Solutions Ph 236a – Week 5

Kevin Barkett, Jonas Lippuner, and Mark Scheel

October 28, 2015

Contents

Problem 1

Part (a)

In Homework 2 we found that

$$\begin{aligned}
\vec{e}_r &= \frac{\partial}{\partial r} = \sin\theta\cos\phi\,\vec{e}_x + \sin\theta\sin\phi\,\vec{e}_y + \cos\theta\,\vec{e}_z \\
\vec{e}_\theta &= \frac{\partial}{\partial\theta} = r\cos\theta\cos\phi\,\vec{e}_x + r\cos\theta\sin\phi\,\vec{e}_y - r\sin\theta\,\vec{e}_z, \\
\vec{e}_\phi &= \frac{\partial}{\partial\phi} = -r\sin\theta\sin\phi\,\vec{e}_x + r\sin\theta\cos\phi\,\vec{e}_y,
\end{aligned} \tag{1.1}$$

and the metric for these basis vectors is

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}. \tag{1.2}$$

Since $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$ is a coordinate basis, the connection coefficients are given by

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{il}(g_{jl,k} + g_{lk,j} - g_{jk,l}). \tag{1.3}$$

We use Mathematica to do all the tedious calculations. First we define the vector of variable `x`, the metric `g`, and its inverse `gInv`.

```
x = {r, t, p}
g = {{1, 0, 0}, {0, r^2, 0}, {0, 0, r^2 Sin[t]^2}}
gInv = Inverse[g]
```

Then we define the function `ConCoef` that calculates $\Gamma^i{}_{jk}$ according to (1.3).

```
ConCoef[i_, j_, k_] := 1/2 Sum[gInv[[i, l]](D[g[[j, l]], x[[k]]]
    + D[g[[l, k]], x[[j]]] - D[g[[j, k]], x[[l]]]), {l, 1, 3}]
```

We then evaluate $\Gamma^i{}_{jk}$ with the `Table` function.

```
Table[ConCoef[1, j, k], {j, 1, 3}, {k, 1, 3}] // MatrixForm
Table[ConCoef[2, j, k], {j, 1, 3}, {k, 1, 3}] // MatrixForm
Table[ConCoef[3, j, k], {j, 1, 3}, {k, 1, 3}] // MatrixForm
```

We find that the only non-zero connection coefficients are

$$\begin{aligned}
\Gamma^r{}_{\theta\theta} &= -r, \\
\Gamma^r{}_{\phi\phi} &= -r\sin^2\theta, \\
\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} &= \frac{1}{r}, \\
\Gamma^\theta{}_{\phi\phi} &= -\sin\theta\,\cos\theta, \\
\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} &= \frac{1}{r}, \\
\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} &= \cot\theta.
\end{aligned} \tag{1.4}$$

Part (b)

Since $g_{ij} = \delta_{ij}$ in an orthonormal basis, all the metric derivatives are 0 and so the connection coefficients are given by

$$\begin{aligned}
\Gamma^{\hat{i}}{}_{\hat{j}\hat{k}} &= \frac{1}{2} g^{\hat{i}\hat{l}}(c_{\hat{l}\hat{j}\hat{k}} + c_{\hat{l}\hat{k}\hat{j}} - c_{\hat{j}\hat{k}\hat{l}}) = \frac{1}{2}(c^{\hat{i}}{}_{\hat{j}\hat{k}} + c^{\hat{i}}{}_{\hat{k}\hat{j}} - c_{\hat{j}\hat{k}}{}^{\hat{i}}) \\
&= \frac{1}{2}(c_{\hat{i}\hat{j}}{}^{\hat{k}} + c_{\hat{i}\hat{k}}{}^{\hat{j}} - c_{\hat{j}\hat{k}}{}^{\hat{i}}),
\end{aligned} \tag{1.5}$$

since $g_{\hat{i}\hat{j}} = \delta_{\hat{i}\hat{j}}$ and so we can raise and lower indices as we please. In Homework 4 we found that the only non-zero commutation coefficients are

