Solutions Ph 236a – Week 3

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Problem 1

Part (a)

We are given $u \wedge v = u \otimes v - v \otimes u$ so we can see that

$$v \wedge u = v \otimes u - u \otimes v = -(u \otimes v - v \otimes u) = -u \wedge v \tag{1.1}$$

so the wedge product is antisymmetric. When looking at p- and q-forms $\mathbf{\Omega_1} = \frac{1}{p!}\Omega_{\alpha\beta...}u_1 \wedge ... \wedge u_p$ and $\mathbf{\Omega_2} = \frac{1}{q!}\omega_{\alpha\beta...}v_1 \wedge ... \wedge v_2$ and so the wedge product will produce an antisymmetric $(p+q)$-form

$$\begin{aligned}\mathbf{\Omega_1} \wedge \mathbf{\Omega_2} &= \left(\frac{1}{p!}\Omega_{\alpha\beta...}(u_1 \wedge ... \wedge u_p)\right) \wedge \left(\frac{1}{q!}\omega_{\mu\nu...}(v_1 \wedge ... \wedge v_q)\right) \\ &= \left(\frac{1}{p!q!}\Omega_{\alpha\beta...}\omega_{\mu\nu...}\right) u_1 \wedge ... \wedge u_p \wedge v_1 \wedge ... \wedge v_q \end{aligned}\tag{1.2}$$

Similarly,

$$\begin{aligned}\mathbf{\Omega_2} \wedge \mathbf{\Omega_1} &= \left(\frac{1}{p!}\Omega_{\alpha\beta...}(v_1 \wedge ... \wedge v_q)\right) \wedge \left(\frac{1}{q!}\omega_{\mu\nu...}(u_1 \wedge ... \wedge u_p)\right) \\ &= \left(\frac{1}{p!q!}\Omega_{\alpha\beta...}\omega_{\mu\nu...}\right) v_1 \wedge ... \wedge v_q \wedge u_1 \wedge ... \wedge u_p \end{aligned}\tag{1.3}$$

so we can see that this equation has all of the same terms as $\mathbf{\Omega_1} \wedge \mathbf{\Omega_2}$ up to a possible minus sign. In order to determine the sign, use the anticommutator relation for the wedge product shown above.

$$\begin{aligned}& u_1 \wedge ... \wedge u_p \wedge v_1 \wedge ... \wedge v_q \\ =& (-1)u_1 \wedge ... \wedge u_{p-1} \wedge v_1 \wedge u_p \wedge v_2 ... \wedge v_q \\ =& (-1)^p v_1 \wedge u_1 \wedge ... \wedge u_p \wedge v_2 \wedge ... \wedge v_q \\ =& ...\text{(repeat this process for each } v_i\text{)}... \\ =& (-1)^{pq} v_1 \wedge ... \wedge v_q \wedge u_1 \wedge ... \wedge u_p \end{aligned}\tag{1.4}$$

which is the same as the term as in $\mathbf{\Omega_2} \wedge \mathbf{\Omega_1}$ up to $(-1)^{pq}$, so because the coefficients are all equivalent, then we can see that $\mathbf{\Omega_1} \wedge \mathbf{\Omega_2} = (-1)^{pq}\mathbf{\Omega_2} \wedge \mathbf{\Omega_1}$

Part (b)

By definition of the exterior derivative, if $\mathbf{\Omega_1}$ is a p-form, then $\tilde{d}\mathbf{\Omega_1}$ is a $(p+1)$-form. To get the desired result, start with equation (3) from the homework and use the commutation relation derived above.

