Solutions Ph 236a – Week 4

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Problem 1

Part (a)

We are given that the first law of thermodynamics for a relativistic fluid can be written as

$$d(\rho V) = -PdV + TdS \tag{1.1}$$

which uses the fact that the total number of baryons, N, is conserved. That means we can write some of the quantities in terms of the number of baryons and the density of baryons. Let n be the baryon number density. Then we can write the volume and entropy as

$$V = N/n \qquad\qquad S = sN \tag{1.2}$$

where s is the entropy per baryon. Then we can rewrite the first law as

$$d\left(\rho \frac{N}{n}\right) = -\, Pd\left(\frac{N}{n}\right) + Td(Ns) \tag{1.3}$$

but since N is constant, we can divide it out.

$$\begin{aligned} d\left(\frac{\rho}{n}\right) &= -\, P\left(\frac{1}{n}\right) + Tds \\ \frac{d\rho}{n} - \frac{\rho dn}{n^2} &= Pd\left(\frac{dn}{n^2}\right) + Tds \\ d\rho &= (P+\rho)\frac{dn}{n} + nTds \end{aligned} \tag{1.4}$$

which is the desired result.

Part (b)

We are given that for a perfect fluid that

$$T_{\mu\nu} = (P+\rho)u_\mu u_\nu + Pg_{\mu\nu} \tag{1.5}$$

and that the conservation of baryons is $(nu^\alpha)_{,\alpha} = 0$. In the fluid's rest frame, write out the 0 component of the equation of motion.

$$\begin{aligned} T^{0\nu}{}_{;\nu} =& P_{,\nu}g^{0\nu} + Pg^{0\nu}{}_{;\nu} + (P+\rho)_{,\nu}u^\nu u^0 + \\ &+ (P+\rho)[u^\nu{}_{;\nu}u^0 + u^0{}_{;\nu}u^\nu] = 0 \end{aligned} \tag{1.6}$$

Now we know that $g^{\mu\nu}{}_{;\nu} = 0$ in general, and also that in the rest frame $u^0 = 1$ and

$$u^0{}_{;\nu} = -u_\alpha u^\alpha{}_{;\nu} = -\frac{1}{2}(u_\alpha u^\alpha)_{;\nu} = 0 \tag{1.7}$$

so that the equation of motion reduces to

$$0 = -\frac{d}{dt}P + \frac{d}{dt}(P+\rho) + (P+\rho)u^{\nu}{}_{;\nu} \tag{1.8}$$

Now let the baryon number density in the fluid's rest frame be n. Now the number-flux vector of baryons is conserved,

$$\begin{aligned}(nu^{\alpha})_{;\alpha} = 0 = n_{,\nu}u^{\nu} + nu^{\nu}{}_{;\nu} &= \frac{dn}{dt} + nu^{\nu}{}_{;\nu} \\ u^{\nu}{}_{;\nu} &= -\frac{1}{n}\frac{dn}{dt} \end{aligned}\tag{1.9}$$

This equation can be used to eliminate $u^{\nu}{}_{;\nu}$ in the equation of motion to give

$$\frac{d\rho}{dt} = \frac{P+\rho}{n}\frac{dn}{dt} \tag{1.10}$$

If we compare this to the first law of thermodynamics from Part (a), it must be that $\frac{ds}{dt} = 0$ which is the condition for isentropy. Therefore the perfect fluid is isentropic.

Problem 2

Part (a)

For each case, we need to find the parameter value which evaluates the curve to the point $P_0 = (0,1,0)$ and then compute the derivative of the function and evaluate at that point.
$x^{\alpha}(\lambda) = (\lambda, 1, \lambda)$:
First, set $(0,1,0) = (\lambda,1,\lambda)$ and we can see that here $\lambda = 0$. Now

$$f(x^{\alpha}(\lambda)) = x^2 - y^2 + z^2 = \lambda^2 - 1 + \lambda^2 = 2\lambda^2 - 1 \tag{2.1}$$

Now take the derivative with respect to λ and evaluate at $\lambda = 0$

$$\left.\frac{df}{d\lambda}\right|_{\lambda=0} = 4\lambda|_{\lambda=0} = 0 \tag{2.2}$$

$x^{\alpha}(\xi) = (\sin\xi, \cos\xi, \xi)$:
Follow the same procedure as above to see that

$$\begin{aligned}(\sin\xi,\cos\xi,\xi) &= (0,1,0) \Rightarrow \xi = 0 \\ f(x^{\xi}(\xi)) = x^2 - y^2 + z^2 = \sin\xi^2 - \cos\xi^2 + \xi^2 &= 1 - 2\cos\xi^2 + \xi^2 \\ \left.\frac{df}{d\xi}\right|_{\xi=0} &= 4\sin\xi\cos\xi + 2\xi|_{\lambda=0} = 0\end{aligned}\tag{2.3}$$