$$\begin{aligned}
c_{\hat{r}\hat{\theta}}{}^{\hat{\theta}} = -c_{\hat{\theta}\hat{r}}{}^{\hat{\theta}} &= -\frac{1}{r}, \\
c_{\hat{r}\hat{\phi}}{}^{\hat{\phi}} = -c_{\hat{\phi}\hat{r}}{}^{\hat{\phi}} &= -\frac{1}{r}, \\
c_{\hat{\theta}\hat{\phi}}{}^{\hat{\phi}} = -c_{\hat{\phi}\hat{\theta}}{}^{\hat{\phi}} &= -\frac{\cos\theta}{r\sin\theta}.
\end{aligned} \tag{1.6}$$

We enter this information in Mathematica as `c`.

```
c = {{{0, 0, 0}, {0, -1/r, 0}, {0, 0, -1/r}},
     {{0, 1/r, 0}, {0, 0, 0}, {0, 0, -Cos[t]/(r Sin[t])}},
     {{0, 0, 1/r}, {0, 0, Cos[t]/(r Sin[t])}, {0, 0, 0}}}
```

And we compute the connection coefficients as follows.

```
ConCoef[i_, j_, k_] :=
    1/2 (c[[i, j, k]] + c[[i, k, j]] - c[[j, k, i]])
Table[ConCoef[1, j, k], {j, 1, 3}, {k, 1, 3}] // MatrixForm
Table[ConCoef[2, j, k], {j, 1, 3}, {k, 1, 3}] // MatrixForm
Table[ConCoef[3, j, k], {j, 1, 3}, {k, 1, 3}] // MatrixForm
```

We find that the only non-zero connection coefficients are

$$\begin{aligned}
\Gamma^{\hat{r}}{}_{\hat{\theta}\hat{\theta}} = \Gamma^{\hat{r}}{}_{\hat{\phi}\hat{\phi}} &= -\frac{1}{r}, \\
\Gamma^{\hat{\theta}}{}_{\hat{r}\hat{\theta}} &= \frac{1}{r}, \\
\Gamma^{\hat{\theta}}{}_{\hat{\phi}\hat{\phi}} &= -\frac{\cot\theta}{r}, \\
\Gamma^{\hat{\phi}}{}_{\hat{r}\hat{\phi}} &= \frac{1}{r}, \\
\Gamma^{\hat{\phi}}{}_{\hat{\theta}\hat{\phi}} &= \frac{\cot\theta}{r}.
\end{aligned} \tag{1.7}$$

Part (c)

Recall that we have

$$A^k{}_{;k} = A^k{}_{,k} + \Gamma^k{}_{ik} A^i. \tag{1.8}$$

In the Cartesian coordinate basis we have $\Gamma^i{}_{jk} = 0$ and so $A^k{}_{;k} = A^k{}_{,k} = \partial_k A^k$.

In the basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$ we use the connection coefficients we have found in (1.4) to find

$$\begin{aligned}
A^k{}_{;k} &= \frac{\partial A^r}{\partial r} + \frac{\partial A^\theta}{\partial \theta} + \frac{\partial A^\phi}{\partial \phi} + A^r \left(0 + \frac{1}{r} + \frac{1}{r}\right) + A^\theta \left(0 + 0 + \cot\theta\right) \\
&\quad + A^\phi \left(0 + 0 + 0\right) \\
&= \frac{\partial A^r}{\partial r} + \frac{\partial A^\theta}{\partial \theta} + \frac{\partial A^\phi}{\partial \phi} + \frac{2}{r} A^r + \cot\theta\, A^\theta \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A^r\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A^\theta\right) + \frac{\partial A^\phi}{\partial \phi}.
\end{aligned} \tag{1.9}$$

