Solutions Ph 236a – Week 2

Kevin Barkett, Jonas Lippuner, and Mark Scheel

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Problem 1

Part (a)

For a particle in its rest frame, we know that its 4-velocity is given by $\vec{u} = (1, \underset{\sim}{0})$ and we are given in the problem statement that its spin vector is $\vec{S} = (0, \underset{\sim}{s})$. Thus

$$\begin{aligned} \vec{S} \cdot \vec{u} &= 0, \\ \frac{d}{d\tau}(\vec{S} \cdot \vec{u}) &= 0 \end{aligned} \tag{1.1}$$

Part (b)

First, let us examine the derivative spatial components of $\vec{S}$ while in the particles rest frame so that

$$\frac{d\underset{\sim}{s}}{d\tau} = A_1 F^i{}_j S^j \tag{1.2}$$

Where i, j are spatial indices that run over $1, 2, 3$. To see what happens, let us pick a spin component and expand (for example, the z-direction)

$$\frac{ds^z}{d\tau} = A_1 F^z{}_j S^j = A_1(s^x B_y - s^y B_x) \tag{1.3}$$

But this is just the form we expect of the usual 3-vector cross product. So using that result with equation (1) from the problem set, we get

$$\begin{aligned} \frac{d\underset{\sim}{s}}{d\tau} &= A_1 \underset{\sim}{s} \times \underset{\sim}{B}, \\ \Rightarrow A_1 &= \frac{ge}{2m} \end{aligned} \tag{1.4}$$

To get A_2, we start with

$$\frac{d}{d\tau}(\vec{S} \cdot \vec{u}) = 0 \Rightarrow \vec{S} \cdot \frac{d\vec{u}}{d\tau} = -\vec{u} \cdot \frac{d\vec{S}}{d\tau} \tag{1.5}$$

Note, that $\frac{d\vec{u}}{d\tau}$ is an acceleration, which is force divided by mass. Since the electromagnetic force is given by $qF^\alpha{}_\beta u^\beta$, we can write

$$\vec{S} \cdot \frac{d\vec{u}}{d\tau} = \frac{q}{m} S_\alpha F^\alpha{}_\beta u^\beta = -A_1 S_\alpha F^\alpha{}_\beta u^\beta + A_2 \tag{1.6}$$

where we used the fact that $A_2 \vec{u} \cdot \vec{u} = -A_2$. Now we can plug in the result of A_1 from above and solve for A_2 and we get

$$\begin{aligned} A_1 &= \frac{ge}{2m}, \\ A_2 &= F^\alpha{}_\beta S_\alpha u^\beta \frac{e}{m}(\frac{g}{2} - 1) \end{aligned} \tag{1.7}$$

Part (c)

Let $\underset{\sim}{E} = 0$ and $\vec{s} = (s^0, \underset{\sim}{S})$. Thus the particle's 4-velocity is given by $\vec{u} = \gamma(1, \underset{\sim}{v})$. Then

$$\vec{S} \cdot \vec{u} = 0 = \gamma(-s^0 + \underset{\sim}{s} \cdot \underset{\sim}{v}) \Rightarrow \underset{\sim}{s} \cdot \underset{\sim}{v} = s^0 \tag{1.8}$$

Now note that since $\underset{\sim}{E} = 0 \Rightarrow F^{\alpha}{}_{\beta} = 0$ if either α or $\beta = 0$. So then

$$\frac{ds^0}{d\tau} = A_1 F^0{}_{\beta} S^{\beta} + A_2 u^0 = \gamma A_2 \tag{1.9}$$

and from that, in the lab frame with time coordinate t, the time rate of change of the helicity is,

$$\begin{aligned}
\frac{d}{dt}(\underset{\sim}{s} \cdot \underset{\sim}{v}) = \frac{ds^0}{dt} = \frac{ds}{d\tau}\frac{d\tau}{dt} &= A_2 \\
&= F^{\alpha}{}_{\beta} S_{\alpha} u^{\beta} \frac{e}{m}(\frac{g}{2} - 1) \\
&= \gamma F^{i}{}_{j} S_i v^j \frac{e}{m}(\frac{g}{2} - 1) \\
&= \gamma v^j (\underset{\sim}{s} \times \underset{\sim}{B})_j \frac{e}{m}(\frac{g}{2} - 1)
\end{aligned} \tag{1.10}$$

In the case where $g = 2$, the quantity $(\frac{g}{2} - 1) = 0$ which implies the time rate of change of the helicity is 0, so the helicity is constant.

Problem 2

Part (a)

Start by evaluating

$$\vec{v}_{\perp} = \mathbf{P}(\vec{v}) = P_{\alpha\beta} v^{\beta} = g_{\alpha\beta} v^{\beta} + u_{\alpha} u_{\beta} v^{\beta} = v_{\alpha} + u_{\alpha} u_{\beta} v^{\beta} \tag{2.1}$$

To show that it is orthogonal to $\vec{u}$,

$$\vec{v}_{\perp} \cdot \vec{u} = (v_{\alpha} + u_{\alpha} u_{\beta} v^{\beta}) u^{\alpha} = v_{\alpha} v^{\alpha} + u_{\alpha} u^{\alpha} u_{\beta} v^{\beta} = v_{\alpha} u^{\alpha}(1 + u_{\alpha} u^{\alpha}) = 0 \tag{2.2}$$

since $\vec{u}$ is a 4-velocity. Thus $\vec{v}_{\perp}$ is orthogonal to $\vec{u}$.