$x^\rho(\xi) = (\sinh\rho, \cosh\rho, \rho + \rho^3)$:

$$\begin{aligned}
(\sinh\rho, \cosh\rho, \rho+\rho^3) = (0,1,0) \Rightarrow \rho = 0 \\
f(x^\rho(\rho)) = x^2 - y^2 + z^2 =& \sinh\rho^2 - \cosh\rho^2 + (\rho+\rho^3)^2 \\
=& \rho^6 + 2\rho^4 + \rho^2 - 1 \\
\left.\frac{df}{d\rho}\right|_{\rho=0} = 6\rho^5 + 8\rho^3 + 2\rho\big|_0 = 0
\end{aligned} \tag{2.4}$$

Part (b)

Computing the components of the vectors $d/d\lambda, d/d\xi$, and $d/d\rho$ is the same thing as asking for the directional derivatives along each of the curves evaluated at the same point as in part (a). The directional derivative at a point P_0 is defined as

$$\left(\frac{d}{d\lambda}\right)_{P_0} = \left(\frac{dx^\alpha}{d\lambda}\right)_{P_0}\left(\frac{d}{dx^\alpha}\right) \tag{2.5}$$

So in the case we are considering with Catesian coordinates (and implied Minkowski metric)

$$\left(\frac{d}{d\lambda}\right)_{P_0} = \left(\frac{dx}{d\lambda}\frac{d}{dx} + \frac{dy}{d\lambda}\frac{d}{dy} + \frac{dz}{d\lambda}\frac{d}{dz}\right)_{P_0} \tag{2.6}$$

Now apply this to each of the curves given in the problem

$$\left(\frac{d}{d\lambda}\right)_{\lambda=0} = \left(\frac{d}{dx} + 0 + \frac{d}{dz}\right)_{\lambda=0} = \frac{d}{dx} + \frac{d}{dz} \tag{2.7}$$

$$\left(\frac{d}{d\xi}\right)_{\xi=0} = \left(\cos\xi\frac{d}{dx} - \sin\xi\frac{d}{dy} + \frac{d}{dz}\right)_{\xi=0} = \frac{d}{dx} + \frac{d}{dz} \tag{2.8}$$

$$\left(\frac{d}{d\rho}\right)_{\rho=0} = \left(\cosh\rho\frac{d}{dx} - \sinh\rho\frac{d}{dy} + (1+3\rho^2)\frac{d}{dz}\right)_{\rho=0} = \frac{d}{dx} + \frac{d}{dz} \tag{2.9}$$

Problem 3

Part (a)

Let $\forall f : M \to \mathbb{R}$ where M is the manifold. Let's expand all of the terms (dropping the vector arrows for convienence),

$$\begin{aligned}
[U,[V,W]] =& U(V(W(f))) - U(W(V(f))) - V(W(U(f))) + W(V(U(f))) \\
[V,[W,U]] =& V(W(U(f))) - V(U(W(f))) - W(U(V(f))) + U(W(V(f))) \\
[W,[U,V]] =& W(U(V(f))) - W(V(U(f))) - U(V(W(f))) + V(U(W(f)))
\end{aligned} \tag{3.1}$$

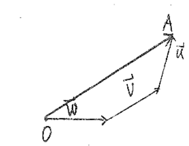
Sum up the above 3 equations and find that all of the terms cancel. Thus we have $[U,[V,W]] + [V,[W,U]] + [W,[U,V]] = 0$

Part(b)

$U(V(W(f)))$ can be denote like below as $\vec{OA}(f) \sim \frac{f(A)-f(O)}{|OA|}$

So $U(V(W(\cdot))) : f \longrightarrow \cancel{\mathbb{R}}$ which means that

f (Samuel Lee, Fall 2009)



$U(V(W(\cdot)))$ is also a vector at O, denoted by $\vec{OA}$

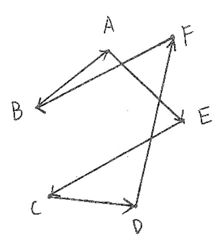
Similary we have $\vec{OB} = U(W(V(\cdot)))$

$\vec{OC} = V(W(U(\cdot)))$

$\vec{OD} = W(V(U(\cdot)))$

$\vec{OE} = V(U(W(\cdot)))$

$\vec{OF} = W(U(V(\cdot)))$



$[U[V,W]] = \vec{OA} - \vec{OB} - \vec{OC} + \vec{OD} = \vec{BA} + \vec{CD}$ (red line)

$[V[W,U]] = \vec{OC} - \vec{OE} - \vec{OF} + \vec{OB} = \vec{EC} + \vec{FB}$ (blue)

$[W[U,V]] = \vec{OF} - \vec{OD} - \vec{OA} + \vec{OE} = \vec{DF} + \vec{AE}$ (black)

No matter which point you begin with, you will come back to it if you follow the arrows drawed above. That means this vector vanishes.