Finally, in the basis $(\vec{e}_{\hat{r}}, \vec{e}_{\hat{\theta}}, \vec{e}_{\hat{\phi}})$ we use the connection coefficients we have found in (1.7) and obtain

$$\begin{aligned}
A^{\hat{k}}{}_{;\hat{k}} &= \frac{\partial A^{\hat{r}}}{\partial r} + \frac{1}{r}\frac{\partial A^{\hat{\theta}}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A^{\hat{\phi}}}{\partial \phi} + A^{\hat{r}} \left(0 + \frac{1}{r} + \frac{1}{r}\right) + A^{\hat{\theta}} \left(0 + 0 + \frac{\cot\theta}{r}\right) \\
&\quad + A^{\hat{\phi}} \left(0 + 0 + 0\right) \\
&= \frac{\partial A^{\hat{r}}}{\partial r} + \frac{1}{r}\frac{\partial A^{\hat{\theta}}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A^{\hat{\phi}}}{\partial \phi} + \frac{2}{r} A^{\hat{r}} + \frac{\cot\theta}{r} A^{\hat{\theta}} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A^{\hat{r}}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A^{\hat{\theta}}\right) + \frac{1}{r\sin\theta}\frac{\partial A^{\hat{\phi}}}{\partial \phi},
\end{aligned} \tag{1.10}$$

which is the familiar divergence in spherical coordinates from (e.g.) Jackson. Note that electromagnetism textbooks use an orthonormal basis, not a coordinate basis, for spherical (and cylindrical) coordinates!

Part (d)

Note that $\det g = r^4 \sin^2\theta$. If we take

$$\epsilon^{ijk} = a\hat{\epsilon}^{ijk}, \tag{1.11}$$

where

$$\hat{\epsilon}^{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of (1,2,3)}, \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of (1,2,3)}, \\ 0 & \text{otherwise} \end{cases} \tag{1.12}$$

is the Levi-Civita tensor in the Cartesian coordinate basis $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$. Note that since we are in 3-dimensional space and there are no minus signs in the metric, we do not pick up a minus sign when changing all covariant indices to contravariant ones. We find

$$\epsilon_{123} = g_{1i}g_{2j}g_{3k}\epsilon^{ijk} = r^4\sin^2\theta\, a\hat{\epsilon}^{123} = r^4\sin^2\theta\, a. \tag{1.13}$$

We want

$$\epsilon_{123} = +\sqrt{\det g} = r^2\sin\theta, \tag{1.14}$$

so it follows that $a = 1/(r^2\sin\theta)$ and we have

$$\epsilon^{ijk} = \begin{cases} +\dfrac{1}{r^2\sin\theta} & \text{if } (i,j,k) \text{ is an even permutation of (1,2,3)}, \\ -\dfrac{1}{r^2\sin\theta} & \text{if } (i,j,k) \text{ is an odd permutation of (1,2,3)}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.15}$$

Part (e)

We have

$$(\operatorname{curl} A)^i = \epsilon^{ijk}A_{k;j} = \epsilon^{ijk}\left(A_{k,j} - \Gamma^l{}_{kj}A_l\right) = \epsilon^{ijk}A_{k,j} - \epsilon^{ijk}\Gamma^l{}_{kj}A_l. \tag{1.16}$$

Note that in the basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$, the connection coefficients are symmetric in the last two indices and thus the second term above vanishes, because it is a contraction of an anti-symmetric with a symmetric object. Thus for the basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$ we find

$$\begin{aligned} (\operatorname{curl} A)^r &= \frac{1}{r^2\sin\theta}\left(\frac{\partial A_\phi}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right), \\ (\operatorname{curl} A)^\theta &= \frac{1}{r^2\sin\theta}\left(\frac{\partial A_r}{\partial\phi} - \frac{\partial A_\phi}{\partial r}\right), \\ (\operatorname{curl} A)^\phi &= \frac{1}{r^2\sin\theta}\left(\frac{\partial A_\theta}{\partial r} - \frac{\partial A_r}{\partial\theta}\right). \end{aligned} \tag{1.17}$$

