Solutions Ph 236a – Week 1

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Problem 1

Part (a)

Consider the interval between two events along a ray of light. Let $\Delta s^2 = 0$ in the Σ frame. Then we have

$$\begin{aligned} 0 = \eta_{\alpha\beta}\Delta x^\alpha \Delta x^\beta &= -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \\ \Rightarrow \quad &(\Delta x^2 + \Delta y^2 + \Delta z^2)/\Delta t^2 = 1 \end{aligned} \tag{1.1}$$

because the speed of light is 1. In a different frame, $\bar{\Sigma}$, since the speed of light is also 1 in this frame we similarly find

$$\begin{aligned} (\Delta\bar{x}^2 + \Delta\bar{y}^2 + \Delta\bar{z}^2)/\Delta\bar{t}^2 = 1 \\ \Rightarrow \quad \Delta\bar{s}^2 = -\Delta\bar{t}^2 + \Delta\bar{x}^2 + \Delta\bar{y}^2 + \Delta\bar{z}^2 = 0 \end{aligned} \tag{1.2}$$

Thus, if $\Delta s^2 = 0$ in one frame, then it is 0 in all frames. □

Part (b)

Assuming linear transformations, $\Delta x^{\bar{\mu}} = L^{\bar{\mu}}_{\nu}\Delta x^\nu$, we get,

$$Q = \eta_{\bar{\alpha}\bar{\beta}}\Delta x^{\bar{\alpha}}\Delta x^{\bar{\beta}} = (\eta_{\bar{\alpha}\bar{\beta}}L^{\bar{\alpha}}_{\mu}L^{\bar{\beta}}_{\nu})\Delta x^\mu \Delta x^\nu \tag{1.3}$$

So thus Q is a quadratic form in the x^α coordinate system. Now if Δx^α is on the light cone, then we have

$$-\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = \Delta s^2 = 0 = \Delta\bar{s}^2 = Q \tag{1.4}$$

□

Part (c)

In the x^α coordinates, the most general form for Q is

$$Q = A_{ij}\Delta x^i \Delta x^j + B_i \Delta x^i \Delta t + c_1 \Delta t^2 \tag{1.5}$$

where i, j run from $1, 2, 3$ and A_{ij}, B_i, and c_1 are constants to be determined. Consider a hypersurface with $t = t_0$ constant so then $\Delta t = 0$. The intersection between the light cone and this surface is a 2-sphere centered at the spatial origin. In this case, $Q = A_{ij}\Delta x^i \Delta x^j$. Because this is a 2-sphere centered on the origin, symmetry demands that $A_{ij} = c_2\delta_{ij}$, for some constant c_2. Thus Q can be written as

$$Q = c_2(\Delta x^2 + \Delta y^2 + \Delta z^2) + c_1\Delta t^2 + B_i \Delta x^i \Delta t. \tag{1.6}$$

Consider now the interval between two events a and b where $\Delta t! = 0$ between a and b. The interval between b and a should be the same as the interval between a and b (reordering the events shouldn't matter). Because reordering the events involves switching the sign on Δt, this shows that $B_i = 0$. Therefore

$$Q = c_1 \Delta t^2 + c_2(\Delta x^2 + \Delta y^2 + \Delta z^2). \tag{1.7}$$

$\square$

Part (d)

Consider the surface with $\Delta y^2 = \Delta z^2 = 0$ intersecting the light cone.

$$Q = c_1 \Delta t^2 + c_2 \Delta x^2 = 0 \Rightarrow c_1 \Delta t^2 = -c_1 \Delta x^2 \tag{1.8}$$

On the light cone, $|\Delta x| = |\Delta t|$ and so we get $c_1 = -c_2$ and then

$$Q = c_2 \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \tag{1.9}$$

$\square$

Part (e)

From the parts above, we have

$$Q = \eta_{\bar{\alpha}\bar{\beta}} \Delta x^{\bar{\alpha}} \Delta x^{\bar{\beta}} = c_2 \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \tag{1.10}$$

