Solutions Ph 236a – Week 6

Kevin Barkett, Jonas Lippuner, and Mark Scheel

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Problem 1

Part (a)

The metric is

$$g_{ij} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{bmatrix}. \tag{1.1}$$

Since $(\vec{e}_\theta, \vec{e}_\phi)$ is a coordinate basis, the connection coefficients are given by

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{il}(g_{jl,k} + g_{lk,j} - g_{jk,l}). \tag{1.2}$$

We use the same Mathematica code as in Homework 5 to do all the tedious calculations.

```
x = {t, p}
g = {{r^2, 0}, {0, r^2 Sin[t]^2}}
gInv = Inverse[g]
ConCoef[i_, j_, k_] := 1/2 Sum[gInv[[i, l]](D[g[[j, l]], x[[k]]]
    + D[g[[l, k]], x[[j]]] - D[g[[j, k]], x[[l]]]), {l, 1, 2}]
Table[ConCoef[1, j, k], {j, 1, 2}, {k, 1, 2}] // MatrixForm
Table[ConCoef[2, j, k], {j, 1, 2}, {k, 1, 2}] // MatrixForm
```

We find that the only non-zero connection coefficients are

$$\begin{aligned} \Gamma^\theta{}_{\phi\phi} &= -\sin\theta\,\cos\theta, \\ \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} &= \cot\theta. \end{aligned} \tag{1.3}$$

Part (b)

Recall that for a curve $\gamma^i(\lambda) = (\theta(\lambda), \phi(\lambda))$ the geodesic equation is

$$0 = \frac{d^2\gamma^i}{d\lambda^2} + \Gamma^i{}_{jk} \frac{d\gamma^j}{d\lambda}\frac{d\gamma^k}{d\lambda}. \tag{1.4}$$

So we get

$$0 = \theta'' - \sin\theta\cos\theta\,(\phi')^2, \tag{1.5}$$

$$0 = \phi'' + 2\cot\theta\,\theta'\phi'. \tag{1.6}$$

Note that we can rewrite (1.6) as

$$\begin{aligned} & 0 = \phi'' + 2\frac{\cos\theta}{\sin\theta}\,\theta'\phi' \\ \Leftrightarrow\ & 0 = \phi''\sin^2\theta + 2\sin\theta\cos\theta\,\theta'\phi' \\ \Leftrightarrow\ & 0 = \frac{d}{d\lambda}\left(\phi'\sin^2\theta\right). \end{aligned} \tag{1.7}$$

Thus we have that

$$\phi' \sin^2\theta = k_1 = \text{const} \tag{1.8}$$

along the entire curve. Note that rotating the sphere in the 3-dimensional space in which it is embedded corresponds to adding constants to θ and ϕ. Thus if $\theta(\lambda = 0) = \theta_0$, we can rotate the sphere so that $\theta_0 = 0$. This means that $k_1 = 0$ at $\lambda = 0$ and since k_1 is constant we have $k_1 = 0$ along the entire curve. Thus we must have that either $\theta = 0, \pi$ or $\phi' = 0$ along the curve. But if $\theta = 0, \pi$ then the curve would just consist of the north or south pole, which are not connected. Thus we have that $\phi' = 0$ and so $\phi(\lambda) = \phi_0$ is constant. Hence (1.2) reduces to $\theta'' = 0$ and so $\theta(\lambda) = k_2\lambda$, because we have $\theta(0) = 0$. Hence, in the rotated coordinate system, the geodesic is

$$\gamma^i(\lambda) = (k_2\lambda, \phi_0), \tag{1.9}$$

which is a great circle passing through the north and south poles. Since rotating the coordinate system preserves great circles, it follows that all geodesics on the sphere are great circles. □

Part (c)

Since we are in a coordinate basis, the Riemann tensor is given by

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\nu\delta}\Gamma^\nu{}_{\beta\gamma}. \tag{1.10}$$