Part(c)

First, let's do this the easy way. If e_α are basis vectors,

$$[e_\alpha, e_\beta] = c_{\alpha\beta}{}^{\gamma} e_\gamma, \tag{3.2}$$

so

$$\begin{aligned}[e_\lambda, [e_\alpha, e_\beta]] = [e_\lambda, c_{\alpha\beta}{}^{\gamma} e_\gamma] &= c_{\alpha\beta}{}^{\gamma}{}_{,\lambda} e_\gamma + c_{\alpha\beta}{}^{\gamma}[e_\lambda, e_\gamma] \\ &= c_{\alpha\beta}{}^{\gamma}{}_{,\lambda} e_\gamma + c_{\alpha\beta}{}^{\gamma} c_{\lambda\gamma}{}^{\rho} e_\rho \\ &= \left(c_{\alpha\beta}{}^{\gamma}{}_{,\lambda} + c_{\alpha\beta}{}^{\rho} c_{\lambda\rho}{}^{\gamma} \right) e_\gamma, \end{aligned} \tag{3.3}$$

where $c_{\alpha\beta}{}^{\gamma}{}_{,\lambda}$ (or equivalently $\partial_\lambda c_{\alpha\beta}{}^{\gamma}$) means e_λ acting on $c_{\alpha\beta}{}^{\gamma}$, i.e. the directional derivative along e_λ of the scalar $c_{\alpha\beta}{}^{\gamma}$.

Then, the Jacobi identity says that the sum of (3.3) for cyclic combinations of λ, α, β is zero. You get

$$c_{\alpha\beta}{}^{\gamma}{}_{,\lambda} + c_{\beta\lambda}{}^{\gamma}{}_{,\alpha} + c_{\lambda\alpha}{}^{\gamma}{}_{,\beta} + c_{\alpha\beta}{}^{\rho} c_{\lambda\rho}{}^{\gamma} + c_{\beta\lambda}{}^{\rho} c_{\alpha\rho}{}^{\gamma} + c_{\lambda\alpha}{}^{\rho} c_{\beta\rho}{}^{\gamma} = 0. \tag{3.4}$$

$\square$

It is instructive to derive this the hard way, using full vector components. Define the vectors as follows,

$$\vec{U} = u^\alpha e_\alpha \qquad \vec{V} = v^\beta e_\beta \qquad \vec{V} = w^\sigma e_\sigma \tag{3.5}$$

where each of the e_α are basis vectors. Then the commutator between two vectors can be written as

$$\begin{aligned}[V, W] = [v^\beta e_\beta, w^\sigma e_\sigma] &= v^\beta w^\sigma [e_\beta, e_\sigma] + v^\beta e_\sigma \partial_\beta w^\sigma - w^\sigma e_\beta \partial_\sigma v^\beta \\ &= v^\beta w^\sigma c_{\beta\sigma}{}^{\gamma} e_\gamma + \left(v^\sigma \partial_\sigma w^\gamma - w^\sigma \partial_\sigma v^\gamma\right) e_\gamma, \\ &= \left(v^\beta w^\sigma c_{\beta\sigma}{}^{\gamma} + v^\sigma \partial_\sigma w^\gamma - w^\sigma \partial_\sigma v^\gamma \right) e_\gamma, \\ &= Q^\gamma e_\gamma, \end{aligned} \tag{3.6}$$

where in the last line I defined Q^γ as a shortcut. The commutator between three vectors is

$$[U, [V, W]] = [u^\alpha e_\alpha, Q^\gamma e_\gamma] = \left(u^\beta Q^\sigma c_{\beta\sigma}{}^{\gamma} + u^\sigma \partial_\sigma Q^\gamma - Q^\sigma \partial_\sigma u^\gamma \right) e_\gamma, \tag{3.7}$$

where I have repeated the steps in Eq. (3.6). Now let's evaluate the derivative of Q^γ, which we need in the above equation:

$$\begin{aligned}\partial_\lambda Q^\gamma = & \, c_{\beta\alpha}{}^{\gamma} (w^\alpha \partial_\lambda v^\beta + v^\beta \partial_\lambda w^\alpha) + v^\beta w^\alpha \partial_\lambda c_{\beta\alpha}{}^{\gamma} \\ & + \partial_\lambda v^\beta \partial_\beta w^\gamma - \partial_\lambda w^\beta \partial_\beta v^\gamma + v^\beta \partial_\lambda \partial_\beta w^\gamma - w^\beta \partial_\lambda \partial_\beta v^\gamma \end{aligned} \tag{3.8}$$

Then

$$\begin{aligned}\langle\tilde{\omega}^\gamma,[U,[V,W]]\rangle = &\left(v^\beta w^\alpha c_{\beta\alpha}{}^\gamma + v^\alpha\partial_\alpha w^\gamma - w^\alpha\partial_\alpha v^\gamma\right)\left(u^\delta c_{\delta\lambda}{}^\gamma - \partial_\lambda u^\gamma\right)\\ &+ u^\lambda\left(c_{\beta\alpha}{}^\gamma(w^\alpha\partial_\lambda v^\beta + v^\beta\partial_\lambda w^\alpha) + v^\beta w^\alpha\partial_\lambda c_{\beta\alpha}{}^\gamma\right.\\ &\left. + \partial_\lambda v^\beta\partial_\beta w^\gamma - \partial_\lambda w^\beta\partial_\beta v^\gamma + v^\beta\partial_\lambda\partial_\beta w^\gamma - w^\beta\partial_\lambda\partial_\beta v^\gamma\right). \quad (3.9)\end{aligned}$$

