\alpha\beta} - \frac{1}{3}\theta^2 - R^\alpha{}_{\beta\alpha\gamma}u^\beta u^\gamma \tag{3.11}$$

Part (c)

From the problem statement, let the hypersurfaces be parameterized by $f =$ constant for a scalar function f. So then let $v_\alpha = hf_{;\alpha}$ for some scalar h. Then

$$v_{\mu;\nu} = h_{;\nu}f_\mu + hf_{;\mu\nu} \tag{3.12}$$

Now consider the quanity of interest.

$$v_{[\lambda}v_{\mu;\nu]} = hf_{[;\lambda}h_{;\nu}f_{;\mu]} + h^2 f_{[;\lambda}f_{;\mu\nu]} \tag{3.13}$$

Now the first term must be zero because $f_{;\lambda}f_{;\mu} = f_{;\mu}f_{;\lambda}$ and so any antisymmetrization of terms of that sort will be zero. The second term vanishes because $f_{;\mu\nu} = f_{;\nu\mu}$ (because f is a scalar). Therefore, it must be that $v_{[\lambda}v_{\mu;\nu]} = 0$.

Part (d)

Consider each of the terms in the differential equation

$$\frac{d\theta}{d\tau} = a^{\alpha}{}_{;\alpha} + w_{\alpha\beta}w^{\alpha\beta} - \sigma_{\alpha\beta}\sigma^{\alpha\beta} - \frac{1}{3}\theta^2 - R^{\alpha}{}_{\beta\alpha\gamma}u^{\beta}u^{\gamma} \tag{3.14}$$

individually. First, the worldlines are all geodesics, so that $a^{\alpha}{}_{;\alpha} = (u^{\alpha}{}_{;\beta}u^{\beta})_{;\alpha} = 0$ by the geodesic equation. Next, u^{α} is a hypersurface orthogonal. Consider the quantity $u^{\gamma}u_{[\gamma}u_{\alpha;\beta]}$. When plugging in the equation for $u_{\alpha;\beta}$ from part (a), the antisymmetrization with u_{γ} will cancel out any symmetric terms, leaving only

$$\begin{aligned} u^{\gamma}u_{[\gamma}u_{\alpha;\beta]} = u^{\gamma}u_{[\gamma}w_{\alpha\beta]} &= 0 \\ \Rightarrow w_{\alpha\beta} &= 0 \end{aligned} \tag{3.15}$$

The strong energy condition imposes an additional constraint for the Ricci tensor,

$$\begin{aligned} T_{\mu\nu}u^{\mu}u^{\nu} + \frac{1}{2}T &\geq 0 \\ \Rightarrow \frac{1}{8\pi}R_{\mu\nu}u^{\mu}u^{\nu} = \left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)u^{\mu}u^{\nu} &\geq 0 \\ \Rightarrow -R_{\mu\nu}u^{\mu}u^{\nu} &\leq 0 \end{aligned} \tag{3.16}$$

Putting these together, along with $-\sigma_{\alpha\beta}\sigma^{\alpha\beta} \leq 0$, the differential equation becomes

$$\begin{aligned} \frac{d\theta}{d\tau} &\leq -\frac{1}{3}\theta^2 \\ \Rightarrow \frac{1}{\theta} - \frac{1}{\theta_0} &\geq \frac{\Delta\tau}{3} \\ \Rightarrow \theta &\leq \frac{\theta_0}{1 + \Delta\tau\theta_0/3} \end{aligned} \tag{3.17}$$

Now if the initial $\theta_0 < 0$, then the RHS of the expression blows up as $\Delta\tau \to -\frac{3}{\theta_0}$, diverging so that $\theta \to -\infty$.

Problem 4

Part (a)

We write the modified Einstein field equation as

$$\begin{aligned} 8\pi T_{\mu\nu} &= R_{\mu\nu} - a g_{\mu\nu} R \\ &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \left(\frac{1}{2} - a\right) g_{\mu\nu} R \\ &= G_{\mu\nu} + \left(\frac{1}{2} - a\right) g_{\mu\nu} R. \end{aligned} \tag{4.1}$$

Raising the indices yields

$$8\pi T^{\mu\nu} = G^{\mu\nu} + \left(\frac{1}{2} - a\right) g^{\mu\nu} R. \tag{4.2}$$

Taking the divergence of the above equation gives

$$\begin{aligned} 8\pi T^{\mu\nu}{}_{;\nu} &= G^{\mu\nu}{}_{;\nu} + \left(\frac{1}{2} - a\right) \left(g^{\mu\nu}{}_{;\nu} R + g^{\mu\nu} R_{,\nu}\right) \\ &= \left(\frac{1}{2} - a\right) R^{,\mu}, \end{aligned} \tag{4.3}$$

since $G^{\mu\nu}{}_{;\nu} = g^{\mu\nu}{}_{;\nu} = 0$. On the other hand, if we contract the modified field equations (4.1), we get

$$\begin{aligned} & 8\pi T^{\nu}{}_{\nu} = R^{\nu}{}_{\nu} - a g^{\nu}{}_{\nu} R \\ \Leftrightarrow \ & 8\pi T = R - 4aR = (1 - 4a) R, \end{aligned} \tag{4.4}$$

and differentiating yields

$$8\pi T^{,\mu} = (1 - 4a) R^{,\mu}. \tag{4.5}$$

Combining the above gives

$$\begin{aligned} & R^{,\mu} = \frac{8\pi T^{\mu\nu}{}_{;\nu}}{1/2 - a} = \frac{8\pi T^{,\mu}}{1 - 4a} \\ \Leftrightarrow \ & T^{\mu\nu}{}_{;\nu} = \frac{1/2 - a}{1 - 4a} T^{,\mu} = \kappa T^{,\mu}, \end{aligned} \tag{4.6}$$

where $\kappa = (1/2 - a)/(1 - 4a)$. The above is the equation of motion for $T^{\mu\nu}$.