If we pick the transformation where we time reverse the coordinates and apply it twice we should get the original Q.

$$Q = \eta_{\bar{\alpha}\bar{\beta}} \Delta x^{\bar{\alpha}} \Delta x^{\bar{\beta}} = (c_2)^2 \eta_{\bar{\alpha}\bar{\beta}} \Delta x^{\bar{\alpha}} \Delta x^{\bar{\beta}} \Rightarrow |c_2| = 1 \tag{1.11}$$

For infinitesimal transformations, we do not expect the quantity to flip signs so this means $c_2 = 1$. Thus, we have shown that

$$\begin{aligned} \eta_{\bar{\alpha}\bar{\beta}} \Delta x^{\bar{\alpha}} \Delta x^{\bar{\beta}} &= \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \\ \Rightarrow \quad \Delta s^2 &= \Delta \bar{s}^2 \end{aligned} \tag{1.12}$$

Problem 2

Part (a)

Let the two frames Σ and $\bar{\Sigma}$ be aligned at the origin when both clocks are at 0. After a time t, an observer at rest in frame Σ will see an object at rest in frame $\bar{\Sigma}$ with a position x. However, the object will see itself as stationary so then $\bar{x} = 0$. Thus our equation is then

$$\begin{aligned} \bar{x} = 0 &= \alpha_{10} t + \alpha_{11} x \\ \Rightarrow \quad x = -\frac{\alpha_{10}}{\alpha_{11}} t \quad &\Rightarrow \quad v = -\frac{\alpha_{10}}{\alpha_{11}} \end{aligned} \tag{2.1}$$

□

Part (b)

For spacetime diagrams of two frames with relative velocity v between them, we know that the angle between the $\bar{t}$ and t axes is the same as the angle between the $\bar{x}$ and x axes. Thus, the slopes of the two barred axes with respect to the unbarred axes are inverses of each other. In part(a), we considered an object on the $\bar{x} = 0$ line which is the $\bar{t}$ axis and found that it had slope $v = -\frac{\alpha_{10}}{\alpha_{11}}$. This implies that the $\bar{x}$ axis, which is the $\bar{t} = 0$ line, has slope $1/v$.

$$\begin{aligned} \bar{t} &= 0 = \alpha_{00} t + \alpha_{01} x \\ \Rightarrow \quad x &= -\frac{\alpha_{00}}{\alpha_{01}} t = \frac{1}{v} t \end{aligned} \tag{2.2}$$

Thus we have $v = -\frac{\alpha_{01}}{\alpha_{00}}$. From this and the result in part (a), we have the equations

$$\begin{aligned} \bar{t} &= \alpha_{00}(t - vx) \\ \bar{x} &= \alpha_{11}(x - vt) \end{aligned} \tag{2.3}$$

□

Part (c)

From the invariance of δs^2 we have

$$\delta s^2 = -t^2 + x^2 = -\bar{t}^2 + \bar{x}^2 \tag{2.4}$$

Substituting in the results from (2.3) we get

$$\begin{aligned} \delta s^2 &= -\alpha_{00}^2 (t - vx)^2 + \alpha_{11}^2 (x - vt)^2 \\ &= -\alpha_{00}^2 (t^2 - 2vx + v^2 x^2) + \alpha_{11}^2 (x^2 - 2vx + v^2 t^2) \\ &= t^2 (v^2 \alpha_{11}^2 - \alpha_{00}^2) + x^2 (\alpha_{11}^2 - \alpha_{00}^2 v^2) + 2vxt(-\alpha_{11}^2 + \alpha_{00}^2) \end{aligned} \tag{2.5}$$