To evaluate the Jacobi identity, we sum the previous equation for cyclic combinations of U, V, and W, and set the result to zero. This is straightforward, but a bit tedious. When we do this, we get

$$\begin{aligned}0 = &\, u^\alpha v^\beta w^\gamma\left(c_{\alpha\beta}{}^\gamma{}_{,\lambda} + c_{\beta\lambda}{}^\gamma{}_{,\alpha} + c_{\lambda\alpha}{}^\gamma{}_{,\beta} + c_{\alpha\beta}{}^\rho c_{\lambda\rho}{}^\gamma + c_{\beta\lambda}{}^\rho c_{\alpha\rho}{}^\gamma + c_{\lambda\alpha}{}^\rho c_{\beta\rho}{}^\gamma\right)\\ &- \left(u^\beta v^\alpha\partial_\lambda w^\gamma + v^\beta w^\alpha\partial_\lambda u^\gamma + w^\beta u^\alpha\partial_\lambda v^\gamma\right)c_{\beta\alpha}{}^\lambda\\ &+ u^\lambda v^\beta\partial_\lambda\partial_\beta w^\gamma + v^\lambda w^\beta\partial_\lambda\partial_\beta u^\gamma + w^\lambda u^\beta\partial_\lambda\partial_\beta v^\gamma\\ &- u^\lambda w^\beta\partial_\lambda\partial_\beta v^\gamma - v^\lambda u^\beta\partial_\lambda\partial_\beta w^\gamma - w^\lambda v^\beta\partial_\lambda\partial_\beta u^\gamma\\ &+ (12 \text{ terms with } c_{\beta\alpha}{}^\gamma \text{ times 1st derivs of } u,\ v, \text{ or } w)\\ &+ (12 \text{ terms with no } c_{\beta\alpha}{}^\gamma \text{ and two 1st derivs of } u,\ v, \text{ or } w).\end{aligned} \quad (3.10)$$

The terms on the last two lines on Eq. (3.10), which I didn't write out explicitly, contain either one or two first derivatives of u, v, or w. These terms cancel out (Just write them out, there are 24 of them. Each term appears twice, but with opposite sign, taking into account the fact that $c_{\beta\alpha}{}^\gamma$ is antisymmetric on the first two indices). However, we need a bit more work to simplify the remaining lines. Let's first look at the two terms that have a second derivative of w: the first term on the 3rd line of Eq. (3.10), and the second term on the 4th line of Eq. (3.10). These terms can be written

$$\begin{aligned}u^\lambda v^\beta\partial_\lambda\partial_\beta w^\gamma - v^\lambda u^\beta\partial_\lambda\partial_\beta w^\gamma &= u^\lambda v^\beta\left(\partial_\lambda\partial_\beta w^\gamma - \partial_\beta\partial_\lambda w^\gamma\right)\\ &= u^\lambda v^\beta\partial_{[e_\lambda,e_\beta]}w^\gamma\\ &= u^\lambda v^\beta\partial_{c_{\lambda\beta}{}^\rho e_\rho}w^\gamma\\ &= u^\lambda v^\beta c_{\lambda\beta}{}^\rho\partial_\rho w^\gamma. \quad (3.11)\end{aligned}$$

Note that this expression is the same as (minus) the first term on line 2 of Eq. (3.10), so it cancels out that term. Likewise, the two terms on line 3 and 4 of Eq. (3.10) that have second derivatives of u cancel out another term on line 2 of Eq. (3.10), and the terms that have second derivatives of v cancel out the remaining term on line 2. So in the end, only the first line of Eq. (3.10) survives, and this is line is the same as Eq. (3.4) since it must hold for arbitrary u, v, and w. □

Problem 4

Part (a)

Expanding the derivative yields

$$\begin{aligned} g^{\alpha\beta}{}_{,\gamma} &= (g^{\mu\alpha}g^{\nu\beta}g_{\mu\nu})_{,\gamma} = g^{\mu\alpha}{}_{,\gamma}g^{\nu\beta}g_{\mu\nu} + g^{\mu\alpha}g^{\nu\beta}{}_{,\gamma}g_{\mu\nu} + g^{\mu\alpha}g^{\nu\beta}g_{\mu\nu,\gamma} \\ &= g^{\mu\alpha}{}_{,\gamma}\delta^{\beta}{}_{\mu} + g^{\nu\beta}{}_{,\gamma}\delta^{\alpha}{}_{\nu} + g^{\mu\alpha}g^{\nu\beta}g_{\mu\nu,\gamma} \\ &= g^{\alpha\beta}{}_{,\gamma} + g^{\alpha\beta}{}_{,\gamma} + g^{\mu\alpha}g^{\nu\beta}g_{\mu\nu,\gamma} \\ \Leftrightarrow \quad -g^{\alpha\beta}{}_{,\gamma} &= g^{\mu\alpha}g^{\nu\beta}g_{\mu\nu,\gamma}, \end{aligned} \tag{4.1}$$

where we have used the fact that $g^{\nu\beta}g_{\mu\nu} = g^{\beta}{}_{\mu} = \delta^{\beta}{}_{\mu}$, which was shown on the second homework. Thus we have shown the desired result. □

Part (b)