$$\begin{aligned}\tilde{d}(\mathbf{\Omega_1} \wedge \mathbf{\Omega_2}) =& \tilde{d}\mathbf{\Omega_1} \wedge \mathbf{\Omega_2} + (-1)^p \mathbf{\Omega_1} \wedge \tilde{d}\mathbf{\Omega_2} \\ \Rightarrow \tilde{d}((-1)^{pq}\mathbf{\Omega_2} \wedge \mathbf{\Omega_1}) =& (-1)^{(p+1)q}\mathbf{\Omega_2} \wedge \tilde{d}\mathbf{\Omega_1} + (-1)^p(-1)^{p(q+1)}\tilde{d}\mathbf{\Omega_2} \wedge \mathbf{\Omega_1} \\ \Rightarrow \tilde{d}(\mathbf{\Omega_2} \wedge \mathbf{\Omega_1}) =& (-1)^q \mathbf{\Omega_2} \wedge \tilde{d}\mathbf{\Omega_1} + (-1)^{2p}\tilde{d}\mathbf{\Omega_2} \wedge \mathbf{\Omega_1} \\ \Rightarrow \tilde{d}(\mathbf{\Omega_2} \wedge \mathbf{\Omega_1}) =& \tilde{d}\mathbf{\Omega_2} \wedge \mathbf{\Omega_1} + (-1)^q \mathbf{\Omega_2} \wedge \tilde{d}\mathbf{\Omega_1} \end{aligned} \tag{1.5}$$

Which is the result we are trying to prove.

Problem 2

Part (a)

Recall that we can always find a frame that locally looks like flat spacetime. In that frame, the Faraday tensor is

$$\begin{aligned} F_{\mu\nu} &= \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{bmatrix}, \\ F^{\mu\nu} &= \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{bmatrix}. \end{aligned} \tag{2.1}$$

Since F is a tensor, if we contract its indices, we get something that is invariant. For example, $F_{\mu\nu}F^{\mu\nu}$ is invariant. Note that this operation corresponds to multiplying the matrices $F_{\mu\nu}$ and $F^{\mu\nu}$ element-wise and then adding up all entries. We find that

$$F_{\mu\nu}F^{\mu\nu} = -2\|\underset{\sim}{E}\|^2 + 2\|\underset{\sim}{B}\|^2 = 2(B^2 - E^2). \tag{2.2}$$

Since $2(B^2 - E^2)$ is invariant, it follows that $B^2 - E^2$ is also invariant. □

Recall that $F_{\mu\nu}$ is antisymmetric, thus its dual ${}^*F_{\alpha\beta}$ exists and is defined as

$$\begin{aligned} {}^*F_{\alpha\beta} = \frac{1}{2!}F^{\mu\nu}\epsilon_{\mu\nu\alpha\beta} = \frac{1}{2}\begin{bmatrix} 0 & 2B^1 & -2(-B^2) & 2B^3 \\ -2B^1 & 0 & 2E^3 & -2E^2 \\ 2(-B^2) & -2E^3 & 0 & 2E^1 \\ -2B^3 & 2E^2 & -2E^1 & 0 \end{bmatrix} = \\ = \begin{bmatrix} 0 & B^1 & B^2 & B^3 \\ -B^1 & 0 & E^3 & -E^2 \\ -B^2 & -E^3 & 0 & E^1 \\ -B^3 & E^2 & -E^1 & 0 \end{bmatrix}. \end{aligned} \tag{2.3}$$

Since ${}^*F_{\alpha\beta}$ is a tensor, the contraction ${}^*F_{\alpha\beta}F^{\alpha\beta}$ is invariant, hence

$$ {}^*F_{\alpha\beta}F^{\alpha\beta} = 4\underset{\sim}{E} \cdot \underset{\sim}{B} \tag{2.4} $$

is also invariant. Thus $\underset{\sim}{E} \cdot \underset{\sim}{B}$ is invariant. □

Part (b)

If a quantity is a spacetime invariant, it also must be an invariant under spatial rotations, i.e. a spatial scalar. In a given Lorentz frame, there are only three ways to make a spatial scalar from $\underset{\sim}{E}$ and $\underset{\sim}{B}$: these are $\underset{\sim}{E} \cdot \underset{\sim}{E}$, $\underset{\sim}{B} \cdot \underset{\sim}{B}$, and $\underset{\sim}{B} \cdot \underset{\sim}{E}$ (and combinations of these scalars).