Thus we use the following Mathematica code to evaluate all components of the Riemann tensor.

```
Riemann[i_, j_, k_, l_] := D[ConCoef[i, j, l], x[[k]]]
  - D[ConCoef[i, j, k], x[[l]]]
  + Sum[ConCoef[i, d, k] ConCoef[d, j, l], {d, 1, 2}]
  - Sum[ConCoef[i, d, l] ConCoef[d, j, k], {d, 1, 2}]
Table[Riemann[1, 1, k, l], {k, 1, 2}, {l, 1, 2}] // MatrixForm
Table[Riemann[1, 2, k, l], {k, 1, 2}, {l, 1, 2}] // MatrixForm
Table[Riemann[2, 1, k, l], {k, 1, 2}, {l, 1, 2}] // MatrixForm
Table[Riemann[2, 2, k, l], {k, 1, 2}, {l, 1, 2}] // MatrixForm
```

We find that the only non-zero components of the Riemann tensor are

$$\begin{aligned} R^\theta{}_{\phi\theta\phi} &= \sin^2\theta, \\ R^\theta{}_{\phi\phi\theta} &= -\sin^2\theta, \\ R^\phi{}_{\theta\theta\phi} &= -1, \\ R^\phi{}_{\theta\phi\theta} &= 1. \end{aligned} \tag{1.11}$$

Problem 2

Part (a)

On Homework 5 we found that the geodesic equation for one-forms is

$$0 = \frac{du_\alpha}{d\lambda} - \Gamma^\mu{}_{\alpha\nu} u_\mu u^\nu, \tag{2.1}$$

so we get

$$\frac{dp_\alpha}{d\lambda} = \Gamma^\mu{}_{\alpha\nu} p_\mu p^\nu. \tag{2.2}$$

Part (b)

The metric is

$$g_{\mu\nu} = \begin{bmatrix} -(1-2M/r) & 0 & 0 & 0 \\ 0 & 1+2M/r & 0 & 0 \\ 0 & 0 & 1+2M/r & 0 \\ 0 & 0 & 0 & 1+2M/r \end{bmatrix}, \tag{2.3}$$

where $r = \sqrt{x^2+y^2+z^2}$. Since we are in a coordinate basis, the connection coefficients are given by

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu}(g_{\beta\mu,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}), \tag{2.4}$$

and so

$$\Gamma^\mu{}_{0\nu} = \frac{1}{2} g^{\mu\lambda}(g_{0\lambda,\nu} + g_{\lambda\nu,0} - g_{0\nu,\lambda}) = \frac{1}{2}(g^{\mu 0} g_{00,\nu} + 0 - g^{\mu\lambda} g_{0\nu,\lambda}). \tag{2.5}$$

Case $\mu = i$ and $\nu = j$: Both terms in (2.5) vanish and we have $\Gamma^i{}_{0j} = 0$.

Case $\mu = i$ and $\nu = 0$: The first term in (2.5) vanishes and we get

$$\Gamma^i{}_{00} = -\frac{1}{2} g^{i\lambda} g_{00,\lambda}. \tag{2.6}$$

Case $\mu = 0$ and $\nu = j$: The second term in (2.5) vanishes and we get

$$\Gamma^0{}_{0j} = \frac{1}{2} g^{00} g_{00,j}. \tag{2.7}$$

Case $\mu = 0$ and $\nu = 0$: Both terms vanish because the metric is independent of t, hence $\Gamma^0{}_{00} = 0$.

Now we find

$$\begin{aligned}\frac{dp_0}{d\lambda} = \Gamma^{\mu}{}_{0\nu} p_\mu p^\nu &= \Gamma^0{}_{00} p_0 p^0 + \Gamma^i{}_{00} p_i p^0 + \Gamma^0{}_{0j} p_0 p^j + \Gamma^i{}_{0j} p_i p^j \\ &= 0 - \frac{1}{2} g^{i\lambda} g_{00,\lambda} p_i p^0 + \frac{1}{2} g^{00} g_{00,j} p_0 p^j + 0 \\ &= \frac{1}{2}\left(-g_{00,\lambda} p^\lambda p^0 + g_{00,j} p^j p^0\right) \\ &= 0, \end{aligned} \tag{2.8}$$

since $g_{00,0} = 0$ and so the sum over $\lambda = 1, 2, 3, 4$ can be replaced with a sum over $k = 1, 2, 3$, just like the sum over j. And we also used $p^0 = g^{0\lambda} p_\lambda = g^{00} p_0$. □

Part (c)

We have $p^0 = g^{0\lambda} p_\lambda = g^{00} p_0$, hence

$$p^0 = -\frac{p_0}{1 - 2M/r}, \tag{2.9}$$

which is not constant in general, because r can vary.