Suppose we have a matrix $C = C(\epsilon)$, for small ϵ, we can Taylor-expand $C(\epsilon)$ around $\epsilon = 0$. We find

$$C(\epsilon) \approx C(0) + \epsilon \left.\frac{dC}{d\epsilon}\right|_{\epsilon=0} + O(\epsilon^2) = A + \epsilon B + O(\epsilon^2), \tag{4.2}$$

where we have defined $A = C(0)$ and $B = \left.\frac{dC}{d\epsilon}\right|_{\epsilon=0}$. To first order in ϵ, we find

$$\begin{aligned} \det(C(\epsilon)) &= \det(A + \epsilon B) = \det(A(1 + \epsilon A^{-1}B)) \\ &= \det(A)\det(1 + \epsilon A^{-1}B) = \det(A)(1 + \epsilon\,\mathrm{tr}(A^{-1}B)) \\ &= \det(A) + \epsilon\,\det(A)\mathrm{tr}(A^{-1}B). \end{aligned} \tag{4.3}$$

And so

$$\begin{aligned} \left.\frac{d(\det(C(\epsilon)))}{d\epsilon}\right|_{\epsilon=0} &= \lim_{\epsilon\to 0}\frac{\det(C(\epsilon)) - \det(C(0))}{\epsilon} \\ &= \det(C(0))\mathrm{tr}\left([C(0)]^{-1}\left.\frac{dC}{d\epsilon}\right|_{\epsilon=0}\right), \end{aligned} \tag{4.4}$$

since $A = C(0)$ and $B = \left.\frac{dC}{d\epsilon}\right|_{\epsilon=0}$. Because we can always shift the argument of the function, we can write the above as

$$(\det C)' = \det C\,\mathrm{tr}(C^{-1}C'). \tag{4.5}$$

Thus for the matrix $C = g_{\mu\nu}$ we have

$$g_{,\alpha} = g\,\mathrm{tr}(g^{\mu\nu}g_{\nu\lambda,\alpha}) = gg^{\mu\nu}g_{\nu\mu,\alpha} = gg^{\mu\nu}g_{\mu\nu,\alpha}, \tag{4.6}$$

because $C^{-1} = g^{\mu\nu}$, $\partial_\alpha C = g_{\gamma\lambda,\alpha}$, matrix multiplication is contraction over inner indices (i.e. $C^{-1}\partial_\alpha C = g^{\mu\nu}g_{\nu\lambda,\alpha}$), taking the trace is contracting the indices of a matrix (i.e. $\mathrm{tr}(A) = A^{\mu}{}_{\mu}$), and finally we used symmetry of $g_{\mu\nu}$ in the last equality. □

Problem 5

Part (a)

On Homework 2 we found that the 1-forms

$$
\begin{aligned}
\tilde{d}r &= \sin\theta\cos\phi\,\tilde{d}x + \sin\theta\sin\phi\,\tilde{d}y + \cos\theta\,\tilde{d}z,\\
\tilde{d}\theta &= \frac{\cos\theta\cos\phi}{r}\,\tilde{d}x + \frac{\cos\theta\sin\phi}{r}\,\tilde{d}y - \frac{\sin\theta}{r}\,\tilde{d}z,\\
\tilde{d}\phi &= -\frac{\sin\phi}{r\sin\theta}\,\tilde{d}x + \frac{\cos\phi}{r\sin\theta}\,\tilde{d}y
\end{aligned} \tag{5.1}
$$

are dual to the basis vectors

$$
\begin{aligned}
\vec{e}_r &= \frac{\partial}{\partial r} = \sin\theta\cos\phi\,\vec{e}_x + \sin\theta\sin\phi\,\vec{e}_y + \cos\theta\,\vec{e}_z\\
\vec{e}_\theta &= \frac{\partial}{\partial\theta} = r\cos\theta\cos\phi\,\vec{e}_x + r\cos\theta\sin\phi\,\vec{e}_y - r\sin\theta\,\vec{e}_z,\\
\vec{e}_\phi &= \frac{\partial}{\partial\phi} = -r\sin\theta\sin\phi\,\vec{e}_x + r\sin\theta\cos\phi\,\vec{e}_y.
\end{aligned} \tag{5.2}
$$

Denote $\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_\phi$ by $\vec{e}_i$, and denote $\vec{e}_{\hat{r}}$, $\vec{e}_{\hat{\theta}}$, $\vec{e}_{\hat{\phi}}$ by $\vec{e}_{\hat{i}}$. We have

$$\vec{e}_{\hat{i}} = L^i{}_{\hat{i}}\vec{e}_i, \tag{5.3}$$

where

$$L^i{}_{\hat{i}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r & 0 \\ 0 & 0 & 1/(r\sin\theta) \end{bmatrix}. \tag{5.4}$$

Denote $\tilde{d}r$, $\tilde{d}\theta$, $\tilde{d}\phi$ by $\tilde{d}x^i$, and define

$$\tilde{\omega}^{\hat{i}} = L^{\hat{i}}{}_i\tilde{d}x^i, \tag{5.5}$$

where

$$L^{\hat{i}}{}_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r\sin\theta \end{bmatrix}. \tag{5.6}$$