Part (b)

For a perfect fluid with density ρ and negligible pressure the stress-energy tensor is

$$T^{\mu\nu} = \rho u^\mu u^\nu. \tag{4.7}$$

Thus in the Newtonian limit, the $\mu = 0$ component of (4.6) is

$$\begin{aligned} T^{0\nu}{}_{;\nu} &= \kappa T^{,0} \\ \Leftrightarrow \quad (\rho u^0 u^\nu)_{,\nu} &= \kappa T^{,0}, \end{aligned} \tag{4.8}$$

because covariant derivatives become ordinary partial derivatives in the Newtonian limit. Note that

$$T = T^\nu{}_\nu = \rho u^\nu u_\nu = -\rho, \tag{4.9}$$

since u^ν is a 4-velocity and so $u^\nu u_\nu = -1$. Recall that $u^\nu = \gamma(1, \underset{\sim}{v})$, and $\gamma = (1-v^2)^{-1/2} \approx 1$, because we are in the Newtonian limit and so $v^2 \ll 1 = c^2$. Thus $u^0 = 1$ and using the Minkowski metric (4.8) becomes

$$\begin{aligned} \kappa(-\partial_t)(-\rho) &= \partial_t \rho + \rho u^0 u^0{}_{,0} + (\rho u^i)_{,i} \\ \Leftrightarrow \quad \kappa \frac{\partial \rho}{\partial t} &= \frac{\partial \rho}{\partial t} + 0 + \nabla \cdot (\rho \underset{\sim}{v}). \end{aligned} \tag{4.10}$$

The non-relativist Euler equation (mass conservation) states that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{v}) = 0, \tag{4.11}$$

and so it follows that $\kappa = 0$, which implies that $a = 1/2$, because $\kappa = (1/2 - a)/(1 - 4a)$. □

Problem 5

Let Newtonian tarjectory be $x^j = x^j(t, n)$ where n tells which trajectory under consideration. Then

$$\underset{\sim}{n} = \frac{\partial}{\partial n} = \frac{\partial x^k}{\partial n} \frac{\partial}{\partial x^k} = n^k \frac{\partial}{\partial x^k} \tag{5.1}$$

which is a connecting vector between neighboring trajectories. The relative acceleration of the neighboring trajectories is then

$$\begin{aligned}
\frac{\partial^2 n^j}{\partial t^2} =& \frac{\partial^2}{\partial t^2}\left(\frac{\partial x^j}{\partial n}\right) \\
=& \frac{\partial}{\partial n}\left(\frac{\partial^2 x^j}{\partial t^2}\right) \\
=& \frac{\partial}{\partial n}\left(-\frac{\partial \Phi}{\partial x^j}\right) \\
=& -n^k \frac{\partial}{\partial x^k}\left(\frac{\partial \Phi}{\partial x^j}\right) \\
=& -n^k \frac{\partial^2 \Phi}{\partial x^k \partial x^j}
\end{aligned} \tag{5.2}$$

where the equation of motion is $\frac{\partial^2 x^j}{\partial t^2} = \frac{\partial \Phi}{\partial x^j}$ for some Newtonian gravitational potential. Now the geodesic deviation equation is

$$\frac{D^2 n^\alpha}{d\tau^2} = R^\alpha{}_{\beta\gamma\delta} u^\beta u^\gamma n^\delta = -R^\alpha{}_{\beta\delta\gamma} u^\beta u^\gamma n^\delta \tag{5.3}$$

Now, in the Newtonian limit, velocities and the curvature is small so expand quanitities in terms of small parameter $\epsilon \ll 1$. This means $u^0 = 1 + \mathcal{O}(\epsilon)$, $u^j = \mathcal{O}(\epsilon)$ and $\Gamma^\alpha{}_{\beta\gamma} = \mathcal{O}(\epsilon^2)$. Also, $\vec{n}$ which connects events of equal proper time now connect events of equal coordinate time, up to order $\mathcal{O}(\epsilon)$. Then

$$\frac{D}{\partial \tau} = \frac{\partial}{\partial t} + \mathcal{O}(\epsilon) \tag{5.4}$$

and therefore, when combined with equation (5.2), this becomes

$$\begin{aligned}
\frac{\partial^2 n^j}{\partial t^2} = -R^j{}_{0k0} n^k + \mathcal{O}(\epsilon) = -n^k \frac{\partial^2 \Phi}{\partial x^k \partial x^j} \\
\Rightarrow R^j{}_{0k0} = \frac{\partial^2 \Phi}{\partial x^k \partial x^j}
\end{aligned} \tag{5.5}$$

in the Newtonian limit. Note, unless the velocities involved approach c, the other components of $R^\alpha{}_{\beta\delta\gamma}$ don't enter into the equation of relative motion of test particles.