Part (d)

The 4-velocity is defined as

$$u_e^\alpha = \frac{dx^\alpha}{d\tau}. \tag{2.10}$$

But since the atom is at rest, we have $dx^i/d\tau = 0$ and so $u_e^\alpha = (u_e^0, 0, 0, 0)$. The normalization condition is $u_e^\alpha (u_e)_\alpha = -1$, so we have

$$\begin{aligned} -1 = u_e^\alpha (u_e)_\alpha &= g_{\alpha\beta} u_e^\alpha u_e^\beta = -(1 - 2M/R)(u_e^0)^2 \\ \Leftrightarrow \quad u_e^0 &= \frac{1}{\sqrt{1 - 2M/R}}, \end{aligned} \tag{2.11}$$

where R is the radius of the sun.

Part (e)

In the rest frame of the emitting atom, we have $E = -\vec{p} \cdot \vec{u}_e = \hbar c/\lambda_e$. But $E = -\vec{p} \cdot \vec{u}_e$ is a scalar and so it is invariant. In the rest frame of the sun, we found in part (d) that $\vec{u}_e = ((1 - 2M/R)^{-1/2}, 0, 0, 0)$ and so at $r = R$

$$\begin{aligned} -\vec{p} \cdot \vec{u}_e &= g_{\alpha\beta} p^\alpha u_e^\beta = g_{00} p^0 u_e^0 \\ &= -\left(1 - \frac{2M}{R}\right)\left(-\frac{p_0}{1 - 2M/R}\right)\frac{1}{\sqrt{1 - 2M/R}} \\ &= \frac{p_0}{\sqrt{1 - 2M/R}}, \end{aligned} \tag{2.12}$$

hence

$$\frac{\hbar c}{\lambda_e} = \frac{p_0}{\sqrt{1-2M/R}}, \tag{2.13}$$

where p_0 is measured in the rest frame of the sun. We found in part (b) that p_0 does not change along the photon's worldline, therefore $p_0(r = R) = p_0(r \to \infty) = p_0$ in the sun's rest frame.

Now consider an observer at rest at $r \to \infty$. Going through the same math as in part (d), we see that $u_r^0 = 1$ in the sun's rest frame. We have $E = -\vec{p} \cdot \vec{u}_r = \hbar c/\lambda_r$ and evaluating this in the sun's rest frame gives

$$\begin{aligned} -\vec{p} \cdot \vec{u}_r &= g_{\alpha\beta} p^\alpha u_r^\beta = g_{00} p^0 u_r^0 \\ &= -(1-0)\left(-\frac{p_0}{1-0}\right)(1) = p_0, \end{aligned} \tag{2.14}$$

so

$$\frac{\hbar c}{\lambda_r} = p_0. \tag{2.15}$$

Combining the above with (2.13) yields

$$\begin{aligned} \frac{\hbar c}{\lambda_e} &= \frac{\hbar c/\lambda_r}{\sqrt{1-2M/R}} \\ \Leftrightarrow \quad \lambda_r &= \frac{\lambda_e}{\sqrt{1-2M/R}}. \end{aligned} \tag{2.16}$$

Thus we find

$$z = \frac{\lambda_r - \lambda_e}{\lambda_e} = \frac{(1-2M/R)^{-1/2}\lambda_e - \lambda_e}{\lambda_e} = \left(1 - \frac{2M}{R}\right)^{-1/2} - 1. \tag{2.17}$$