We find that

$$
\begin{aligned}
\langle\tilde{\omega}^{\hat{i}}, \vec{e}_{\hat{j}}\rangle &= \langle L^{\hat{i}}{}_i\tilde{d}x^i, L^j{}_{\hat{j}}\vec{e}_j\rangle = L^{\hat{i}}{}_i L^j{}_{\hat{j}}\langle\tilde{d}x^i, \vec{e}_j\rangle = L^{\hat{i}}{}_i L^j{}_{\hat{j}}\delta^i{}_j = L^{\hat{i}}{}_i L^i{}_{\hat{j}}\\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \delta^{\hat{i}}{}_{\hat{j}},
\end{aligned} \tag{5.7}
$$

where we used the fact that $\tilde{d}x^i$ are dual to $\vec{e}_j$. We have thus shown that the basis 1-forms $\tilde{\omega}^{\hat{i}}$defined in (5.5) are dual to the orthonormal basis vectors $\vec{e}_{\hat{j}}$. Evaluating (5.5) yields

$$\begin{aligned}
\tilde{\omega}^{\hat{r}} &= \tilde{d}r = \sin\theta\cos\phi\,\tilde{d}x + \sin\theta\sin\phi\,\tilde{d}y + \cos\theta\,\tilde{d}z, \\
\tilde{\omega}^{\hat{\theta}} &= r\tilde{d}\theta = \cos\theta\cos\phi\,\tilde{d}x + \cos\theta\sin\phi\,\tilde{d}y - \sin\theta\,\tilde{d}z, \\
\tilde{\omega}^{\hat{\phi}} &= r\sin\theta\,\tilde{d}\phi = -\sin\phi\,\tilde{d}x + \cos\phi\,\tilde{d}y.
\end{aligned} \tag{5.8}$$

Part (b)

Recall that $c_{\hat{i}\hat{j}}{}^{\hat{k}}$ is defined as

$$[\vec{e}_{\hat{i}}, \vec{e}_{\hat{j}}] = c_{\hat{i}\hat{j}}{}^{\hat{k}}\vec{e}_{\hat{k}}, \tag{5.9}$$

where

$$[\vec{e}_{\hat{i}}, \vec{e}_{\hat{j}}] = \partial_{\vec{e}_{\hat{i}}}\partial_{\vec{e}_{\hat{j}}} - \partial_{\vec{e}_{\hat{j}}}\partial_{\vec{e}_{\hat{i}}} = \vec{e}_{\hat{i}}\vec{e}_{\hat{j}} - \vec{e}_{\hat{j}}\vec{e}_{\hat{i}}. \tag{5.10}$$

Note that $[\vec{e}_{\hat{i}}, \vec{e}_{\hat{j}}]$ is antisymmetric and so $c_{\hat{i}\hat{j}}{}^{\hat{k}}$ is antisymmetric. Hence $c_{\hat{i}\hat{j}}{}^{\hat{k}} = 0$ for $\hat{\imath} = \hat{\jmath}$ and $c_{\hat{i}\hat{j}}{}^{\hat{k}} = -c_{\hat{j}\hat{i}}{}^{\hat{k}}$. Thus

$$0 = c_{\hat{r}\hat{r}}{}^{\hat{r}} = c_{\hat{r}\hat{r}}{}^{\hat{\theta}} = c_{\hat{r}\hat{r}}{}^{\hat{\phi}} = c_{\hat{\theta}\hat{\theta}}{}^{\hat{r}} = c_{\hat{\theta}\hat{\theta}}{}^{\hat{\theta}} = c_{\hat{\theta}\hat{\theta}}{}^{\hat{\phi}} = c_{\hat{\phi}\hat{\phi}}{}^{\hat{r}} = c_{\hat{\phi}\hat{\phi}}{}^{\hat{\theta}} = c_{\hat{\phi}\hat{\phi}}{}^{\hat{\phi}}. \tag{5.11}$$

We get

$$\begin{aligned}
[\vec{e}_{\hat{r}}, \vec{e}_{\hat{\theta}}] = \vec{e}_{\hat{r}}\vec{e}_{\hat{\theta}} - \vec{e}_{\hat{\theta}}\vec{e}_{\hat{r}} &= \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial r}\right) \\
&= -\frac{1}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial\theta\partial r} = -\frac{1}{r}\vec{e}_{\hat{\theta}},
\end{aligned} \tag{5.12}$$

which means that

$$\begin{aligned}
c_{\hat{r}\hat{\theta}}{}^{\hat{r}} &= -c_{\hat{\theta}\hat{r}}{}^{\hat{r}} = 0, \\
c_{\hat{r}\hat{\theta}}{}^{\hat{\theta}} &= -c_{\hat{\theta}\hat{r}}{}^{\hat{\theta}} = -\frac{1}{r}, \\
c_{\hat{r}\hat{\theta}}{}^{\hat{\phi}} &= -c_{\hat{\theta}\hat{r}}{}^{\hat{\phi}} = 0.
\end{aligned} \tag{5.13}$$