Now suppose that there were three independent *spacetime* invariants that can be made from $\underset{\sim}{E} \cdot \underset{\sim}{E}$, $\underset{\sim}{B} \cdot \underset{\sim}{B}$, and $\underset{\sim}{B} \cdot \underset{\sim}{E}$. This would imply that each of these spatial scalars is actually a spacetime scalar. However, it is easy to verify (e.g. by carrying out an explicit Lorentz transformation) that $\underset{\sim}{E} \cdot \underset{\sim}{E}$ and $\underset{\sim}{B} \cdot \underset{\sim}{B}$ are not spacetime scalars.

Alternatively, one can show that all combinations of $F^{\mu\nu}$ and ${}^*F^{\mu\nu}$, where all indices are contracted, involve only $B^2 - E^2$ or $\underset{\sim}{E} \cdot \underset{\sim}{B}$. This is more difficult.

Since both tensors are antisymmetric we have

$$ F^{\mu}{}_{\mu} = {}^*F^{\mu}{}_{\mu} = 0, \tag{2.5} $$

which is not a combination of $\underset{\sim}{E}$ and $\underset{\sim}{B}$. Thus we need to consider expressions with two tensors. The possible combinations are

$$ \begin{aligned} F_{\mu\nu}F^{\mu\nu} &= -F_{\mu\nu}F^{\nu\mu} = 2(B^2 - E^2), \\ {}^*F_{\mu\nu}{}^*F^{\mu\nu} &= -{}^*F_{\mu\nu}{}^*F^{\nu\mu} = 2(E^2 - B^2), \\ F_{\mu\nu}{}^*F^{\mu\nu} &= -F_{\mu\nu}{}^*F^{\nu\mu} = 4\underset{\sim}{E} \cdot \underset{\sim}{B}, \end{aligned} \tag{2.6} $$

which are just multiples of $B^2 - E^2$ or $\underset{\sim}{E} \cdot \underset{\sim}{B}$. Note that inserting a Levi-Civita tensor would simply convert one of the tensors to its dual and it would not give a different combination of $F^{\mu\nu}$ and ${}^*F^{\mu\nu}$ that what was considered above.

Unfortunately, the above is not quite enough to show that the only invariants are $B^2 - E^2$ and $\underset{\sim}{E} \cdot \underset{\sim}{B}$. One can also construct scalars with more than just two tensors. For example,

$$ \begin{aligned} F^{\mu}{}_{\nu}F^{\nu}{}_{\lambda}F^{\lambda}{}_{\mu} &= (-F_{\nu}{}^{\mu})(-F_{\lambda}{}^{\nu})(-F_{\mu}{}^{\lambda}) = -F^{\nu}{}_{\mu}F^{\lambda}{}_{\nu}F^{\mu}{}_{\lambda} \\ &= -F^{\nu}{}_{\mu}F^{\mu}{}_{\lambda}F^{\lambda}{}_{\nu}, \end{aligned} \tag{2.7} $$

since we can relabel contracted indices, the above says $x = -x$ and thus $x = F^{\mu}{}_{\nu}F^{\nu}{}_{\lambda}F^{\lambda}{}_{\mu} = 0$, in this particular example. There seems to be no easy argument that one could apply to *any* higher order combination of F and ${}^*F$ with all indices contracted.