Restoring factors of c and G, we find that

$$\frac{2M}{R} = \frac{2M_\odot G}{R_\odot c^2} \sim 4.24 \times 10^{-6}, \tag{2.18}$$

since this is a small number, we can expand z above as follows

$$\begin{aligned} z = \left(1 - \frac{2M}{R}\right)^{-1/2} - 1 &= 1 - \left(-\frac{1}{2}\right)\frac{2M}{R} + O\left(\left(\frac{M}{R}\right)^2\right) - 1 \\ &= \frac{M}{R} + O\left(\left(\frac{M}{R}\right)^2\right), \end{aligned} \tag{2.19}$$

which is the desired result. □

Problem 3

Consider the quantity $T^{\alpha}{}_{\beta}\, w_\alpha v^\beta$ which is a scalar since it is the inner product of a tensor with a 1-form and vector. The second covariant derivative of this quantity is *not* zero, but it is independent of the order of differentiation,

$$(T^{\alpha}{}_{\beta}\, w_\alpha v^\beta)_{;\mu\nu} = (T^{\alpha}{}_{\beta}\, w_\alpha v^\beta)_{;\nu\mu}. \tag{3.1}$$

Then

$$\begin{aligned} 0 =&(T^{\alpha}{}_{\beta}\, w_\alpha v^\beta)_{;\mu\nu} - (T^{\alpha}{}_{\beta}\, w_\alpha v^\beta)_{;\nu\mu} \\ =&T^{\alpha}{}_{\beta;\mu\nu}\, w_\alpha v^\beta + T^{\alpha}{}_{\beta}\, w_{\alpha;\mu\nu} v^\beta + T^{\alpha}{}_{\beta}\, w_\alpha v^\beta{}_{;\mu\nu} \\ &- T^{\alpha}{}_{\beta;\nu\mu}\, w_\alpha v^\beta - T^{\alpha}{}_{\beta}\, w_{\alpha;\nu\mu} v^\beta - T^{\alpha}{}_{\beta}\, w_\alpha v^\beta{}_{;\nu\mu} \\ =&(T^{\alpha}{}_{\beta;\mu\nu} - T^{\alpha}{}_{\beta;\nu\mu})w_\alpha v^\beta + T^{\alpha}{}_{\beta}\,(w_{\alpha;\mu\nu} - w_{\alpha;\nu\mu})v^\beta + T^{\alpha}{}_{\beta}\, w_\alpha(v^\beta{}_{;\mu\nu} - v^\beta{}_{;\nu\mu}) \\ =&(T^{\alpha}{}_{\beta;\mu\nu} - T^{\alpha}{}_{\beta;\nu\mu})w_\alpha v^\beta + T^{\alpha}{}_{\beta}\,(-R^{\lambda}{}_{\alpha\nu\mu} w_\lambda)v^\beta + T^{\alpha}{}_{\beta}\, w_\alpha(R^{\beta}{}_{\lambda\nu\mu} v^\lambda) \end{aligned} \tag{3.2}$$

Relabeling the indices above, and moving the covariant derivatives of $T^{\alpha}{}_{\beta}$ to the other side, then we get

$$\begin{aligned} (T^{\alpha}{}_{\beta;\mu\nu} - T^{\alpha}{}_{\beta;\nu\mu})w_\alpha v^\beta =&(T^{\lambda}{}_{\beta}\, R^{\alpha}{}_{\lambda\nu\mu} - T^{\alpha}{}_{\lambda}\, R^{\alpha}{}_{\beta\nu\mu})w_\alpha v^\beta \\ \Rightarrow T^{\alpha}{}_{\beta;\mu\nu} - T^{\alpha}{}_{\beta;\nu\mu} =&R^{\alpha}{}_{\lambda\nu\mu} T^{\lambda}{}_{\beta} - R^{\alpha}{}_{\beta\nu\mu} T^{\alpha}{}_{\lambda} \end{aligned} \tag{3.3}$$

Problem 4

Part (a)