In the basis $(\vec{e}_{\hat{r}}, \vec{e}_{\hat{\theta}}, \vec{e}_{\hat{\phi}})$ the connection coefficients are not symmetric in the last two indices and so we the second term in (1.16) does not vanish. Also note that the Levi-Civita tensor in the basis $(\vec{e}_{\hat{r}}, \vec{e}_{\hat{\theta}}, \vec{e}_{\hat{\phi}})$ is the same as in 3-dimensional Cartesian coordinates, because the metric is the identity. Thus we find

$$\begin{aligned}
(\mathrm{curl}\, A)^{\hat{r}} &= \frac{1}{r}\frac{\partial A_{\hat{\phi}}}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial A_{\hat{\theta}}}{\partial \phi} + \frac{\cot\theta}{r} A_{\hat{\phi}} \\
&= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta\, A_{\hat{\phi}}) - \frac{\partial A_{\hat{\theta}}}{\partial\phi}\right), \\
(\mathrm{curl}\, A)^{\hat{\theta}} &= \frac{1}{r\sin\theta}\frac{\partial A_{\hat{r}}}{\partial\phi} - \frac{\partial A_{\hat{\phi}}}{\partial r} - \frac{1}{r}A_{\hat{\phi}} \\
&= \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_{\hat{r}}}{\partial\phi} - \frac{\partial}{\partial r}(rA_{\hat{\phi}})\right), \\
(\mathrm{curl}\, A)^{\hat{\phi}} &= \frac{\partial A_{\hat{\theta}}}{\partial r} - \frac{1}{r}\frac{\partial A_{\hat{r}}}{\partial\theta} + \frac{1}{r}A_{\hat{\theta}} \\
&= \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_{\hat{\theta}}) - \frac{\partial A_{\hat{r}}}{\partial\theta}\right), \qquad (1.18)
\end{aligned}$$

which is the usual curl in spherical coordinates found in electromagnetism textbooks (these textbooks, as we see, use an orthonormal basis and not a coordinate basis).

Problem 2

Part (a)

First note that from the notes that $g_{\lambda\mu,\gamma} = \Lambda_{\lambda\mu\gamma} + \Gamma_{\mu\lambda\gamma}$. So then we can write

$$\begin{aligned}
{g^{\alpha\beta}}_{,\gamma} &= -\, g_{\lambda\mu,\gamma} g^{\mu\beta} g^{\lambda\alpha} \\
&= -\, (\Gamma_{\lambda\mu\gamma} + \Gamma_{\mu\lambda\gamma}) g^{\mu\beta} g^{\lambda\alpha} \\
&= -\, {\Gamma^{\alpha}}_{\mu\gamma} g^{\mu\beta} - {\Gamma^{\beta}}_{\lambda\gamma} g^{\lambda\alpha} \qquad (2.1)
\end{aligned}$$

Part (b)

In coordinate frame, ${\Gamma^{\mu}}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu})$. in the case where $\alpha = \mu$ as in this problem, then the last two terms will cancel, leaving ${\Gamma^{\alpha}}_{\alpha\beta} = \frac{1}{2} g^{\alpha\nu} g_{\nu\alpha,\beta}$. From the previous set, we have that $\frac{g_{,\beta}}{g} = g^{\alpha\nu} g_{\nu\alpha,\beta}$ so then

$${\Gamma^{\alpha}}_{\alpha\beta} = \frac{1}{2}\frac{g_{,\beta}}{g} = \frac{1}{2}\left(\log(-g)\right)_{,\beta} = \left(\log(-g)^{\frac{1}{2}}\right)_{,\beta} \qquad (2.2)$$

Part (c)

We will use the results from parts (a) and (b) above.

$$
\begin{aligned}
g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu} &= -g^{\alpha\beta}{}_{,\beta} - \Gamma^{\beta}{}_{\lambda\beta}g^{\lambda\alpha} && \text{by part (a)} \\
&= -g^{\alpha\beta}{}_{,\beta} - \left(\log(-g)^{\frac{1}{2}}\right)_{,\lambda} g^{\lambda\alpha} && \text{by part (b)} \\
&= -g^{\alpha\nu}{}_{,\nu} - \left(\log(-g)^{\frac{1}{2}}\right)_{,\nu} g^{\nu\alpha} && \text{relabeling indices} \\
&= -g^{\alpha\nu}{}_{,\nu} - (-g)^{\frac{1}{2}}{}_{,\nu} g^{\alpha\nu}(-g)^{-\frac{1}{2}} \\
&= -\frac{1}{(-g)^{\frac{1}{2}}}(g^{\alpha\nu}(-g)^{\frac{1}{2}})_{,\nu}
\end{aligned} \tag{2.3}
$$