Another proof that the only invariants formed by $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are $B^2 - E^2$ and $\underset{\sim}{E} \cdot \underset{\sim}{B}$ goes as follows. Consider the vector

$$\underset{\sim}{F} = \underset{\sim}{E} + i\underset{\sim}{B} = (E_x + iB_x, E_y + iB_y, E_z + iB_z), \tag{2.8}$$

where i is the imaginary unit. Now consider a Lorentz boost in the (t, x), plane, i.e. let $\underset{\sim}{v} = (v_x, 0, 0)$. Applying the usual Lorentz transformations for $\underset{\sim}{E}$ and $\underset{\sim}{B}$, namely,

$$\begin{aligned} \underset{\sim}{E}'_{\parallel} &= \underset{\sim}{E}_{\parallel} \\ \underset{\sim}{E}'_{\perp} &= \gamma(\underset{\sim}{E}_{\perp} + \underset{\sim}{v} \times \underset{\sim}{B}_{\perp}) \\ \underset{\sim}{B}'_{\parallel} &= \underset{\sim}{B}_{\parallel} \\ \underset{\sim}{B}'_{\perp} &= \gamma(\underset{\sim}{B}_{\perp} - \underset{\sim}{v} \times \underset{\sim}{E}_{\perp}), \end{aligned} \tag{2.9}$$

where $\underset{\sim}{E}_{\parallel} = (E_x, 0, 0)$, $\underset{\sim}{E}_{\perp} = (0, E_y, E_z)$ and similarly for $\underset{\sim}{B}$, we find

$$\begin{aligned} F'_x &= E_x + iB_x = F_x \\ F'_y &= \gamma(E_y - v_x B_z) + i\gamma(B_y + v_x E_z) \\ &= \gamma(E_y + iB_y) + \gamma v_x(iE_z - B_z) = \gamma F_y + \gamma v_x i F_z \\ F'_z &= \gamma(E_z + v_x B_y) + i\gamma(B_z - v_x E_y) \\ &= \gamma(E_z + iB_z) + \gamma v_x(-iE_y + B_y) = \gamma F_z - \gamma v_x i F_y. \end{aligned} \tag{2.10}$$

We can write the above as

$$\begin{aligned} F'_x &= F_x \\ F'_y &= \cosh\psi\, F_y + i \sinh\psi\, F_z \\ F'_z &= -i\sinh\psi\, F_y + \cosh\psi\, F_z, \end{aligned} \tag{2.11}$$

where $\tanh\psi = v_x$ and so

$$\begin{aligned} \cosh\psi &= \sqrt{\frac{1}{1 - \tanh^2\psi}} = \sqrt{\frac{1}{1 - v_x^2}} = \gamma \\ \sinh\psi &= \sqrt{\cosh^2\psi - 1} = \sqrt{\gamma^2 - 1} = \sqrt{\frac{1}{1 - v_x^2} - 1} = \sqrt{\frac{1 - 1 + v_x^2}{1 - v_x^2}} \\ &= \sqrt{\frac{v_x^2}{1 - v_x^2}} = \frac{v_x}{\sqrt{1 - v_x^2}} = v_x\gamma. \end{aligned} \tag{2.12}$$

Recall that $\cosh\psi = \cos i\psi$ and $\sinh\psi = -i\sin i\psi$, so we get

$$\begin{aligned} F'_x &= F_x \\ F'_y &= \cos i\psi\, F_y + \sin i\psi\, F_z \\ F'_z &= -\sin i\psi\, F_y + \cos i\psi\, F_z, \end{aligned} \tag{2.13}$$

and so $\underset{\sim}{F}'$ is obtained by rotating $\underset{\sim}{F}$ by the imaginary angle $-i\psi$ in 3-space in the (y,z) plane. Similarly, the Lorentz boosts in the (t,y) and (t,z) planes correspond to rotations by imaginary angles in the (x,z) and (x,y) planes in 3-space. The only other Lorentz transformations in 4-space are regular rotations in 3-space, which obviously correspond to rotations by real angles in 3-space.