Now we are given that $\nabla_{\vec{u}}\vec{e}_\alpha = -\vec{e}_\beta \Omega^{\beta}{}_{\alpha}$, where $\Omega^{\beta\alpha} = a^\beta u^\alpha - u^\beta a^\alpha + u_\lambda w_\rho \epsilon^{\lambda\rho\beta\alpha}$ as well as that $\vec{u} = \vec{e}_0$. We also know that $\nabla_{\vec{e}_\beta}\vec{e}_\alpha = \Gamma^{\lambda}{}_{\alpha\beta}\vec{e}_\lambda$. Now consider

$$\begin{aligned} \Gamma^{\beta}{}_{\alpha 0}\vec{e}_\beta = \nabla_{\vec{e}_0}\vec{e}_\alpha = \nabla_{\vec{u}}\vec{e}_\alpha = -\vec{e}_\beta \Omega^{\beta}{}_{\alpha} \\ \Rightarrow \Gamma^{\beta}{}_{\alpha 0} = -\Omega^{\beta}{}_{\alpha} = -a^\beta u_\alpha + u^\beta a_\alpha - u^\lambda w^\rho \epsilon_{\lambda\rho}{}^{\beta}{}_{\alpha} \end{aligned} \tag{4.1}$$

Now, because we know that $\vec{u} = \vec{e}_0$, we can say that $u^i = u_i = 0, u^0 = -u_0 = 1$ for $i = 1, 2, 3$ the spatial components. This implies that $u^0 \epsilon_{0\rho}{}^{\beta}{}_{\alpha} \Rightarrow \epsilon_{0k}{}^{i}{}_{j}$ is nonzero for i, j, k all spatial and unique. Now choosing α and β, we can get some for the connection coefficients:

$$\begin{aligned} \Gamma^{i}{}_{00} =&a^i & \Gamma^{0}{}_{00} = 0 \\ \Gamma^{0}{}_{i0} =&a_i & \Gamma^{i}{}_{j0} = -w^k \epsilon_{0k}{}^{i}{}_{j} \end{aligned} \tag{4.2}$$

Part (b)

We are given a spatial geodesic curve parameterized by $x^0 = \tau$ and $x^i = sn^i$ where τ is the proper time and s is the proper distance. Now the geodesic equation is

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0 \tag{4.3}$$

However, given the curves above, $\frac{dx^i}{ds} = n^i$ and $\frac{dx^0}{ds} = 0$ so $\frac{d^2x^\alpha}{ds^2} = 0$. Also, $n^0 = 0$ because $\vec{n}$ is spatial (meaning $n^\alpha u_\alpha = 0$). Then the geodesic equation is

$$\begin{aligned} 0 = 0 + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} &= \Gamma^\alpha{}_{jk}n^j n^k \\ \Rightarrow \Gamma^i{}_{jk} &= \Gamma^0{}_{jk} = 0 \end{aligned} \tag{4.4}$$

Part (c)

Let the 3-velocity be given by $\underset{\sim}{v} = \left(dx^i/dx^0\right)\vec{e}_i$. Now the geodesic equation for a freely falling particle is

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0 \tag{4.5}$$

Break up the equation into the temporal component ($\alpha = 0$) and the spatial components ($\alpha = i, i = 1, 2, 3$) and use the results for the connection coefficients found above.

$$\begin{aligned} \frac{d^2x^0}{d\lambda^2} &= -\left(\Gamma^0{}_{j0} + \Gamma^0{}_{0j}\right)\frac{dx^0}{d\lambda}\frac{dx^j}{d\lambda} \\ &= -2a_j\frac{dx^0}{d\lambda}\frac{dx^j}{d\lambda} \\ &= -2a_j\frac{dx^j}{dx^0}\left(\frac{dx^0}{d\lambda}\right)^2 \\ \frac{d^2x^0}{d\lambda^2} &= -2(\underset{\sim}{a}\cdot\underset{\sim}{v})\left(\frac{dx^0}{d\lambda}\right)^2 \end{aligned} \tag{4.6}$$