Now we can equate terms of appropriate powers of x and t. Starting with the cross term, we get

$$0 = 2vxt(-\alpha_{11}^2 + \alpha_{00}^2) \Rightarrow \alpha_{11}^2 = \alpha_{00}^2 \tag{2.6}$$

so now we have,

$$t^2 = t^2\alpha_{00}^2(v^2-1),\ x^2 = x^2\alpha_{00}^2(1-v^2)$$
$$\Rightarrow \alpha_{00}^2 = \frac{1}{1-v^2} = \alpha_{11} \tag{2.7}$$

If we were to choose the negative root, that would correspond to flipping the direction of the axis between the frames so we pick the positive root. Thus $\alpha_{00} = \alpha_{11} = \frac{1}{\sqrt{1-v^2}}$. □

Problem 3

Part (a)

A vector $\vec{v}$ is spacelike if and only if $\eta_{\mu\nu}v^\mu v^\nu > 0$. We find

$$\begin{aligned}\eta_{\mu\nu}(A^\mu + B^\mu)(A^\nu + B^\nu) &= \eta_{\mu\nu}A^\mu A^\nu + \eta_{\mu\nu}A^\mu B^\nu + \eta_{\mu\nu}B^\mu A^\nu + \eta_{\mu\nu}B^\mu B^\nu \\ &= \eta_{\mu\nu}A^\mu A^\nu + 2\eta_{\mu\nu}A^\mu B^\nu + \eta_{\mu\nu}B^\mu B^\nu, \end{aligned} \tag{3.1}$$

because the metric $\eta_{\mu\nu}$ is symmetric (i.e. $\eta_{\mu\nu} = \eta_{\nu\mu}$). Since $\vec{A}$ and $\vec{B}$ are spacelike, we have $\eta_{\mu\nu}A^\mu A^\nu > 0$ and $\eta_{\mu\nu}B^\mu B^\nu > 0$. And we also have $\eta_{\mu\nu}A^\mu B^\nu = 0$, because $\vec{A}$ and $\vec{B}$ are orthogonal. Thus we find that

$$\eta_{\mu\nu}A^\mu A^\nu + 2\eta_{\mu\nu}A^\mu B^\nu + \eta_{\mu\nu}B^\mu B^\nu > 0, \tag{3.2}$$

and so by (3.1) we have shown that $\vec{A} + \vec{B}$ is spacelike. □

Part (b)

Recall that for a vector $\vec{v}$, we have

$$\eta_{\mu\nu}v^\mu v^\nu = -(v^0)^2 + (v^1)^2 + (v^2)^2 + (v^3)^2 = -(v^0)^2 + \|\underset{\sim}{v}\|^2, \tag{3.3}$$

where $\|\underset{\sim}{v}\|$ is the usual Euclidean norm of the 3-vector $\underset{\sim}{v}$. Let $\vec{N}$ be a null vector, thus

$$\begin{aligned} & 0 = \eta_{\mu\nu}N^\mu N^\nu = -(N^0)^2 + \|\underset{\sim}{N}\|^2 \\ \Leftrightarrow\ & (N^0)^2 = \|\underset{\sim}{N}\|^2 \\ \Leftrightarrow\ & |N^0| = \|\underset{\sim}{N}\|. \end{aligned} \tag{3.4}$$

And let $\vec{T}$ be a timelike vector, hence

$$\begin{aligned} & 0 > \eta_{\mu\nu}T^\mu T^\nu = -(T^0)^2 + \|\underset{\sim}{T}\|^2 \\ \Leftrightarrow\ & (T^0)^2 > \|\underset{\sim}{T}\|^2 \\ \Leftrightarrow\ & |T^0| > \|\underset{\sim}{T}\|. \end{aligned} \tag{3.5}$$

Now suppose that $\vec{N}$ and $\vec{T}$ are orthogonal. It follows that

$$0 = \eta_{\mu\nu} N^\mu T^\nu = -N^0 T^0 + N^1 T^1 + N^2 T^2 + N^3 T^3 = -N^0 T^0 + \underset{\sim}{N} \cdot \underset{\sim}{T}. \quad (3.6)$$