We also find

$$\begin{aligned}
[\vec{e}_{\hat{r}}, \vec{e}_{\hat{\phi}}] = \vec{e}_{\hat{r}}\vec{e}_{\hat{\phi}} - \vec{e}_{\hat{\phi}}\vec{e}_{\hat{r}} &= \frac{\partial}{\partial r}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial}{\partial r}\right) \\
&= -\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\phi} + \frac{1}{r\sin\theta}\frac{\partial^2}{\partial r\partial\phi} - \frac{1}{r\sin\theta}\frac{\partial^2}{\partial\phi\partial r} = -\frac{1}{r}\vec{e}_{\hat{\phi}},
\end{aligned} \tag{5.14}$$

which gives

$$\begin{aligned}
c_{\hat{r}\hat{\phi}}{}^{\hat{r}} &= -c_{\hat{\phi}\hat{r}}{}^{\hat{r}} = 0, \\
c_{\hat{r}\hat{\phi}}{}^{\hat{\theta}} &= -c_{\hat{\phi}\hat{r}}{}^{\hat{\theta}} = 0, \\
c_{\hat{r}\hat{\phi}}{}^{\hat{\phi}} &= -c_{\hat{\phi}\hat{r}}{}^{\hat{\phi}} = -\frac{1}{r}.
\end{aligned} \tag{5.15}$$

Finally, we obtain

$$\begin{aligned}
[\vec{e}_{\hat{\theta}}, \vec{e}_{\hat{\phi}}] = \vec{e}_{\hat{\theta}}\vec{e}_{\hat{\phi}} - \vec{e}_{\hat{\phi}}\vec{e}_{\hat{\theta}} &= \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right) \\
&= -\frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial}{\partial\phi} + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi} - \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\phi\partial\theta} = -\frac{\cos\theta}{r\sin\theta}\vec{e}_{\hat{\phi}},
\end{aligned} \tag{5.16}$$

and so

$$\begin{aligned}
c_{\hat{\theta}\hat{\phi}}{}^{\hat{r}} &= -c_{\hat{\phi}\hat{\theta}}{}^{\hat{r}} = 0, \\
c_{\hat{\theta}\hat{\phi}}{}^{\hat{\theta}} &= -c_{\hat{\phi}\hat{\theta}}{}^{\hat{\theta}} = 0, \\
c_{\hat{\theta}\hat{\phi}}{}^{\hat{\phi}} &= -c_{\hat{\phi}\hat{\theta}}{}^{\hat{\phi}} = -\frac{\cos\theta}{r\sin\theta}.
\end{aligned} \tag{5.17}$$

To show that $c_{ij}{}^k$ is not a tensor, consider its transformation under a change of basis. We find

$$\begin{aligned}
c_{\bar{i}\bar{j}}{}^{\bar{k}}\vec{e}_{\bar{k}} &= [\vec{e}_{\bar{i}}, \vec{e}_{\bar{j}}] = [\partial_{\bar{i}}, \partial_{\bar{k}}] = \partial_{\bar{i}}\partial_{\bar{j}} - \partial_{\bar{j}}\partial_{\bar{i}} \\
&= L^i{}_{\bar{i}}\partial_i L^j{}_{\bar{j}}\partial_j - L^j{}_{\bar{j}}\partial_j L^i{}_{\bar{i}}\partial_i \\
&= L^i{}_{\bar{i}}\left(L^j{}_{\bar{j},i}\partial_j + L^j{}_{\bar{j}}\partial_i\partial_j\right) - L^j{}_{\bar{j}}\left(L^i{}_{\bar{i},j}\partial_i + L^i{}_{\bar{i}}\partial_j\partial_i\right) \\
&= L^i{}_{\bar{i}}L^j{}_{\bar{j}}(\partial_i\partial_j - \partial_j\partial_i) + \left(L^i{}_{\bar{i}}L^k{}_{\bar{j},i} - L^j{}_{\bar{j}}L^k{}_{\bar{i},j}\right)\partial_k \\
&= L^i{}_{\bar{i}}L^j{}_{\bar{j}}c_{ij}{}^k\partial_k + \left(L^i{}_{\bar{i}}L^k{}_{\bar{j},i} - L^j{}_{\bar{j}}L^k{}_{\bar{i},j}\right)L^{\bar{k}}{}_k\partial_{\bar{k}} \\
&= L^i{}_{\bar{i}}L^j{}_{\bar{j}}L^{\bar{k}}{}_k c_{ij}{}^k\vec{e}_{\bar{k}} + \left(L^i{}_{\bar{i}}L^k{}_{\bar{j},i} - L^j{}_{\bar{j}}L^k{}_{\bar{i},j}\right)L^{\bar{k}}{}_k\vec{e}_{\bar{k}},
\end{aligned} \tag{5.18}$$

and thus

$$c_{\bar{i}\bar{j}}{}^{\bar{k}} = L^i{}_{\bar{i}}L^j{}_{\bar{j}}L^{\bar{k}}{}_k c_{ij}{}^k + \left(L^i{}_{\bar{i}}L^k{}_{\bar{j},i} - L^j{}_{\bar{j}}L^k{}_{\bar{i},j}\right)L^{\bar{k}}{}_k, \tag{5.19}$$

which means that the commutation coefficient $c_{ij}{}^k$ does not transform like a tensor because of the extra second term.