Part (d)

Again, we will use the result from part (b) to find

$$
\begin{aligned}
A^{\alpha}{}_{;\alpha} =& A^{\alpha}{}_{,\alpha} + \Gamma^{\alpha}{}_{\beta\alpha}A^{\beta} \\
=& A^{\alpha}{}_{,\alpha} + \frac{1}{(-g)^{\frac{1}{2}}}((-g)^{\frac{1}{2}})_{,\beta}A^{\beta} \\
=& \frac{1}{(-g)^{\frac{1}{2}}}((-g)^{\frac{1}{2}}A^{\alpha})_{,\alpha}
\end{aligned} \tag{2.4}
$$

Part (e)

Start with the definition of the Levi-Civita tensor as $\epsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\hat{\epsilon}_{\mu\nu\rho\sigma}$ where we know that $\hat{\epsilon}_{0123} = 1$ and each permutation results in a sign change. Now let us expand out the covariant derivative and use the fact that for coordinate basis that $\Gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\gamma\beta}$ so that

$$
\begin{aligned}
\epsilon_{\mu\nu\rho\sigma;\alpha} = (\sqrt{-g}\hat{\epsilon}_{\mu\nu\rho\sigma})_{,\alpha} &- \Gamma^{\beta}{}_{\mu\alpha}\sqrt{-g}\hat{\epsilon}_{\beta\nu\rho\sigma} - \Gamma^{\beta}{}_{\nu\alpha}\sqrt{-g}\hat{\epsilon}_{\mu\beta\rho\sigma} \\
&- \Gamma^{\beta}{}_{\rho\alpha}\sqrt{-g}\hat{\epsilon}_{\mu\nu\beta\sigma} - \Gamma^{\beta}{}_{\sigma\alpha}\sqrt{-g}\hat{\epsilon}_{\mu\nu\rho\beta}
\end{aligned} \tag{2.5}
$$

First treat the case where μ, ν, ρ, σ are not all unique. Then the first term in Eq. (2.5) vanishes. But what about the other terms? Suppose $\mu = \nu$. Then the last two terms of Eq. (2.5) vanish individually because of the antisymmetry of Levi-Civita, and the remaining terms can be written

$$
-\Gamma^{\beta}{}_{\mu\alpha}\sqrt{-g}\hat{\epsilon}_{\beta\nu\rho\sigma} + \Gamma^{\beta}{}_{\nu\alpha}\sqrt{-g}\hat{\epsilon}_{\beta\mu\rho\sigma}, \tag{2.6}
$$

which is antisymmetric in μ and ν, and thus vanishes when $\mu = \nu$. Therefore $\epsilon_{\mu\nu\rho\sigma;\alpha} = 0$ if $\mu = \nu$. The same argument can be used if any of the components are equal, and shows that $\epsilon_{\mu\nu\rho\sigma;\alpha} = 0$ if μ, ν, ρ, σ are not all unique.

So now consider the case where μ, ν, ρ, σ are all unique. Examine the first connection term in Eq. (2.5): $-\Gamma^{\beta}{}_{\mu\alpha}\sqrt{-g}\hat{\epsilon}_{\beta\nu\rho\sigma}$. The only value for β that can make a nonzero contribution to this term is $\beta = \mu$, since any other value would result in a duplicated index in the Levi-Civita tensor. So then this term becomes $-\Gamma^{\mu}{}_{\mu\alpha}\sqrt{-g}\hat{\epsilon}_{\mu\nu\rho\sigma}$ (where there is no sum on the μ index). Similarly, the second connection term is $-\Gamma^{\nu}{}_{\nu\alpha}\sqrt{-g}\hat{\epsilon}_{\mu\nu\rho\sigma}$ (where there is no sum on the ν index), and so on. Summing the four connection terms therefore yields $-\Gamma^{\beta}{}_{\beta\alpha}\sqrt{-g}\hat{\epsilon}_{\mu\nu\rho\sigma}$ (where there *is* a sum on β). Therefore, for μ, ν, ρ, σ all unique,

$$\epsilon_{\mu\nu\rho\sigma;\alpha} = \left((\sqrt{-g})_{,\alpha} - \Gamma^{\beta}{}_{\beta\alpha}\sqrt{-g}\right)\hat{\epsilon}_{\mu\nu\rho\sigma} \tag{2.7}$$