From the above we get

$$|\underset{\sim}{N} \cdot \underset{\sim}{T}| = |N^0 T^0| = |N^0||T^0| > \|\underset{\sim}{N}\|\|\underset{\sim}{T}\|, \quad (3.7)$$

where we used (3.4) and (3.5). Since $\underset{\sim}{N}$ and $\underset{\sim}{T}$ are ordinary 3-vectors, we can use the Cauchy–Schwarz inequality, which states that

$$\|\underset{\sim}{N}\|\|\underset{\sim}{T}\| \geq |\underset{\sim}{N} \cdot \underset{\sim}{T}|. \quad (3.8)$$

Combining this with (3.7) yields

$$|\underset{\sim}{N} \cdot \underset{\sim}{T}| > |\underset{\sim}{N} \cdot \underset{\sim}{T}|, \quad (3.9)$$

which is a contradiction and thus our assumption that $\vec{N}$ and $\vec{T}$ are orthogonal must be wrong. □

Part (c)

Given a null vector $\vec{k}$, without loss of generality, we can choose a coordinate system such that

$$\vec{k} = \hat{t} + \hat{x}. \quad (3.10)$$

Now let $\vec{v} = A\hat{t} + B\hat{x} + C\hat{y} + D\hat{z}$ be a vector orthogonal to $\vec{k}$, so

$$0 = \eta_{\mu\nu} k^\mu v^\nu = -A + B, \quad (3.11)$$

hence $A = B$. If $\vec{v}$ is not spacelike, then we have

$$0 \geq \eta_{\mu\nu} v^\mu v^\nu = -A^2 + B^2 + C^2 + D^2 = C^2 + D^2, \quad (3.12)$$

because $A = B$. Thus it follows that $C = D = 0$. And therefore

$$\vec{v} = A(\hat{t} + \hat{x}) = A\vec{k}, \quad (3.13)$$

and so we have shown that any non-spacelike vector orthogonal to the null vector $\vec{k}$ is a multiple of $\vec{k}$. □

Problem 4

First consider frame 1. In frame 1, the 4-velocity of frame 1 is just $u_1^{\bar{\alpha}} = (1, \underset{\sim}{0})$ and the 4-velocity of frame 2 (as measured in frame 1) is $u_2^{\bar{\alpha}} = \gamma(1, \underset{\sim}{v})$, where

$\gamma = (1 - v^2)^{-1/2}$ and $v = \|\underset{\sim}{v}\|$ is the speed of frame 2 with respect to frame 1. Since $\vec{u}_1$ and $\vec{u}_2$ are 4-vectors, their dot product $\vec{u}_1 \cdot \vec{u}_2 = -\gamma$ is Lorentz invariant.

So now we consider some general frame where frame 1 and 2 have 4-velocities $u_1^\alpha = \gamma_1(1, \underset{\sim}{v}_1)$ and $u_2^\alpha = \gamma_2(1, \underset{\sim}{v}_2)$, respectively. Where we used $\gamma_i = (1 - v_i^2)^{-1/2}$ and $v_i = \|\underset{\sim}{v}_i\|$ for $i = 1, 2$. Using the invariance of $\vec{u}_1 \cdot \vec{u}_2$ we now find

$$\begin{aligned} & -\gamma = \vec{u}_1 \cdot \vec{u}_2 = \gamma_1\gamma_2(-1 + \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2) \\ \Leftrightarrow\ & \gamma^2 = \gamma_1^2\gamma_2^2(1 - \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2)^2 \\ \Leftrightarrow\ & \frac{1}{1 - v^2} = \frac{(1 - \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2)^2}{(1 - v_1^2)(1 - v_2^2)}. \end{aligned} \tag{4.1}$$