And then also

$$\begin{aligned} \frac{d^2x^i}{d\lambda^2} &= -\Gamma^i{}_{00}\left(\frac{dx^0}{d\lambda}\right)^2 - \left(\Gamma^i{}_{k0} + \Gamma^i{}_{0k}\right)\frac{dx^k}{d\lambda}\frac{dx^0}{d\lambda} \\ &= -a^i\left(\frac{dx^0}{d\lambda}\right)^2 + 2w^j\epsilon_{0j}{}^i{}_k\frac{dx^k}{d\lambda}\frac{dx^0}{d\lambda} \\ &= -a^i\left(\frac{dx^0}{d\lambda}\right)^2 - \left(\underset{\sim}{w}\times\frac{d\underset{\sim}{x}}{d\lambda}\right)^i\frac{dx^0}{d\lambda} \\ \frac{d^2x^i}{d\lambda^2} &= \left(-a^i - 2(\underset{\sim}{w}\times\underset{\sim}{v}^i)\right)\left(\frac{dx^0}{d\lambda}\right)^2 \end{aligned} \tag{4.7}$$

Now look at the derivatives of x^i with respect to x^0. We know that $\frac{dx^i}{dx^0} = (\frac{dx^i}{d\lambda})/(\frac{dx^0}{d\lambda})$ and since $\frac{d}{d\tau} = \frac{d}{dx^0}$, then we can write

$$\begin{aligned}\frac{d^2x^i}{d\tau^2} = \left(\frac{d}{dx^0}\right)^2 x^i &= \frac{1}{\frac{dx^0}{d\lambda}}\frac{d}{d\lambda}\left((\frac{dx^i}{d\lambda})\Big/(\frac{dx^0}{d\lambda})\right)\\ &= \left(\frac{d^2x^i}{d\lambda^2}\Big/\left(\frac{dx^0}{d\lambda}\right)^2\right) - \left(\frac{dx^i}{d\lambda}\frac{d^2x^0}{d\lambda^2}\Big/\left(\frac{dx^0}{d\lambda}\right)^3\right)\\ &= \left(\frac{d^2x^i}{d\lambda^2} - v^i\frac{d^2x^0}{d\lambda^2}\right)\Big/\left(\frac{dx^0}{d\lambda}\right)^2 \end{aligned} \tag{4.8}$$

Substituting in the expressions above for $\frac{d^2x^i}{d\lambda^2}$ and $\frac{d^2x^0}{d\lambda^2}$ and contracting with $\vec{e}_i$, the expression becomes

$$\frac{d^2x^i}{d\tau^2}\vec{e}_i = -\underset{\sim}{a} - 2(\underset{\sim}{w}\times\underset{\sim}{v}) + 2\underset{\sim}{v}(\underset{\sim}{a}\cdot\underset{\sim}{v}) \tag{4.9}$$

Problem 5

Part (a)

From class, the equation of geodesic deviation in general is

$$\frac{D^2x^\alpha}{dt} = R^\alpha{}_{\beta\gamma\delta}u^\beta u^\gamma x^\delta \tag{5.1}$$

Since we are in a local Lorentz frame comoving with the center of mass, the 4-velocity is $\vec{u} = (1,0,0,0)$ and in this case x^α is purely spatial so can be written as x^i. So then the equation becomes

$$\frac{d^2x^j}{dt^2} = R^j{}_{00k}x^k = -R^j{}_{0k0}x^k \tag{5.2}$$

Part (b)

Torque is a length crossed with a force and force is mass times acceleration. Let $f^j = \rho\frac{d^2x^j}{dt^2} = -\rho R^j{}_{0k0}x^k$ be the force per unit volume with density ρ. The torque is then

$$\tau_i = \int \epsilon_{ilj}x^l f^j d^3x = -\epsilon_{ilj}\int x^l\rho R^j{}_{0k0}x^k d^3x = -\epsilon_{ilj}R^j{}_{0k0}\int \rho x^l x^k d^3x \tag{5.3}$$

where the fact that R is nearly constant was used to take it out of the integral. Now to get the quadrupole tensor, consider the quantity $\delta^{lk}\epsilon_{ilj}R^j{}_{0k0}$ which is the same as $\epsilon_i{}^{kj}R_{j0k0}$:

$$\epsilon_i{}^{kj}R_{j0k0} = -\epsilon_i{}^{jk}R_{j0k0} = -\epsilon_i{}^{jk}R_{k0j0} = -\epsilon_i^{kj}R_{j0k} = 0 \tag{5.4}$$

where the indices were relabeled in the last step. This means that any term that has the form of $\delta^{lk}\epsilon_{ilj}R^j{}_{0k0}$ can be added to Eq. (5.3) without changing its value.