From above in part (b), we can use $\Gamma^{\beta}{}_{\beta\alpha} = \left(\log(-g)^{(\frac{1}{2})}\right)_{,\alpha}$ and then the equation reduces to

$$\epsilon_{\mu\nu\rho\sigma;\alpha} = \left((\sqrt{-g})_{,\alpha} - \left(\log(-g)^{(\frac{1}{2})}\right)_{,\alpha}\sqrt{-g}\right)\hat{\epsilon}_{\mu\nu\rho\sigma} = 0 \tag{2.8}$$

Problem 3

Part (a)

To write out the components of $\nabla_{\vec{u}}\tilde{u} = 0$, use the property of taking the covariant derivative on a lower index,

$$\begin{aligned}
\nabla_{\vec{u}}\tilde{u} = 0 =& u^{\alpha}u_{\beta;\alpha} \\
=& u^{\alpha}(u_{\beta,\alpha} - \Gamma^{\gamma}{}_{\beta\alpha}u_{\gamma}) \\
=& \frac{du_{\beta}}{d\lambda} - \Gamma^{\gamma}{}_{\beta\alpha}u_{\gamma}u^{\alpha} = 0
\end{aligned} \tag{3.1}$$

which is the parallel transport equation for $\tilde{u}$.

Part (b)

To show this is equivalent to the geodesic equation for vector components, start with the 1-form geodesic equation and manipulate it into the form of the vector geodesic equation we are given. Specfically,

$$\begin{aligned}
\frac{du^{\mu}g_{\mu\beta}}{d\lambda} - \Gamma^{\gamma}{}_{\beta\alpha}u_{\gamma}u^{\alpha} =& \frac{du^{\mu}}{d\lambda}g_{\mu\beta} + u^{\mu}u^{\nu}g_{\mu\beta,\nu} - \Gamma^{\gamma}{}_{\beta\alpha}u_{\gamma}u^{\alpha} = \\
=& \frac{du^{\mu}}{d\lambda}g_{\mu\beta} + u^{\mu}u^{\nu}(\Gamma_{\mu\beta\nu} + \Gamma_{\beta\mu\nu}) - \Gamma^{\gamma}{}_{\beta\alpha}u_{\gamma}u^{\alpha} = \\
=& \frac{du^{\mu}}{d\lambda}g_{\mu\beta} + u^{\mu}u^{\nu}(g_{\mu\delta}\Gamma^{\delta}{}_{\beta\nu} + g_{\beta\delta}\Gamma^{\delta}{}_{\mu\nu}) - \Gamma^{\gamma}{}_{\beta\alpha}u_{\gamma}u^{\alpha} \\
=& \frac{du^{\mu}}{d\lambda}g_{\mu\beta} + u_{\delta}u^{\nu}\Gamma^{\delta}{}_{\beta\nu} + u^{\mu}u^{\nu}g_{\beta\delta}\Gamma^{\delta}{}_{\mu\nu} - \Gamma^{\gamma}{}_{\beta\alpha}u_{\gamma}u^{\alpha}
\end{aligned} \tag{3.2}$$

where in going from line 1 to line 2 the relation $g_{\mu\beta,\nu} = \Gamma_{\mu\beta\nu} + \Gamma_{\beta\mu\nu}$ was used and then the definition of $\Gamma_{\mu\beta\nu} = g_{\mu\delta}\Gamma^{\delta}{}_{\beta\nu}$ was used to get to line 3. On the last line of the equation above, note that the second and last terms cancel after a relabeling of indicies. After that, what's left is