Part (b)

We want to show that $\vec{v}_\perp = \mathbf{P}(\vec{v_\perp})$ and we know from above that $\vec{v_\perp} = v_\alpha + u_\alpha u_\beta v^\beta$ so then

$$\begin{aligned}\mathbf{P}(\vec{v_\perp}) &= (g_{\alpha\beta} + u_\alpha u_\beta)(v^\beta + u^\beta u_\gamma v^\gamma)\\ &= g_{\alpha\beta}v^\beta + g_{\alpha\beta}u^\beta u_\gamma v^\gamma + u_\alpha u_\beta v^\beta + u_\alpha u_\beta u^\beta u_\gamma v^\gamma\\ &= v_\alpha + u_\alpha u_\gamma v^\gamma + u_\alpha u_\beta v^\beta - u_\alpha u_\gamma v^\gamma\\ &= v_\alpha + u_\alpha u_\beta v^\beta\\ &= \vec{v_\perp} \end{aligned} \tag{2.3}$$

Thus we have shown that $\mathbf{P}$ is unique.

Part (c)

Since we are given a non-null vector $\vec{q}$ we can define $q^2 = \vec{q}\cdot\vec{q}$ and we don't need to be afraid of dividing by q^2. Now define our projection tensor for $\vec{q}$ as

$$\mathbf{P}_{\vec{q}} = g_{\alpha\beta} - \frac{1}{q^2}q_\alpha q_\beta \tag{2.4}$$

It can be shown using the same arguments as in parts (a) and (b) that $\mathbf{P}_{\vec{q}}$ is indeed the projection tensor and that it is unique.

Part (d)

For a null vector $\vec{k}$, construct its projection tensor as $\mathbf{P}_{\vec{k}} = ck_\alpha k_\beta$ for an arbitrary real number c. Thus, for an arbitrary vector $\vec{v}$, we have

$$\vec{k}\cdot\mathbf{P}_{\vec{k}}(\vec{v}) = k^\alpha c k_\alpha k_\beta v^\beta = ck_\beta v^\beta(0) = 0 \tag{2.5}$$

Thus $\mathbf{P}_{\vec{k}}$ is indeed the projection operator, however it is not unique because c is arbitrary.

Problem 3

Part (a)

Using the usual spherical coordinates, the Cartesian coordinates are given by

$$\begin{aligned}x &= r\sin\theta\cos\phi,\\ y &= r\sin\theta\sin\phi,\\ z &= r\cos\theta.\end{aligned} \tag{3.1}$$

Since $\vec{e}_r$ is the tangent vector of the curves parametrized by r with constant θ and ϕ, we find

$$\vec{e}_r = \frac{\partial}{\partial r} = \frac{\partial x^i}{\partial r}\frac{\partial}{\partial x^i} = \sin\theta\cos\phi\,\vec{e}_x + \sin\theta\sin\phi\,\vec{e}_y + \cos\theta\,\vec{e}_z, \tag{3.2}$$

where $i = 1, 2, 3$ and we used $x^1 = x$, $x^2 = y$, and $x^3 = z$. Similarly, we find

$$\begin{aligned}
\vec{e}_\theta &= \frac{\partial}{\partial \theta} = \frac{\partial x^i}{\partial \theta}\frac{\partial}{\partial x^i} = r\cos\theta\cos\phi\,\vec{e}_x + r\cos\theta\sin\phi\,\vec{e}_y - r\sin\theta\,\vec{e}_z, \\
\vec{e}_\phi &= \frac{\partial}{\partial \phi} = \frac{\partial x^i}{\partial \phi}\frac{\partial}{\partial x^i} = -r\sin\theta\sin\phi\,\vec{e}_x + r\sin\theta\cos\phi\,\vec{e}_y.
\end{aligned} \tag{3.3}$$

Part (b)

Recall that the spherical coordinates are given by

$$\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2}, \\
\theta &= \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right), \\
\phi &= \begin{cases} \operatorname{arccot}(x/y) & \text{if } y \geq 0 \\ \operatorname{arccot}(x/y) + \pi & \text{if } y < 0, \end{cases}
\end{aligned} \tag{3.4}$$

where arccos returns a value in $[0, \pi]$ and arccot returns a value in $[0, \pi]$ (when extending its domain to include $\pm\infty$). We find that