Solving for v^2 (recall that v is the magnitude of the relative velocity between the two frames) yields

$$\begin{aligned} v^2 &= 1 - \frac{(1 - v_1^2)(1 - v_2^2)}{(1 - \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2)^2} = \frac{(1 - \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2)^2 - (1 - v_1^2)(1 - v_2^2)}{(1 - \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2)^2} \\ &= \frac{1 - 2\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2 + (\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2)^2 - 1 + v_1^2 + v_2^2 - v_1^2 v_2^2}{(1 - \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2)^2} \\ &= \frac{\|\underset{\sim}{v}_1 - \underset{\sim}{v}_2\|^2 + (\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2)^2 - v_1^2 v_2^2}{(1 - \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2)^2}. \end{aligned} \tag{4.2}$$

Now recall Lagrange's identity in 3 dimension:

$$\|\underset{\sim}{a} \times \underset{\sim}{b}\|^2 = \|\underset{\sim}{a}\|^2 \|\underset{\sim}{b}\|^2 - (\underset{\sim}{a} \cdot \underset{\sim}{b})^2. \tag{4.3}$$

Using this identity, we can write (4.2) as

$$v^2 = \frac{\|\underset{\sim}{v}_1 - \underset{\sim}{v}^2\|^2 - \|\underset{\sim}{v}_1 \times \underset{\sim}{v}_2\|^2}{(1 - \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2)^2}, \tag{4.4}$$

since $v_i = \|\underset{\sim}{v}_i\|$. Thus we have what we need to show. □

Problem 5

Part (a)

Without loss of generality, take all motions to be in the x-direction. We have the following equations as constraints for the 4-velocity $\vec{u}$ and 4-acceleration $\vec{a}$,

$$\begin{aligned} \vec{u} \cdot \vec{u} &= -1 = -(u^t)^2 + (u^x)^2 \\ \vec{u} \cdot \vec{a} &= 0 = -a^t u^t + a^x u^x \\ \vec{a} \cdot \vec{a} &= g^2 = -(a^t)^2 + (a^x)^2 \end{aligned} \tag{5.1}$$

From these we can obtain the following relations,

$$a^t = \frac{u^x}{u^t} a^1 \tag{5.2}$$

$$g^2 = (a^x)^2 (1 - (\frac{u^x}{u^t})^2) \tag{5.3}$$

$$-1 = -(u^t)^2 (1 - (\frac{u^x}{u^t})^2) \tag{5.4}$$

Plugging (5.3) into (5.4) gives us $g^2 = (a^x)^2/(u^t)^2$ which can then be used in (5.2) to yield,

$$a^x = gu^t, \quad a^t = gu^x \tag{5.5}$$

Now we can differentiate with respect to the proper time, τ,

$$gu^x = a^t = \frac{du^t}{d\tau} = \frac{da^x}{d\tau}\frac{1}{g} = \frac{1}{g}\frac{d^2u^x}{d\tau^2}$$

$$\Rightarrow \quad \frac{d^2u^x}{d\tau^2} = g^2 u^x \tag{5.6}$$

The solution to this differential equation is $u^x = A\cosh g\tau + B\sinh g\tau$. The initial conditions are $u^x(t) = 0 \Rightarrow A = 0$ and $\frac{du^x}{d\tau}(0) = g \Rightarrow B = 1$. Then, using the properties of the hyperbolic functions and $\vec{u}\cdot\vec{u} = -1$, we get,

$$u^x = \sinh g\tau, \quad u^t = \cosh g\tau \tag{5.7}$$

To get the equations of motion for x and t, we can integrate these equations. With conditions that $x = t = 0$ when $\tau = 0$ (and plugging back in relevant factors of c),

$$x = c^2 g^{-1}(\cosh\frac{g\tau}{c} - 1), \quad t = cg^{-1}\sinh\frac{g\tau}{c} \tag{5.8}$$

Thus, 30 years on earth gives a proper time of $\tau = \frac{c}{g}\sinh^{-1}\frac{gt}{c} \approx 4.1$yr. Plug that into the equation for x to get a distance of $x \approx 29$ly.