Alternatively, one can use the following (simpler) argument. In some bases (e.g. $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$) it is obvious that all commutation coefficients vanish, i.e. $c_{ij}{}^k = 0$. If $c_{ij}{}^k$ were a tensor, then we would have

$$c_{\bar{i}\bar{j}}{}^{\bar{k}} = L^i{}_{\bar{i}}L^j{}_{\bar{j}}L^{\bar{k}}{}_k c_{ij}{}^k = 0, \tag{5.20}$$

in all other coordinate systems. But we have seen that this is not true in the orthonormal spherical coordinate system, thus $c_{ij}{}^k$ is not a tensor.

Problem 6

Let $L^{\alpha}{}_{\bar{\alpha}}$ be the transformation matrix from the unbarred into the barred frame, i.e. $\vec{e}_{\bar{\alpha}} = L^{\alpha}{}_{\bar{\alpha}}\vec{e}_{\alpha}$. Then we find

$$\Gamma^{\bar{\gamma}}{}_{\bar{\alpha}\bar{\beta}}\vec{e}_{\bar{\gamma}} = \nabla_{\vec{e}_{\bar{\beta}}}\vec{e}_{\bar{\alpha}} = \nabla_{L^{\beta}{}_{\bar{\beta}}\vec{e}_{\beta}}(L^{\alpha}{}_{\bar{\alpha}}\vec{e}_{\alpha}) = L^{\beta}{}_{\bar{\beta}}\nabla_{\vec{e}_{\beta}}(L^{\alpha}{}_{\bar{\alpha}}\vec{e}_{\alpha}), \tag{6.1}$$

where we have used linearity of the covariant derivative. Using the chain rule, we get

$$\Gamma^{\bar{\gamma}}{}_{\bar{\alpha}\bar{\beta}}\vec{e}_{\bar{\gamma}} = L^{\beta}{}_{\bar{\beta}}\left((\nabla_{\vec{e}_{\beta}}L^{\alpha}{}_{\bar{\alpha}})\vec{e}_{\alpha} + L^{\alpha}{}_{\bar{\alpha}}\nabla_{\vec{e}_{\beta}}\vec{e}_{\alpha}\right), \tag{6.2}$$

where we have treated $L^{\alpha}{}_{\bar{\alpha}}$ like a scalar field, because it is not a tensor in the sense that it transforms like a tensor under a coordinate change. Therefore,

$$\nabla_{\vec{e}_{\beta}}L^{\alpha}{}_{\bar{\alpha}} = \partial_{\vec{e}_{\beta}}L^{\alpha}{}_{\bar{\alpha}} = L^{\alpha}{}_{\bar{\alpha},\beta}, \tag{6.3}$$

and so

$$\begin{aligned}
\Gamma^{\bar{\gamma}}{}_{\bar{\alpha}\bar{\beta}}\vec{e}_{\bar{\gamma}} &= L^{\beta}{}_{\bar{\beta}}\left(L^{\alpha}{}_{\bar{\alpha},\beta}\vec{e}_{\alpha} + L^{\alpha}{}_{\bar{\alpha}}\Gamma^{\gamma}{}_{\alpha\beta}\vec{e}_{\gamma}\right) \\
&= L^{\beta}{}_{\bar{\beta}}L^{\alpha}{}_{\bar{\alpha},\beta}L^{\bar{\gamma}}{}_{\alpha}\vec{e}_{\bar{\gamma}} + L^{\beta}{}_{\bar{\beta}}L^{\alpha}{}_{\bar{\alpha}}\Gamma^{\gamma}{}_{\alpha\beta}L^{\bar{\gamma}}{}_{\gamma}\vec{e}_{\bar{\gamma}} \\
&= \left(L^{\beta}{}_{\bar{\beta}}L^{\alpha}{}_{\bar{\alpha}}L^{\bar{\gamma}}{}_{\gamma}\Gamma^{\gamma}{}_{\alpha\beta} + L^{\beta}{}_{\bar{\beta}}L^{\bar{\gamma}}{}_{\alpha}L^{\alpha}{}_{\bar{\alpha},\beta}\right)\vec{e}_{\bar{\gamma}},
\end{aligned} \tag{6.4}$$

where we have just rearranged terms and expanded unbarred vectors in terms of barred ones. We have thus found

$$\Gamma^{\bar{\gamma}}{}_{\bar{\alpha}\bar{\beta}} = L^{\beta}{}_{\bar{\beta}}L^{\alpha}{}_{\bar{\alpha}}L^{\bar{\gamma}}{}_{\gamma}\Gamma^{\gamma}{}_{\alpha\beta} + L^{\beta}{}_{\bar{\beta}}L^{\bar{\gamma}}{}_{\alpha}L^{\alpha}{}_{\bar{\alpha},\beta}. \tag{6.5}$$

If the second term was not there, $\Gamma^{\gamma}{}_{\alpha\beta}$ would indeed transform as a tensor, but because the second term is not 0 in general, $\Gamma^{\gamma}{}_{\alpha\beta}$ does not transform like a tensor. $\square$