Thus we have a correspondence between all Lorentz transformations in 4-space (rotations in the (t,x), (t,y), (t,z), (x,y), (x,z), and (y,z) planes) to rotations in 3-space by imaginary or real angles. Note that $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are independent in $\underset{\sim}{F}$ and thus by considering all Lorentz transformations of $\underset{\sim}{F}$ in 4-space, we are actually considering all possible Lorentz transformations of $\underset{\sim}{E}$ and $\underset{\sim}{B}$. All these transformations correspond to rotations of $\underset{\sim}{F}$ in 3-space (by imaginary or complex angles). But rotations only preserve the length of a vector, thus $\|\underset{\sim}{F}\|$ is the *only* quantity conserved by rotations and thus it is the *only* quantity conserved by Lorentz transformations of $\underset{\sim}{E}$ and $\underset{\sim}{B}$. Therefore, the only conserved quantity is

$$\|\underset{\sim}{F}\| = E^2 - B^2 + 2i\underset{\sim}{E} \cdot \underset{\sim}{B}, \tag{2.14}$$

hence the real numbers $E^2 - B^2$ and $\underset{\sim}{E} \cdot \underset{\sim}{B}$ are the *only* quantities that are independently conserved under Lorentz transformations of $\underset{\sim}{E}$ and $\underset{\sim}{B}$. □

Also see `http://arxiv.org/abs/1309.4185` for a more complicated proof.

Problem 3

Part (a)

We are given that $A_{\alpha\beta}u^\alpha u^\beta = 0$ for a timelike vector u^α. Now let $\vec{u} = \vec{e}_t + \epsilon\vec{e}_i, 0 < \epsilon < 1$ for $i = 1, 2, 3$. Expanding $A_{\alpha\beta}u^\alpha u^\beta = 0$, we get

$$\begin{aligned} A_{00} + \epsilon A_{0i} + \epsilon A_{i0} + \epsilon^2 A_{ii} = 0 \\ \Rightarrow A_{00} = 0, A_{ii} = 0, A_{i0} = -A_{0i} \end{aligned} \tag{3.1}$$

Now let $\vec{u} = \vec{e}_t + \epsilon\vec{e}_i + \delta\vec{e}_j, 0 < \epsilon, \delta < \frac{1}{2}$ for $(i,j) = 1, 2, 3$ and $i \neq j$. Then we get

$$\begin{aligned} A_{00} &+ \epsilon A_{0i} + \epsilon A_{i0} + \epsilon^2 A_{ii} + \\ &+ \delta A_{0j} + \delta A_{j0} + \epsilon^2 A_{jj} + \epsilon\delta A_{ij} + \delta\epsilon A_{ji} = 0 \\ \Rightarrow 0 &+ \epsilon A_{0i} - \epsilon A_{0i} + \epsilon^2 0 + \delta A_{0j} - \delta A_{0j} + \epsilon^2 0 + \epsilon\delta A_{ij} + \delta\epsilon A_{ji} = 0 \\ \Rightarrow A_{ij} &= - A_{ji} \end{aligned} \tag{3.2}$$

thus we have $A_{\alpha\beta} = -A_{\beta\alpha} \forall \alpha, \beta = 0, 1, 2, 3$.

Part (b)

Lets start with the 4-momentum relationship $\vec{p}\cdot\vec{p} = -m^2$ where m is the rest mass of an arbitrary particle. Now take the derivative of both sides,

$$\frac{d}{d\tau}(-m^2) = 0 = \frac{d}{d\tau}(\vec{p}\cdot\vec{p}) = 2\vec{p}\cdot\frac{d}{d\tau}\vec{p} \tag{3.3}$$

However, we know that the Lorentz force is given by the equation,

$$\left(\frac{d}{d\tau}\vec{p}\right)^{\alpha} = qF^{\alpha}{}_{\beta}\,u^{\beta} \tag{3.4}$$

where q is the charge and F is the Faraday tensor. Substituting that back in, we find that

$$0 = p_\alpha\left(\frac{d}{d\tau}p\right)^{\alpha} = q\vec{p}_\alpha F^{\alpha}{}_{\beta}\,u^{\beta} = mqF^{\alpha}{}_{\beta}\,u_\alpha u^{\beta}$$
$$\Rightarrow F_{\alpha\beta}u^{\alpha}u^{\beta} = 0 \tag{3.5}$$

Therefore, using the result from Part (a), we can say that F is antisymmetric.