$$\begin{aligned}\tau_i =& -\epsilon_{ilj}R^j{}_{0k0}\int \rho x^l x^k d^3x + \epsilon_{ilj}R^j{}_{0k0}\int \rho\frac{1}{3}r^2\delta^{lk}d^3x \\ =& -\epsilon_{ilj}R^j{}_{0k0}\int \rho(x^l x^k - \frac{1}{3}r^2\delta^{lk})d^3x \\ =& -\epsilon_{ilj}R^j{}_{0k0}t^{lk}\end{aligned} \tag{5.5}$$

where $t^{lk} = \int \rho(x^l x^k - \frac{1}{3}r^2\delta^{lk})d^3x$.

Part (c)

Torque is the change of angular momentum with time so in the local Lorentz frame $dS^i/dt = -\epsilon^i{}_{lj}R^j{}_{0k0}t^{lk}$. Define the spin vector S^μ such that there are only spatial components governed by the equation above. Since u^μ is the 4-velocity of the center of mass, $S^\mu u_\mu = 0$. Define $t^{\alpha\beta}$ similarly so then $t^{\alpha\beta}u_\beta = 0$. Now $u^i = 0$ and $u^0 = 1$ in the LLF, so we can write in this frame $R^j{}_{0k0} = R^j{}_{\sigma k\lambda}u^\sigma u^\lambda$. Similarly, in the LLF,

$$\epsilon_{ijk} = \epsilon_{0ijk} = u^\mu\epsilon_{\mu ijk}, \tag{5.6}$$

so

$$\begin{aligned}\frac{dS^i}{dt} = -\epsilon^i{}_{lj}R^j{}_{0k0}t^{lk} &= -u^\mu\epsilon_\mu{}^i{}_{lj}R^j{}_{0k0}t^{lk} \\ &= -u^\mu\epsilon_\mu{}^{ilj}R_{j0k0}t_l{}^k \\ &= -u^\mu\epsilon_\mu{}^{i\beta\alpha}R_{\alpha 0\eta 0}t_\beta{}^\eta \\ &= -u^\mu\epsilon_\mu{}^{i\beta\alpha}R_{\alpha\lambda\eta\sigma}u^\lambda u^\sigma t_\beta{}^\eta \\ &= u_\mu\epsilon^{i\beta\alpha\mu}R_{\alpha\lambda\eta\sigma}u^\lambda u^\sigma t_\beta{}^\eta \\ &= u_\mu\epsilon^{i\beta\alpha\mu}R_{\eta\sigma\alpha\lambda}u^\lambda u^\sigma t_\beta{}^\eta \\ &= \epsilon^{i\beta\alpha\mu}u_\mu u^\lambda u^\sigma t_{\beta\eta}R^\eta{}_{\sigma\alpha\lambda},\end{aligned} \tag{5.7}$$

where on the 3rd line we have used $t^{\alpha\beta}u_\beta = 0$ (i.e. t with any zero index is zero) and the antisymmetry of Riemann to write 3-d sums as 4-d sums, and on the 4th line we have used $R_{j0k0} = R_{j\sigma k\lambda}u^\sigma u^\lambda$.

In the LLF, dS^i/dt is the same as $DS^i/d\tau$. Because $S^0 = 0$ and $\epsilon^{0\beta\alpha\mu}u_\mu = 0$, we can change the i index to a 4-d μ index, and write

$$\frac{DS^\nu}{d\tau} = \epsilon^{\nu\beta\alpha\mu}u_\mu u^\sigma u^\lambda t_{\beta\eta}R^\eta{}_{\sigma\alpha\lambda}. \tag{5.8}$$

We have now constructed a tensor equation, so even though we derived this equation in the LLF, it is true in all frames.