To get the distance as viewed by the observer on the rocket, plug $\tau = 30$yr into the equation for x and we find $x \approx 5.3 \times 10^{12}$ly. □

Part (b)

Traveling halfway corresponds to $x = 15000$ly. Invert the equation for x to get

$$\tau = cg^{-1}\cosh^{-1}(xgc^{-2} + 1) \approx 10.309\text{yr} \tag{5.9}$$

Plugging this in for the equation for $t(\tau)$ gives us $t_{\frac{1}{2}} \approx 15001$ y for half the trip and thus $t \approx 30002$ y for the entire trip. □

Part (c)

Let the mass of the rocket ship be M. The change in mass-energy of the rocket matches the energy radiated away so $d(Mu^t) = -dE_{rad}$. Because the energy radiated away is in the form of photons, $dE_{rad} = dP_{rad}$ which is equal to the momentum change in the rocket. Thus we have,

$$\begin{aligned} d(Mu^t) &= -dP = -d(Mu^x) \\ (dM)u^t + M(du^t) &= -(dM)u^x - M(du^x) \\ \frac{dM}{M} &= -\frac{d(u^t+u^x)}{(u^t+u^x)} \\ \Rightarrow \quad \ln M/M_0 &= -\ln\left(u^t+u^x\right) \\ \Rightarrow \quad M &= \frac{M_0}{(u^t+u^x)} \end{aligned} \tag{5.10}$$

From (5.7), we get

$$M = \frac{M_0}{(u^t+u^x)} = M_0 e^{-g\tau} \tag{5.11}$$

Plugging in $\tau = 10.3$yr for half the trip gives us $M_{\frac{1}{2}} \approx \frac{M_0}{30000}$. Thus, for a full trip we find,

$$M_{final} \approx 10^{-9} M_0 \tag{5.12}$$

□

Problem 6

Part (a)

The observer is at rest in the observer's rest frame, hence $u^\alpha = (1, \underset{\sim}{0})$. Let $p^\alpha = (p^0, \underset{\sim}{p})$ in the observer's rest frame. Then we have

$$p^\alpha_{(3)} = (p^0, \underset{\sim}{p}) + (-p^0 + 0)(1, \underset{\sim}{0}) = (p^0, \underset{\sim}{p}) + (-p^0, \underset{\sim}{p}) = (0, \underset{\sim}{p}). \tag{6.1}$$

So the time component of $\vec{p}_{(3)}$ is indeed zero in the observer's rest frame and the spatial components are $\underset{\sim}{p}$, which is the 3-vector part of $\vec{p}$ in the observer's rest frame and thus this is the 3-momentum of the particle as measured by the observer. □

Part (b)

We have shown in part (a) that for a single particle with 4-momentum $\vec{p}_i$, we have $\vec{p}_i + (\vec{p}_i \cdot \vec{u}_{\text{cm}})\vec{u}_{\text{cm}} = (0, \underset{\sim}{p}_i)$, where $\underset{\sim}{p}_i$ is the 3-momentum of particle i measured in the rest frame of the observer with 4-velocity $\vec{u}_{\text{cm}}$. The total 4-momentum of the system of particles is simply the sum of the 4-momenta of the individual particles, thus we have

$$\vec{p} = \sum_i \vec{p}_i, \tag{6.2}$$

and so

$$\begin{aligned}
\vec{p} + (\vec{p} \cdot \vec{u}_{\text{cm}})\vec{u}_{\text{cm}} &= \sum_i \vec{p}_i + \left[\left(\sum_i \vec{p}_i\right) \cdot \vec{u}_{\text{cm}}\right] \vec{u}_{\text{cm}} = \sum_i \vec{p}_i + \left[\sum_i \vec{p}_i \cdot \vec{u}_{\text{cm}}\right] \vec{u}_{\text{cm}} \\
&= \sum_i \vec{p}_i + \left[\sum_i (\vec{p}_i \cdot \vec{u}_{\text{cm}})\vec{u}_{\text{cm}}\right] = \sum_i \left[\vec{p}_i + (\vec{p}_i \cdot \vec{u}_{\text{cm}})\vec{u}_{\text{cm}}\right] \\
&= \sum_i (0, \underset{\sim}{p}_i) = \left(0, \sum_i \underset{\sim}{p}_i\right) = (0, \underset{\sim}{0}),
\end{aligned} \tag{6.3}$$