$$\begin{aligned}
\tilde{d}r = \partial_i r\,\tilde{d}x^i = \frac{\partial r}{\partial x^i}\tilde{d}x^i &= \frac{1}{2\sqrt{x^2 + y^2 + z^2}}\left(2x\tilde{d}x + 2y\tilde{d}y + 2z\tilde{d}z\right) \\
&= \sin\theta\cos\phi\,\tilde{d}x + \sin\theta\sin\phi\,\tilde{d}y + \cos\theta\,\tilde{d}z,
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
\tilde{d}\theta = \partial_i\theta\tilde{d}x^i &= -\frac{1}{\sqrt{1-\dfrac{z^2}{x^2+y^2+z^2}}}\Bigg(-\frac{2xz\tilde{d}x}{2(x^2+y^2+z^2)^{3/2}} \\
&\quad -\frac{2yz\tilde{d}y}{2(x^2+y^2+z^2)^{3/2}}+\frac{\tilde{d}z}{\sqrt{x^2+y^2+z^2}}-\frac{2z^2\tilde{d}z}{2(x^2+y^2+z^2)^{3/2}}\Bigg) \\
&= -\sqrt{\frac{x^2+y^2+z^2}{x^2+y^2}}\frac{-xz\tilde{d}x-zy\tilde{d}y+(x^2+y^2)\tilde{d}z}{(x^2+y^2+z^2)^{3/2}} \\
&= \frac{xz}{\sqrt{x^2+y^2}(x^2+y^2+z^2)}\tilde{d}x+\frac{zy}{\sqrt{x^2+y^2}(x^2+y^2+z^2)}\tilde{d}y \\
&\quad -\frac{\sqrt{x^2+y^2}}{x^2+y^2+z^2}\tilde{d}z \\
&= \frac{r^2\sin\theta\cos\phi\cos\theta}{r^3\sin\theta}\tilde{d}x+\frac{r^2\sin\theta\sin\phi\cos\theta}{r^3\sin\theta}\tilde{d}y-\frac{r\sin\theta}{r^2}\tilde{d}z \\
&= \frac{\cos\theta\cos\phi}{r}\tilde{d}x+\frac{\cos\theta\sin\phi}{r}\tilde{d}y-\frac{\sin\theta}{r}\tilde{d}z,
\end{aligned} \tag{3.6}$$

and finally

$$\begin{aligned}
\tilde{d}\phi = \partial_i\phi\tilde{x}^i &= -\frac{1}{1+x^2/y^2}\left(\frac{\tilde{d}x}{y}-\frac{x\tilde{d}y}{y^2}\right)=\frac{1}{x^2+y^2}\left(-y\tilde{d}x+x\tilde{d}y\right) \\
&= \frac{1}{r^2\sin^2\theta}\left(-r\sin\theta\sin\phi\,\tilde{d}x+r\sin\theta\cos\phi\,\tilde{d}y\right) \\
&= -\frac{\sin\phi}{r\sin\theta}\tilde{d}x+\frac{\cos\phi}{r\sin\theta}\tilde{d}y.
\end{aligned} \tag{3.7}$$

Part (c)

We have found in part (a) that

$$\begin{aligned}
\vec{e}_r &= \sin\theta\cos\phi\,\vec{e}_x+\sin\theta\sin\phi\,\vec{e}_y+\cos\theta\,\vec{e}_z, \\
\vec{e}_\theta &= r\cos\theta\cos\phi\,\vec{e}_x+r\cos\theta\sin\phi\,\vec{e}_y-r\sin\theta\,\vec{e}_z, \\
\vec{e}_\phi &= -r\sin\theta\sin\phi\,\vec{e}_x+r\sin\theta\cos\phi\,\vec{e}_y,
\end{aligned} \tag{3.8}$$

and in part (b) we found that

$$\begin{aligned}
\tilde{d}r &= \sin\theta\cos\phi\,\tilde{d}x+\sin\theta\sin\phi\,\tilde{d}y+\cos\theta\,\tilde{d}z, \\
\tilde{d}\theta &= \frac{\cos\theta\cos\phi}{r}\tilde{d}x+\frac{\cos\theta\sin\phi}{r}\tilde{d}y-\frac{\sin\theta}{r}\tilde{d}z, \\
\tilde{d}\phi &= -\frac{\sin\phi}{r\sin\theta}\tilde{d}x+\frac{\cos\phi}{r\sin\theta}\tilde{d}y.
\end{aligned} \tag{3.9}$$

Recall that $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ are basis vectors and thus they are linearly independent. It can be shown (but is not necessary) that $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$ are also linearly independent. And since these are 3 vectors in a 3-dimensional space, it follows that they form a basis. Similarly, it can be shown that $(\tilde{d}r, \tilde{d}\theta, \tilde{d}\phi)$ are linearly independent and since the space of 1-forms is also 3-dimensional, it follows that they form a basis too.

Thus it remains to be shown that $(\tilde{d}r, \tilde{d}\theta, \tilde{d}\phi)$ are dual to $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$. Since $(\tilde{d}x, \tilde{d}y, \tilde{d}z)$ are dual to $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$, we have that

$$\langle \tilde{d}x^i, \vec{e}_j \rangle = \delta^i{}_j, \tag{3.10}$$

where $\tilde{d}x^1 = \tilde{d}x$, $\tilde{d}x^2 = \tilde{d}y$, $\tilde{d}x^3 = \tilde{d}z$, $\vec{e}_1 = \vec{e}_x$, $\vec{e}_2 = \vec{e}_y$, and $\vec{e}_3 = \vec{e}_z$. We find