$$\begin{aligned}\frac{du^\mu}{d\lambda} g_{\mu\beta} + u^\mu u^\nu g_{\beta\delta}\Gamma^{\delta}{}_{\mu\nu} =& \frac{du^\mu}{d\lambda} g_{\mu\beta} + g_{\mu\beta}\Gamma^{\mu}{}_{\gamma\alpha} u^\gamma u^\alpha \\ =& g_{\mu\beta}\left(\frac{du^\mu}{d\lambda} + \Gamma^{\mu}{}_{\gamma\alpha} u^\gamma u^\alpha\right) \\ =& 0\end{aligned} \tag{3.3}$$

as the equation in the parenthesis is simply the the geodesic equation for vectors. Note that the geodesic equation for vectors is *not* just the "naively raised" geodesic equation for 1-forms.

Problem 4

Part (a)

A geodesic with tangent vector $\vec{u}$ is spacelike, timelike or null according to whether $\vec{u}\cdot\vec{u}$ is $> 0, < 0, = 0$. But $\vec{u}\cdot\vec{u}$ is conserved along the geodesic since $\nabla_{\vec{u}}(\vec{u}\cdot\vec{u}) = 2\vec{u}\cdot\nabla_{\vec{u}}\vec{u} = 0$ since the geodesic equation is $\nabla_{\vec{u}}\vec{u} = 0$.

Part (b)

The general case gives Euler-Lagrange equation, using the definitions of F, y and $\dot{x}$ in the problem statement

$$\begin{aligned}0 =& \frac{\partial F}{\partial x} - \frac{d}{ds}\frac{\partial F}{\partial \dot{x}} \\ =& \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} - \frac{d}{ds}\left(\frac{\partial F}{\partial y}\frac{\partial y}{\partial \dot{x}}\right) \\ =& \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} - \frac{\partial y}{\partial \dot{x}}\frac{d}{ds}\left(\frac{\partial F}{\partial y}\right) - \frac{\partial F}{\partial y}\frac{d}{ds}\left(\frac{\partial y}{\partial \dot{x}}\right)\end{aligned} \tag{4.1}$$

Now the partial can be reexpressed in the following manner

$$\frac{d}{ds}\left(\frac{\partial F}{\partial y}\right) = \left(\frac{\partial}{\partial s} + \frac{\partial y}{\partial s}\frac{\partial}{\partial y} + \frac{\partial x}{\partial s}\frac{\partial}{\partial x} + \frac{\partial \dot{x}}{\partial s}\frac{\partial}{\partial \dot{x}}\right)\left(\frac{\partial F}{\partial y}\right) = \frac{\partial^2 F}{\partial y^2}\frac{\partial y}{\partial s} \tag{4.2}$$

where the fact that $F(y)$ only explicitly depends on y to produce the last equality. Then the Euler-Lagrange equation can be simplified as

$$-\frac{\partial y}{\partial \dot{x}}\frac{\partial^2 F}{\partial y^2}\frac{\partial y}{\partial s} + \frac{\partial F}{\partial y}\left[\frac{\partial y}{\partial x} - \frac{d}{ds}\left(\frac{\partial y}{\partial \dot{x}}\right)\right] = 0 \tag{4.3}$$

Now, because s is an affine parameter $\frac{\partial y}{\partial s} = 0$ and since F is a monotonic function of y, its derivative is nonzero, or $\frac{\partial F}{\partial y} \neq 0$. The equation then becomes

$$\begin{aligned} 0 =& -\frac{\partial y}{\partial \dot{x}}\frac{\partial^2 F}{\partial y^2}(0) + \frac{\partial F}{\partial y}\left[\frac{\partial y}{\partial x} - \frac{d}{ds}\left(\frac{\partial y}{\partial \dot{x}}\right)\right] \\ =& \frac{\partial F}{\partial y}\left[\frac{\partial y}{\partial x} - \frac{d}{ds}\left(\frac{\partial y}{\partial \dot{x}}\right)\right] \\ \Rightarrow 0 =& \frac{\partial y}{\partial x} - \frac{d}{ds}\left(\frac{\partial y}{\partial \dot{x}}\right) \end{aligned} \tag{4.4}$$

which gives the Euler-Lagrange equation for geodesics we are looking for.