Problem 4

Part (a)

See (2.1) for the Faraday tensor. From it, we can compute

$$F^{\mu}{}_{\nu} = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{bmatrix} \tag{4.1}$$

Now the contraction of F with itself gives us

$$F^{\mu\nu}F_{\mu\nu} = 2(\underset{\sim}{B}^2 - \underset{\sim}{E}^2) \tag{4.2}$$

where $\underset{\sim}{B}^2 = (B^1)^2 + (B^2)^2 + (B^3)^2$. So then, using equation (7) from the problem set, the energy density is

$$\begin{aligned} T^{00} =& \frac{1}{4\pi}\left(F^{0\mu}F^{0}{}_{\mu} - \frac{2}{4}g^{00}(\underset{\sim}{B}^2 - \underset{\sim}{E}^2)\right) \\ =& \frac{1}{4\pi}\left(\underset{\sim}{E}^2 + \frac{1}{2}(\underset{\sim}{B}^2 - \underset{\sim}{E}^2)\right) \\ =& \frac{1}{8\pi}(\underset{\sim}{B}^2 + \underset{\sim}{E}^2) \end{aligned} \tag{4.3}$$

which should be a familiar result. To get the momentum density in general, let us first look at a particular case, the density in the $x-$direction

$$T^{0x} = \frac{1}{4\pi}\left(F^{0\mu}F^{x}{}_{\mu} - \frac{1}{2}g^{0x}(\underset{\sim}{B}^2 - \underset{\sim}{E}^2)\right) = \quad \frac{1}{4\pi}(E^yB^z - E^zB^y + 0) \tag{4.4}$$

The momentum density in the $i-$direction is given by

$$\begin{aligned} T^{0i} =& \frac{1}{4\pi}\left(F^{0\mu}F^{i}{}_{\mu}\right) \\ =& \frac{1}{4\pi}\left(\epsilon_{ijk}E^jB^k\right) \end{aligned} \tag{4.5}$$

Which can be written in 3-vector form as the Poynting vector $\vec{S} = \frac{1}{4\pi}\vec{E}\times\vec{B}$. Finally, for the 3D stress tensor, if $i = j$, then $g^{ii} = 1$ and otherwise $g^{ij} = 0$ so we can rewrite the 3D spatial metric as $g^{ij} = \delta^{ij}$. To get the rest of the tensor, consider the case where $i = x$ and $j = y$. Then

$$F^{x\mu}F^{y}{}_{\mu} = -E^xE^y - B^xB^y$$

and in the case where $i = j = x$ then

$$F^{x\mu}F^{y}{}_{\mu} = -(E^x)^2 + B^2 - (B^x)^2$$

so in general

$$\begin{aligned} F^{i\mu}F^{j}{}_{\mu} &= -E^iE^j - B^iB^j + \delta^{ij}B^2 \\ \Rightarrow T^{ij} &= \frac{1}{4\pi}\left(-E^iE^j - B^iB^j + \frac{1}{2}\delta^{ij}(\underset{\sim}{B}^2 + \underset{\sim}{E}^2)\right) \end{aligned} \tag{4.6}$$

Part (b)

Given the general Faraday, we know that part of Maxwell's equations read

$$\partial_\beta F^{\alpha\beta} = F^{\alpha\beta}{}_{,\beta} = 4\pi J^\alpha \tag{4.7}$$