$$\begin{aligned}
\langle \tilde{d}r, \vec{e}_r \rangle &= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1, \\
\langle \tilde{d}r, \vec{e}_\theta \rangle &= r\sin\theta\cos\theta\cos^2\phi + r\sin\theta\cos\theta\sin^2\phi - r\sin\theta\cos\theta = 0, \\
\langle \tilde{d}r, \vec{e}_\phi \rangle &= -r\sin^2\theta\sin\phi\cos\phi + r\sin^2\theta\sin\phi\cos\phi = 0,
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
\langle \tilde{d}\theta, \vec{e}_r \rangle &= \frac{\sin\theta\cos\theta\cos^2\phi}{r} + \frac{\sin\theta\cos\theta\sin^2\phi}{r} - \frac{\sin\theta\cos\theta}{r} = 0, \\
\langle \tilde{d}\theta, \vec{e}_\theta \rangle &= \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta = 1, \\
\langle \tilde{d}\theta, \vec{e}_\phi \rangle &= -\sin\theta\cos\theta\sin\phi\cos\phi + \sin\theta\cos\theta\sin\phi\cos\phi = 0,
\end{aligned} \tag{3.12}$$

and finally

$$\begin{aligned}
\langle \tilde{d}\phi, \vec{e}_r \rangle &= -\frac{\sin\phi\cos\phi}{r} + \frac{\sin\phi\cos\phi}{r} = 0, \\
\langle \tilde{d}\phi, \vec{e}_\theta \rangle &= -\frac{\cos\theta\sin\phi\cos\phi}{\sin\theta} + \frac{\cos\theta\sin\phi\cos\phi}{\sin\theta} = 0, \\
\langle \tilde{d}\phi, \vec{e}_\phi \rangle &= \sin^2\phi + \cos^2\phi = 1.
\end{aligned} \tag{3.13}$$

Thus we have shown that

$$\langle \tilde{d}x^{\bar{i}}, \vec{e}_{\bar{j}} \rangle = \delta^{\bar{i}}{}_{\bar{j}}, \tag{3.14}$$

where $\tilde{d}x^{\bar{1}} = \tilde{d}r$, $\tilde{d}x^{\bar{2}} = \tilde{d}\theta$, $\tilde{d}x^{\bar{3}} = \tilde{d}\phi$, $\vec{e}_{\bar{1}} = \vec{e}_r$, $\vec{e}_{\bar{2}} = \vec{e}_\theta$, and $\vec{e}_{\bar{3}} = \vec{e}_\phi$, and so we have shown that $(\tilde{d}r, \tilde{d}\theta, \tilde{d}\phi)$ are dual to $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$. □

Part (d)

Recall that the components of the metric tensor are given by

$$g_{\bar{i}\bar{j}} = \vec{e}_{\bar{i}} \cdot \vec{e}_{\bar{j}}. \tag{3.15}$$

Using (3.8) we find

$$\begin{aligned}
g_{\bar{1}\bar{1}} &= \vec{e}_r\cdot\vec{e}_r = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1,\\
g_{\bar{1}\bar{2}} = g_{\bar{2}\bar{1}} &= \vec{e}_r\cdot\vec{e}_\theta = r\sin\theta\cos\theta\cos^2\phi + r\sin\theta\cos\theta\sin^2\phi\\
&\qquad - r\sin\theta\cos\theta\\
&= 0,\\
g_{\bar{1}\bar{3}} = g_{\bar{3}\bar{1}} &= \vec{e}_r\cdot\vec{e}_\phi = -r\sin^2\theta\sin\phi\cos\phi + r\sin^2\theta\sin\phi\cos\phi\\
&= 0,\\
g_{\bar{2}\bar{2}} &= \vec{e}_\theta\cdot\vec{e}_\theta = r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta\\
&= r^2,\\
g_{\bar{2}\bar{3}} = g_{\bar{3}\bar{2}} &= -r^2\sin\theta\cos\theta\sin\phi\cos\phi + r^2\sin\theta\cos\theta\sin\phi\cos\phi = 0,\\
g_{\bar{3}\bar{3}} &= r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi = r^2\sin^2\theta. \qquad (3.16)
\end{aligned}$$

Hence the components of the metric tensor in spherical coordinates are

$$g_{\bar{i}\bar{j}} = \begin{bmatrix} 1 & 0 & 0\\ 0 & r^2 & 0\\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}. \qquad (3.17)$$

The full metric tensor in all its glory is

$$\mathbf{g} = g_{\bar{i}\bar{j}}\,\tilde{d}x^{\bar{i}}\tilde{\otimes}dx^{\bar{j}} = \tilde{d}r\otimes\tilde{d}r + r^2\,\tilde{d}\theta\otimes\tilde{d}\theta + r^2\sin^2\theta\,\tilde{d}\phi\otimes\tilde{d}\phi. \qquad (3.18)$$

Part (e)