Problem 5

Let $v^\alpha = S^{\alpha\beta}{}_\gamma M_\beta{}^\gamma$, then the left-hand side is

$$\begin{aligned} (S^{\alpha\beta}{}_\gamma M_\beta{}^\gamma)_{;\alpha} &= v^\alpha{}_{;\alpha} = v^\alpha{}_{,\alpha} + \Gamma^\alpha{}_{\mu\alpha} v^\mu \\ &= (S^{\alpha\beta}{}_\gamma M_\beta{}^\gamma)_{,\alpha} + \Gamma^\alpha{}_{\mu\alpha} S^{\mu\beta}{}_\gamma M_\beta{}^\gamma \\ &= S^{\alpha\beta}{}_{\gamma,\alpha} M_\beta{}^\gamma + S^{\alpha\beta}{}_\gamma M_\beta{}^\gamma{}_{,\alpha} + \Gamma^\alpha{}_{\mu\alpha} S^{\mu\beta}{}_\gamma M_\beta{}^\gamma. \end{aligned} \tag{5.1}$$

Expanding the right-hand side yields

$$\begin{aligned} & S^{\alpha\beta}{}_{\gamma;\alpha} M_\beta{}^\gamma + S^{\alpha\beta}{}_\gamma M_\beta{}^\gamma{}_{;\alpha} \\ =& M_\beta{}^\gamma (S^{\alpha\beta}{}_{\gamma,\alpha} + \Gamma^\alpha{}_{\mu\alpha} S^{\mu\beta}{}_\gamma + \Gamma^\beta{}_{\nu\alpha} S^{\alpha\nu}{}_\gamma - \Gamma^\lambda{}_{\gamma\alpha} S^{\alpha\beta}{}_\lambda) \\ & + S^{\alpha\beta}{}_\gamma (M_\beta{}^\gamma{}_{,\alpha} - \Gamma^\delta{}_{\beta\alpha} M_\delta{}^\gamma + \Gamma^\gamma{}_{\epsilon\alpha} M_\beta{}^\epsilon) \\ =& S^{\alpha\beta}{}_{\gamma,\alpha} M_\beta{}^\gamma + S^{\alpha\beta}{}_\gamma M_\beta{}^\gamma{}_{,\alpha} + \Gamma^\alpha{}_{\mu\alpha} S^{\mu\beta}{}_\gamma M_\beta{}^\gamma \\ & + (\Gamma^\beta{}_{\nu\alpha} S^{\alpha\nu}{}_\gamma M_\beta{}^\gamma - \Gamma^\delta{}_{\beta\alpha} S^{\alpha\beta}{}_\gamma M_\delta{}^\gamma) \\ & + (-\Gamma^\lambda{}_{\gamma\alpha} S^{\alpha\beta}{}_\lambda M_\beta{}^\gamma + \Gamma^\gamma{}_{\epsilon\alpha} S^{\alpha\beta}{}_\gamma M_\beta{}^\epsilon) \\ =& S^{\alpha\beta}{}_{\gamma,\alpha} M_\beta{}^\gamma + S^{\alpha\beta}{}_\gamma M_\beta{}^\gamma{}_{,\alpha} + \Gamma^\alpha{}_{\mu\alpha} S^{\mu\beta}{}_\gamma M_\beta{}^\gamma + (0) + (0), \end{aligned} \tag{5.2}$$

since the terms inside the parenthesis exactly cancel because all indices are dummy indices. Thus we have shown that

$$(S^{\alpha\beta}{}_\gamma M_\beta{}^\gamma)_{;\alpha} = S^{\alpha\beta}{}_{\gamma;\alpha} M_\beta{}^\gamma + S^{\alpha\beta}{}_\gamma M_\beta{}^\gamma{}_{;\alpha}. \tag{5.3}$$

□