Now let's take the derivative of the Maxwell stress-energy tensor, (noting that $F^{\mu\nu}\partial_\beta F_{\mu\nu} = F_{\mu\nu}\partial_\beta F^{\mu\nu}$),

$$\begin{aligned} \partial_\beta T^{\alpha\beta} =& \frac{1}{4\pi}\left(F^{\beta}{}_{\mu}\partial_\beta F^{\alpha\mu} + F^{\alpha\mu}\partial_\beta F^{\beta}{}_{\mu} - \frac{1}{4}g^{\alpha\beta}(2F^{\mu\nu}\partial_\beta F_{\mu\nu})\right) \\ =& \frac{1}{4\pi}\left(-4\pi F^{\alpha\mu}J_\mu + \left(F^{\beta}{}_{\mu}\partial_\beta F^{\alpha\mu} - \frac{1}{2}F^{\mu\beta}\partial^\alpha F_{\mu\beta}\right)\right) \end{aligned} \tag{4.8}$$

Clearly we will have the desired result $\partial_\beta T^{\alpha\beta} = -F^{\alpha\beta}J_\beta$ if the other terms vanish. We can show that it does by using the antisymmetry of $F^{\mu\nu}$ and the Maxwell equation

$$\partial_\beta F_{\alpha\mu} + \partial_\alpha F_{\mu\beta} + \partial_\mu F_{\beta\alpha} = 0. \tag{4.9}$$

Using this equation and maniuplating upper and lower indices we can show that

$$\begin{aligned} 0^\alpha =& F^{\beta\mu}\left(\partial_\beta F^\alpha{}_\mu + \partial^\alpha F_{\mu\beta} + \partial_\mu F_\beta{}^\alpha\right) \\ =& F^\beta{}_\mu \partial_\beta F^{\alpha\mu} - F^{\beta\mu}\partial^\alpha F_{\mu\beta} + F^{\mu\beta}\partial_\beta F_\mu{}^\alpha \\ =& 2\left(F^\beta{}_\mu \partial_\beta F^{\alpha\mu} - \frac{1}{2}F^{\beta\mu}\partial^\alpha F_{\mu\beta}\right) \end{aligned} \tag{4.10}$$

Thus the extra terms vanish so we get the result $\partial_\beta T^{\alpha\beta} = -F^{\alpha\beta}J_\beta$.

Part (c)

If the total stress energy tensor is conserved, then $\partial_\beta\left(T^{\alpha\beta}_{EM} + T^{\alpha\beta}_{matter}\right) = 0$ and $\partial_\beta T^{\alpha\beta}_{EM} = -F^{\alpha\beta}J_\beta$ give $\partial_\beta T^{\alpha\beta}_{matter} = F^{\alpha\beta}J_\beta$. Examine the time and spatial components seperately with $\vec{J} = (\rho, \underset{\sim}{j})$.

First, note that by definition of $T^{\alpha\beta}_{matter}$, T^{00}_{matter} is the energy density and T^{0i}_{matter} is the energy flux in the i-th direction. Then we can say

$$\alpha = 0 : \partial_t T^{00}_{matter} + \partial_i T^{0i}_{matter} = \frac{d\text{Energy}}{dtdV} = F^{0i}J_i = \underset{\sim}{j}\cdot\underset{\sim}{E} \tag{4.11}$$

which is simply Ohmic heating.

For the spatial components, T^{i0}_{matter} is the density of the i-th component of momentum and T^{ij}_{matter} is the flux of the i-th component of momentum in the j-th direction. Then

$$\alpha = i : \partial_t T^{i0}_{matter} + \partial_i T^{ij}_{matter} = \frac{dp^i}{dtdV} = F^{i0}J_0 + F^{ij}J_j = \rho E^i + \epsilon^{ijk}J_j B_k \tag{4.12}$$

which is just the Lorentz force equation.