Since $g_{\hat{i}\hat{j}} = \vec{e}_{\hat{i}}\cdot\vec{e}_{\hat{j}}$, it is obvious that $g_{\hat{i}\hat{j}} = \delta_{\hat{i}\hat{j}}$ if the vector $\vec{e}_{\hat{i}}$ are orthonormal. We have already found in (3.16) that $\vec{e}_{\bar{i}}$ are orthogonal, so we can make them orthonormal by scaling them with the inverse of their lengths. Thus we obtain

$$\begin{aligned}
\vec{e}_{\hat{r}} &= \frac{\vec{e}_r}{\sqrt{\vec{e}_r\cdot\vec{e}_r}} = \frac{\vec{e}_r}{1} = \sin\theta\cos\phi\,\vec{e}_x + \sin\theta\sin\phi\,\vec{e}_y + \cos\theta\,\vec{e}_z,\\
\vec{e}_{\hat{\theta}} &= \frac{\vec{e}_\theta}{\sqrt{\vec{e}_\theta\cdot\vec{e}_\theta}} = \frac{\vec{e}_\theta}{r} = \cos\theta\cos\phi\,\vec{e}_x + \cos\theta\sin\phi\,\vec{e}_y - \sin\theta\,\vec{e}_z,\\
\vec{e}_{\hat{\phi}} &= \frac{\vec{e}_\phi}{\sqrt{\vec{e}_\phi\cdot\vec{e}_\phi}} = \frac{\vec{e}_\phi}{r\sin\theta} = -\sin\phi\,\vec{e}_x + \cos\phi\,\vec{e}_y. \qquad (3.19)
\end{aligned}$$

Problem 4

Part (a)

If $A_{\mu\nu}$ is antisymmetric and $S_{\mu\nu}$ is symmetric, then

$$S^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}S_{\alpha\beta} = g^{\nu\beta}g^{\mu\alpha}S_{\beta\alpha} = S^{\nu\mu} \qquad (4.1)$$

so we know that $S^{\mu\nu}$ is also symmetric.

So finally

$$A_{\mu\nu}S^{\mu\nu} = -A_{\nu\mu}S^{\mu\nu} = -A_{\nu\mu}S^{\nu\mu} = -A_{\mu\nu}S^{\mu\nu}. \tag{4.2}$$

Here the first step used antisymmetry of $A_{\mu\nu}$, the second step used symmetry of $S^{\mu\nu}$, and the third step relabeled dummy indices. Since $A_{\mu\nu}S^{\mu\nu}$ is equal to minus itself, it must be zero.

Part (b)

Let $A_{\mu\nu}$ and $S_{\mu\nu}$ be as above and let $V_{\mu\nu}$ be an arbitrary tensor. For equation (4) in the problem set, we can see that

$$\begin{aligned} V^{\mu\nu}A_{\mu\nu} &= \frac{1}{2}(V^{\mu\nu}A_{\mu\nu} + V^{\mu\nu}A_{\mu\nu}) \\ &= \frac{1}{2}(V^{\mu\nu}A_{\mu\nu} - V^{\mu\nu}A_{\nu\mu}) \\ &= \frac{1}{2}(V^{\mu\nu}A_{\mu\nu} - V^{\nu\mu}A_{\mu\nu}) \\ &= \frac{1}{2}(V^{\mu\nu} - V^{\nu\mu})A_{\mu\nu} \end{aligned} \tag{4.3}$$

Similarly to get equation (5) in the problem set

$$\begin{aligned} V^{\mu\nu}S_{\mu\nu} &= \frac{1}{2}(V^{\mu\nu}S_{\mu\nu} + V^{\mu\nu}S_{\mu\nu}) \\ &= \frac{1}{2}(V^{\mu\nu}S_{\mu\nu} - V^{\mu\nu}S_{\nu\mu}) \\ &= \frac{1}{2}(V^{\mu\nu}S_{\mu\nu} + V^{\nu\mu}S_{\mu\nu}) \\ &= \frac{1}{2}(V^{\mu\nu} + V^{\nu\mu})S_{\mu\nu} \end{aligned} \tag{4.4}$$

Part (c)

To show that these transformation matricies are inverses of each other, simply operate with both of them in succession.

$$\vec{e}_{\bar{\mu}} = \Lambda^{\mu}{}_{\bar{\mu}}\vec{e}_{\mu} = \Lambda^{\mu}{}_{\bar{\mu}}(\Lambda^{\bar{\nu}}{}_{\mu}\vec{e}_{\bar{\nu}}) \tag{4.5}$$

Since we are transforming back into the barred frame, that means

$$\begin{aligned} \vec{e}_{\bar{\mu}} &= \delta^{\bar{\nu}}{}_{\bar{\mu}}\vec{e}_{\bar{\nu}} \\ \Rightarrow \Lambda^{\mu}{}_{\bar{\mu}}\Lambda^{\bar{\nu}}{}_{\mu} &= \delta^{\bar{\nu}}{}_{\bar{\mu}} \end{aligned} \tag{4.6}$$

The reverse can be shown be doing the same thing except with 1-forms instead. Now we know by definition that tensors can be written as

$$T^{\alpha}{}_{\beta}{}^{\gamma} = \mathbf{T}(\tilde{w}^{\alpha}, \vec{e}_{\beta}, \tilde{w}^{\gamma}) \tag{4.7}$$