since the frame with 4-velocity $\vec{u}_{\text{cm}}$ is by definition that frame where the total 3-momentum of the system of particles is zero, hence where $\sum_i \underset{\sim}{p}_i = \underset{\sim}{0}$.

Since $\vec{p}$ and $\vec{u}_{\text{cm}}$ are 4-vectors, $\vec{p} \cdot \vec{u}_{\text{cm}}$ is a Lorentz-invariant scalar (i.e. it is the same in every Lorentz frame). If we now transform to another Lorentz frame, we get

$$\begin{aligned}
p^{\bar{\mu}} &= \Lambda^{\bar{\mu}}{}_{\nu} p^{\nu} \\
u^{\bar{\mu}}_{\text{cm}} &= \Lambda^{\bar{\mu}}{}_{\nu} u^{\nu}_{\text{cm}},
\end{aligned} \tag{6.4}$$

and thus we obtain

$$p^{\bar{\mu}} + (\vec{p} \cdot \vec{u}_{\text{cm}}) u^{\bar{\mu}}_{\text{cm}} = \Lambda^{\bar{\mu}}{}_{\nu}(p^{\nu} + (\vec{p} \cdot \vec{u}_{\text{cm}}) u^{\nu}_{\text{cm}}) = \Lambda^{\bar{\mu}}{}_{\nu}(\vec{0}) = \vec{0}, \tag{6.5}$$

and so (6.3) holds in every Lorentz frame. □

Part (c)

Since E_{tot} and $\underset{\sim}{p}_{\text{tot}}$ are the energy and 3-momentum of the system of particles measured in the lab frame, we have that

$$p^{\alpha}_{\text{lab}} = (E_{\text{tot}}, \underset{\sim}{p}_{\text{tot}}) \tag{6.6}$$

is the 4-momentum of the system measured in the lab frame. Let the 4-velocity of center-of-momentum frame measured in the lab frame be

$$u^{\alpha}_{\text{cm,lab}} = \gamma(1, \underset{\sim}{v}_{\text{cm,lab}}). \tag{6.7}$$

We have shown in part (b) that $\vec{p} + (\vec{p} \cdot \vec{u}_{\rm cm})\vec{u}_{\rm cm} = 0$ in every Lorentz frame, so in particular, this also holds in the lab frame, which gives

$$0 = E_{\rm tot} + (\vec{p}_{\rm lab} \cdot \vec{u}_{\rm cm,lab})\gamma \tag{6.8}$$
$$0 = \underset{\sim}{p}_{\rm tot} + (\vec{p}_{\rm lab} \cdot \vec{u}_{\rm cm,lab})\gamma \underset{\sim}{v}_{\rm cm}. \tag{6.9}$$

From (6.8) we get

$$\vec{p}_{\rm lab} \cdot \vec{u}_{\rm cm,lab} = -\frac{E_{\rm tot}}{\gamma}, \tag{6.10}$$

and then using this in (6.9) we find

$$\underset{\sim}{v}_{\rm cm} = -\frac{\underset{\sim}{p}_{\rm tot}}{(\vec{p}_{\rm lab} \cdot \vec{u}_{\rm cm,lab})\gamma} = \frac{\underset{\sim}{p}_{\rm tot}}{E_{\rm tot}}, \tag{6.11}$$

which is what we need to show. $\square$