Physics 236a assignment 9:

(Dec 01, 2015. Due on December 10, 2015)

1. **Dragging of inertial frames** [20 points]

   A rigid infinitesimally-thin spherical shell of (uniformly-distributed) mass M and radius R rotates with angular velocity Ω with respect to observers at infinity.

   (a) Assume linearized gravity in Lorentz gauge, and assume that the rotation is about the z axis. Show that

   $$T_{0y} = -\frac{Mr\Omega}{4\pi R^2} \sin\theta \cos\phi \, \delta(r - R). \tag{1}$$

   (b) By solving the (linearized) Einstein equation, show that

   $$\bar{h}_{0y} = f(r) \sin\theta \cos\phi, \tag{2}$$

   where

   $$f(r) = \begin{cases} -\dfrac{4M\Omega r}{3R} & r < R \\ -\dfrac{4M\Omega R^2}{3r^2} & r > R \end{cases} \tag{3}$$

   (Hint: the Poisson equation simplifies since T_{0y} is proportional to the spherical harmonic $Y_{11}(\theta, \phi)$).

   (c) Define $\omega = -g_{0\phi}/g_{\phi\phi}$. Show that ω at a given point in spacetime can be interpreted as the angular velocity of an inertial observer at that point, relative to inertial observers at infinity (this part does not depend on the previous parts of the problem).

   (d) Compute $g_{0\phi}$ by transforming the results of part 1b into spherical coordinates.

(e) Show that $\omega = 4M\Omega/3R$ everywhere inside the spherical shell. Remember we assume linearized gravity, so we can neglect terms of order h^2.

Thus we see that a uniform rotating shell "drags all inertial frames" inside the shell with a uniform angular velocity ω.

2. **Clocks near a neutron star** [15 points]

   Paul is in a circular orbit around a spherical nonspinning neutron star at r=6M. Peter is fired from a cannon, radially, from the surface of the neutron star (which has radius less than 6M but greater than 2M), at less than escape velocity. Peter passes near Paul on his way out, reaches some maximum radius, and then passes near Paul on the way back down. Paul has completed 10 orbits during this time. Peter and Paul compare their clocks both times they pass. How much time has elapsed on Paul's clock? On Peter's clock?

3. **So you really want to go inside a black hole?** [10 points]

   Show that a rocket ship that crosses inside the horizon of a Schwarzschild black hole will reach $r = 0$ in a proper time $\tau < \pi M$, no matter how the engines are fired.

4. **More general orbits in Schwarzschild geometry** [10 points]

   (a) Show that
   $$L^2 = p_\theta^2 + \sin^{-2}\theta p_\phi^2 \tag{4}$$
   is a constant of the motion along any Schwarzschild geodesic.

   (b) Show that all geodesics in Schwarzschild spacetime are planar. (Hint, to simplify, rotate coordinates so that at the geodesic is initially at $\theta = \pi/2$ with $\dot{\theta} = 0$).

   (c) Show that all geodesics in Schwarzschild are *stably* planar (that is, a small nonplanar perturbation results in an orbit that results in small oscillations about a planar orbit). (Hint: use Eq. (4)).

5. **Rindler metric** [15 points]

The Rindler spacetime is described by the metric

$$ds^2 = -g^2 z^2 dt^2 + dx^2 + dy^2 + dz^2, \tag{5}$$

where g is a constant. Note the coordinate singularity at $z = 0$.

(a) Show that Rindler spacetime is flat. Hint: Find a coordinate transformation that reduces Eq. (5) to

$$ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2. \tag{6}$$

(b) In a spacetime diagram (Z, T), illustrate the relationship between the two coordinate systems (t, z) and (T, Z). A point particle is dropped at $t = 0$, at a distance $z = z_0$ above the $z = 0$ plane. Show its trajectory in your diagram. What are the equations of motion of the particle in Rindler coordinates? At what Rindler time t will the particle reach the $z = 0$ plane?

(c) Show that near the event horizon $r = 2M$ of the Schwarzschild metric, the Schwarzschild geometry can be closely approximated by the Rindler geometry. Find the corresponding value of the Rindler constant g and describe the motion of a particle as it approaches the event horizon.