We also know that it a linear operator and so plugging in the transformation matrices we have

$$\begin{aligned} T^{\bar{\alpha}}{}_{\bar{\beta}}{}^{\bar{\gamma}} &= \mathbf{T}(\tilde{w}^{\bar{\alpha}}, \vec{e}_{\bar{\beta}}, \tilde{w}^{\bar{\gamma}}) \\ &= \mathbf{T}(\Lambda^{\bar{\alpha}}{}_{\alpha}\tilde{w}^{\alpha}, \Lambda^{\beta}{}_{\bar{\beta}}\vec{e}_{\beta}, \Lambda^{\bar{\gamma}}{}_{\gamma}\tilde{w}^{\gamma}) \\ &= \Lambda^{\bar{\alpha}}{}_{\alpha}\Lambda^{\beta}{}_{\bar{\beta}}\Lambda^{\bar{\gamma}}{}_{\gamma}\mathbf{T}(\tilde{w}^{\alpha}, \vec{e}_{\beta}, \tilde{w}^{\gamma}) \\ &= \Lambda^{\bar{\alpha}}{}_{\alpha}\Lambda^{\beta}{}_{\bar{\beta}}\Lambda^{\bar{\gamma}}{}_{\gamma}T^{\alpha}{}_{\beta}{}^{\gamma} \end{aligned} \tag{4.8}$$

Part (d)

Since $g_{\mu\nu}$ is a tensor, its indices can be raised and lowered as any other tensor.

$$g_{\alpha\beta} = g_{\alpha\mu}g_{\beta\nu}g^{\mu\nu} \tag{4.9}$$

$$g^{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}g_{\mu\nu} \tag{4.10}$$

To show that $g^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta}$, just raise or lower one of the indices and use equation (8) from the homework set.

$$g^{\alpha}{}_{\beta} = g^{\alpha\gamma}g_{\gamma\beta} = \delta^{\alpha}{}_{\beta} \tag{4.11}$$

Problem 5

Part (a)

Case 1: First consider the case where α, β, and γ are not unique. In that case $\epsilon^{\alpha\beta\gamma\rho} = 0$ for all ρ, which means the right-hand side will be 0. If α, β, and γ are not unique, it makes no sense to talk about even or odd permutations of them and so this case falls under "otherwise". The same argument holds if μ, ν, and λ are not unique.

Case 2: Thus we are left with the cases where α, β, and γ are unique and also μ, ν, and λ are unique. However, since each index can be 0, 1, 2, or 3, the two sets of indices do not necessarily have to be the same (e.g. we could have $\alpha, \beta, \gamma = 0, 1, 2$ and $\mu, \nu, \lambda = 1, 2, 3$). Consider a case like this (i.e. the set $\{\alpha, \beta, \gamma\}$ is different from the set $\{\mu, \nu, \lambda\}$). Since the Levi-Civita tensor is only non-zero if all four indices are distinct, there is only one value of ρ, say ρ_1, that makes $\epsilon^{\alpha\beta\gamma\rho}$ non-zero. Similarly, there is only one value of ρ, say ρ_2, that makes

$\epsilon_{\mu\nu\lambda\rho}$ non-zero. Since $\{\mu, \nu, \lambda\}$ is different from $\{\alpha, \beta, \gamma\}$, it follows that $\rho_1 \neq \rho_2$ and thus all four terms in the sum over ρ are 0 and so the total sum is 0. Since the two sets of indices are not the same, one is not a permutation of the other and so this case also falls under "otherwise".

Thus we are left with the cases where the two sets of indices are the same. In that case (α, β, γ) is always either an even or odd permutation of (μ, ν, λ). Recall that the Levi-Civita tensor does not change under an even permutation of its indices.

Case 3: Consider the case where (α, β, γ) is an even permutation of (μ, ν, λ). Since we can perform any even permutations we like without changing the Levi-Civita tensor, we can consider the specific case $(\alpha, \beta, \gamma) = (\mu, \nu, \lambda)$ without loss of generality. There is only one value of ρ, say ρ_0 that is different from α, β, and γ. This is the only value of ρ for which $\epsilon^{\alpha\beta\gamma\rho}$ is non-zero. Thus the sum over ρ reduces to one non-zero term, namely

$$\delta^{\alpha\beta\gamma}{}_{\mu\nu\lambda} = -\epsilon^{\alpha\beta\gamma\rho}\epsilon_{\mu\nu\lambda\rho} = -\epsilon^{\alpha\beta\gamma\rho_0}\epsilon_{\alpha\beta\gamma\rho_0} = -(\pm 1)(\mp 1) = +1, \tag{5.1}$$

since $\epsilon_{\alpha\beta\gamma\delta} = -\epsilon^{\alpha\beta\gamma\delta}$.

Case 4: The only remaining case now is where (α, β, γ) is an odd permutation of (μ, ν, λ). In that case we can take $(\alpha, \beta, \gamma) = (\nu, \mu, \lambda)$ without loss of generality, because even permutations do not change the Levi-Civita tensors. Again, we only have one value of ρ, say ρ_0 for which $\epsilon^{\alpha\beta\gamma\rho}$ is non-zero. The sum over ρ again reduces to one term, namely

$$\begin{aligned}\delta^{\alpha\beta\gamma}{}_{\mu\nu\lambda} &= -\epsilon^{\alpha\beta\gamma\rho}\epsilon_{\mu\nu\lambda\rho} = -\epsilon^{\alpha\beta\gamma\rho_0}\epsilon_{\beta\alpha\gamma\rho_0} = +\epsilon^{\alpha\beta\gamma\rho_0}\epsilon_{\alpha\beta\gamma\rho_0} \\ &= +(\pm 1)(\mp 1) = -1.\end{aligned} \tag{5.2}$$