Problem 5

Part (a)

From $T^{\mu\nu}{}_{,\nu} = 0$ we obtain $0 = T^{\mu\nu}{}_{,\nu} = T^{\mu 0}{}_{,0} + T^{\mu i}{}_{,i}$ and so $T^{\mu 0}{}_{,0} = -T^{\mu i}{}_{,i}$. Recall that $\partial_0 = \partial/\partial t$ and so we find

$$\frac{\partial}{\partial t}\int T^{0\alpha}\, d^3x = \int T^{\alpha 0}{}_{,0}\, d^3x = -\int T^{\alpha i}{}_{,i}\, d^3x, \tag{5.1}$$

where we have assumed that $T^{0\alpha}$ is a smooth function of x^μ so that we can interchange differentiation and integration, and we have used that T is symmetric. Applying the divergence theorem to the last integral yields

$$\frac{\partial}{\partial t}\int T^{0\alpha}\,d^3x = -\int_S T^{\alpha i} n_i\,dS = 0, \tag{5.2}$$

where n_i is the normal vector to the surface S. Since $T^{\mu\nu} = 0$ outside some bounded region of space, take the surface S to be a surface that is outside this region and encloses it. Thus on the surface S, $T^{\mu\nu} = 0$ and so the integral over S is zero. Note that the contributions to the volume integral from points outside S is zero, because $T^{\mu\nu}$ vanishes there. Thus our result does indeed apply to the volume integral over all space. □

Part (b)

We find

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\int T^{00}x^i x^j\,d^3x &= \frac{\partial}{\partial t}\int T^{00}{}_{,0}\,x^i x^j\,d^3x = -\frac{\partial}{\partial t}\int T^{0k}{}_{,k}\,x^i x^j\,d^3x \\ &= -\frac{\partial}{\partial t}\left(\int_S T^{0k}x^i x^j n_k\,d^3x - \int T^{0k}(\delta^i{}_k x^j + x^i \delta^j{}_k)\right) \\ &= 0 + \frac{\partial}{\partial t}\int (T^{0i}x^j + T^{0j}x^i)\,d^3x,\end{aligned} \tag{5.3}$$

where we have used integration by parts and the surface term vanishes by the same argument used in part (a). Finally we get

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\int T^{00}x^i x^j\,d^3x &= \int (T^{0i}{}_{,0}\,x^j + T^{0j}{}_{,0}\,x^i)\,d^3x = -\int (T^{ki}{}_{,k}\,x^j + T^{kj}{}_{,k}\,x^i)\,d^3x \\ &= -0 + \int (T^{ki}\delta^j{}_k + T^{kj}\delta^i{}_k)\,d^3x = 2\int T^{ij}\,d^3x,\end{aligned} \tag{5.4}$$

where we have again used integration by parts and the fact that the surface terms are 0. □

Part (c)

We obtain

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\int T^{00}(x^i x_i)^2\,d^3x &= \frac{\partial}{\partial t}\int T^{00}{}_{,0}\,(x^i x_i)^2\,d^3x = -\frac{\partial}{\partial t}\int T^{0k}{}_{,k}\,(x^i x_i)^2\,d^3x \\
&= -0 + \frac{\partial}{\partial t}\int T^{0k} 2(x^j x_j)(\delta^i{}_k x_i + x^i\delta_{ik})\,d^3x \\
&= 4\int T^{k0}{}_{,0}\,(x^j x_j)x_k\,d^3x = -4\int T^{kl}{}_{,l}\,(x^j x_j)x_k\,d^3x \\
&= -0 + 4\int T^{kl}(\delta^j{}_l x_j x_k + x^j\delta_{jl}x_k + x^j x_j\delta_{kl})\,d^3x \\
&= 4\int (T^{kj}x_j x_k + T^k{}_j\,x^j x_k + T^k{}_k\,x^j x_j)\,d^3x \\
&= 8\int T^{ij}x_i x_j\,d^3x + 4\int T^k{}_k\,x^i x_i\,d^3x, \qquad (5.5)
\end{aligned}$$

where we have used integration by parts twice. □