Thus we have shown that

$$\delta^{\alpha\beta\gamma}{}_{\mu\nu\lambda} = \begin{cases} +1 & \text{if } (\alpha, \beta, \gamma) \text{ is an even permutation of } (\mu, \nu, \lambda) \\ -1 & \text{if } (\alpha, \beta, \gamma) \text{ is an odd permutation of } (\mu, \nu, \lambda) \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

Note that

$$\delta^{\alpha\beta}{}_{\mu\nu} = \frac{1}{2}\delta^{\alpha\beta\gamma}{}_{\mu\nu\gamma} \tag{5.4}$$

From what we found above, it is obvious that $\delta^{\alpha\beta}{}_{\mu\nu} = 0$ if $\alpha = \beta$, $\mu = \nu$, or $\{\alpha, \beta\} \neq \{\mu, \nu\}$, which fall under "otherwise".

If (α, β) is an even permutation of (μ, ν), then we must have that $(\alpha, \beta) = (\mu, \nu)$, and there are only two values of γ, say γ_1 and γ_2, for which $\delta^{\alpha\beta\gamma}$ is non-zero. Thus the sum over γ reduces to two terms, namely

$$\delta^{\alpha\beta}{}_{\mu\nu} = \frac{1}{2}\delta^{\alpha\beta\gamma}{}_{\mu\nu\gamma} = \frac{1}{2}\left(\delta^{\alpha\beta\gamma_1}{}_{\alpha\beta\gamma_1} + \delta^{\alpha\beta\gamma_2}{}_{\alpha\beta\gamma_2}\right) = \frac{1}{2}(1+1) = +1. \tag{5.5}$$

Finally, if (α,β) is an odd permutation of (μ,ν), then we must have that $(\alpha,\beta) = (\nu,\mu)$, and there are again only two values of γ, say γ_1 and γ_2, for which $\delta^{\alpha\beta\gamma}$ is non-zero. Thus the sum over γ again reduces to two terms, namely

$$\begin{aligned}\delta^{\alpha\beta}{}_{\mu\nu} &= \frac{1}{2}\delta^{\alpha\beta\gamma}{}_{\mu\nu\gamma} = \frac{1}{2}\left(\delta^{\alpha\beta\gamma_1}{}_{\beta\alpha\gamma_1} + \delta^{\alpha\beta\gamma_2}{}_{\beta\alpha\gamma_2}\right) = \frac{1}{2}(-1-1)\\ &= -1.\end{aligned} \tag{5.6}$$

Thus we have shown that

$$\delta^{\alpha\beta}{}_{\mu\nu} = \begin{cases} +1 & \text{if } (\alpha,\beta) \text{ is an even permutation of } (\mu,\nu)\\ -1 & \text{if } (\alpha,\beta) \text{ is an odd permutation of } (\mu,\nu)\\ 0 & \text{otherwise.}\end{cases} \tag{5.7}$$

Part (b)

Since J is a 3-index antisymmetric tensor, ${}^*J$ is a 1-index tensor. We find

$$\begin{aligned}{}^{**}J_{\alpha\beta\gamma} &= {}^*J^{\mu}\epsilon_{\mu\alpha\beta\gamma} = {}^*J_{\mu}\epsilon^{\mu}{}_{\alpha\beta\gamma} = \frac{1}{3!}J^{\nu\lambda\rho}\epsilon_{\nu\lambda\rho\mu}\epsilon^{\mu}{}_{\alpha\beta\gamma} = \frac{1}{3!}J_{\nu\lambda\rho}\epsilon^{\nu\lambda\rho\mu}\epsilon_{\mu\alpha\beta\gamma}\\ &= -\frac{1}{3!}J_{\nu\lambda\rho}\epsilon^{\nu\lambda\rho\mu}\epsilon_{\alpha\beta\gamma\mu} = -\frac{1}{3!}J_{\nu\lambda\rho}(-\delta^{\nu\lambda\rho}{}_{\alpha\beta\gamma})\\ &= \frac{1}{6}J_{\nu\lambda\rho}\delta^{\nu\lambda\rho}{}_{\alpha\beta\gamma}.\end{aligned} \tag{5.8}$$

Now recall that J is an antisymmetric tensor. Thus $J_{\alpha\beta\gamma} = 0$ if the 3 indices are not distinct. Similarly, $\delta^{\nu\lambda\rho}{}_{\alpha\beta\gamma} = 0$ if α, β, and γ are not distinct. Similarly, the terms in the sum above are 0 if ν, λ, and ρ are not distinct. Also $\delta^{\nu\lambda\rho}{}_{\alpha\beta\gamma} = 0$ if $\{\nu,\lambda,\rho\} \neq \{\alpha,\beta,\gamma\}$. Thus there are only 6 terms in the sum for which $\delta^{\nu\lambda\rho}{}_{\alpha\beta\gamma} \neq 0$, namely the terms where (ν,λ,ρ) is a permutation of (α,β,γ). We find

$$\begin{aligned}{}^{**}J_{\alpha\beta\gamma} &= \frac{1}{6}J_{\nu\lambda\rho}\delta^{\nu\lambda\rho}{}_{\alpha\beta\gamma} = \frac{1}{6}\left(J_{\alpha\beta\gamma} - J_{\alpha\gamma\beta} + J_{\gamma\alpha\beta} - J_{\gamma\beta\alpha} + J_{\beta\gamma\alpha} - J_{\beta\alpha\gamma}\right)\\ &= \frac{1}{6}\left(J_{\alpha\beta\gamma} + J_{\alpha\beta\gamma} + J_{\alpha\beta\gamma} + J_{\alpha\beta\gamma} + J_{\alpha\beta\gamma} + J_{\alpha\beta\gamma}\right)\\ &= \frac{1}{6}\left(6J_{\alpha\beta\gamma}\right) = J_{\alpha\beta\gamma},\end{aligned} \tag{5.9}$$

where we have used that J is antisymmetric, hence $J_{\alpha\gamma\beta} - J_{\alpha\beta\gamma}$, for example. □

For the 2-index antisymmetric tensor F we find

$$\begin{aligned}{}^{**}F_{\alpha\beta} &= \frac{1}{2!}{}^*F^{\mu\nu}\epsilon_{\mu\nu\alpha\beta} = \frac{1}{2!}{}^*F_{\mu\nu}\epsilon^{\mu\nu}{}_{\alpha\beta} = \frac{1}{2!}\frac{1}{2!}F^{\lambda\rho}\epsilon_{\lambda\rho\mu\nu}\epsilon^{\mu\nu}{}_{\alpha\beta}\\ &= \frac{1}{4}F_{\lambda\rho}\epsilon^{\lambda\rho\mu\nu}\epsilon_{\mu\nu\alpha\beta} = \frac{1}{4}F_{\lambda\rho}\epsilon^{\lambda\rho\mu\nu}\epsilon_{\alpha\beta\mu\nu} = \frac{1}{4}F_{\lambda\rho}\left(-2\delta^{\lambda\rho}{}_{\alpha\beta}\right)\\ &= -\frac{1}{2}F_{\lambda\rho}\delta^{\lambda\rho}{}_{\alpha\beta}.\end{aligned} \tag{5.10}$$

From part (a) we know that ${\delta^{\lambda\rho}}_{\alpha\beta} = 0$ unless $(\lambda, \rho) = (\alpha, \beta)$ or $(\lambda, \rho) = (\beta, \alpha)$. Thus we find

$$\begin{aligned} {}^{**}F_{\alpha\beta} &= -\frac{1}{2} F_{\lambda\rho} {\delta^{\lambda\rho}}_{\alpha\beta} = -\frac{1}{2} \left(F_{\alpha\beta} - F_{\beta\alpha} \right) = -\frac{1}{2} \left(F_{\alpha\beta} + F_{\alpha\beta} \right) \\ &= -F_{\alpha\beta}, \end{aligned} \tag{5.11}$$

where we have used that F is antisymmetric, hence $F_{\alpha\beta} = -F_{\beta\alpha}$. □

Finally, for the 1-index antisymmetric tensor B we find

$$\begin{aligned} {}^{**}B_\alpha &= \frac{1}{3!} {}^*B^{\mu\nu\lambda} \epsilon_{\mu\nu\lambda\alpha} = \frac{1}{3!} {}^*B_{\mu\nu\lambda} {\epsilon^{\mu\nu\lambda}}_\alpha = \frac{1}{3!} B^\rho \epsilon_{\rho\mu\nu\lambda} {\epsilon^{\mu\nu\lambda}}_\alpha \\ &= \frac{1}{3!} B_\rho \epsilon^{\rho\mu\nu\lambda} \epsilon_{\mu\nu\lambda\alpha} = -\frac{1}{3!} B_\rho \epsilon^{\rho\mu\nu\lambda} \epsilon_{\alpha\mu\nu\lambda} = -\frac{1}{6} B_\rho \left(-2 {\delta^{\rho\mu}}_{\alpha\mu} \right) \\ &= \frac{1}{3} B_\rho {\delta^{\rho\mu}}_{\alpha\mu}. \end{aligned} \tag{5.12}$$

Note that ${\delta^{\rho\mu}}_{\alpha\mu} = 0$ if $\rho \neq \alpha$ because in that case the two sets of indices are always different. Thus ${\delta^{\rho\mu}}_{\alpha\mu}$ is only non-zero if $\rho = \alpha$, in that case we find

$${\delta^{\rho\mu}}_{\alpha\mu} = {\delta^{\rho 0}}_{\rho 0} + {\delta^{\rho 1}}_{\rho 1} + {\delta^{\rho 2}}_{\rho 2} + {\delta^{\rho 3}}_{\rho 3} = 3, \tag{5.13}$$

because ρ will be 0, 1, 2, or 3. Thus we found that

$${\delta^{\rho\mu}}_{\alpha\mu} = 3 {\delta^\rho}_\alpha, \tag{5.14}$$

where ${\delta^\rho}_\alpha$ is the usual Kronecker delta. Thus (5.12) becomes

$${}^{**}B_\alpha = \frac{1}{3} B_\rho {\delta^{\rho\mu}}_{\alpha\mu} = \frac{1}{3} B_\rho 3 {\delta^\rho}_\alpha = B_\alpha, \tag{5.15}$$

which is what we need to show. □