Physics 236a assignment, Week 8:

(November 19, 2015. Due on December 01, 2015)

1. **More on the Euler equation** [15 points]

   (a) If $\vec{\xi}$ is a timelike Killing vector and $\vec{u} = \vec{\xi}/|\vec{\xi} \cdot \vec{\xi}|^{1/2}$ is a 4-velocity, show that

   $$\vec{a} = \nabla_{\vec{u}}\vec{u} = \frac{1}{2}\nabla \log |\vec{\xi} \cdot \vec{\xi}| \tag{1}$$

   (b) Hydrostatic equilibrium means that one can find a frame in which all the fluid variables are constant in time. Hydrostatic equilibrium implies that a timelike Killing vector field exists, and that the 4-velocity of the fluid is parallel to this Killing vector. Using the Euler equation

   $$(p + \rho)\nabla_{\vec{u}}\vec{u} = -\nabla p - \vec{u}\nabla_{\vec{u}}p \tag{2}$$

   and Eq. (1), derive the equation of hydrostatic equilibrium

   $$-p_{,\nu} = (p + \rho)\frac{\partial}{\partial x^{\nu}} \log(-g_{00})^{1/2}. \tag{3}$$

   Here the 0 direction is defined by the timelike Killing vector field $\partial/\partial x^0$.

2. **Effect of gravitational waves on matter** [15 points]

   Consider a short rod with two beads that are free to slide along the rod. Let the separation between the beads be described by a vector $\vec{\xi}$, and let $\vec{n} = \vec{\xi}/|\xi|$ be a unit vector that points along the rod. Then the proper distance between the beads ℓ is given by $\ell = \vec{\xi} \cdot \vec{n}$. Assume that the center of mass of the rod moves along a geodesic, and that the rod is attached to a gyroscope, so that $\vec{n}$ is Fermi-Walker transported.

(a) Show that
$$\frac{d^2\ell}{d\tau^2} = \vec{n} \cdot \nabla_{\vec{u}} \nabla_{\vec{u}} \vec{\xi} \tag{4}$$
where τ is the proper time measured by the rod, and $\vec{u}$ is $d/d\tau$.

(b) Write down an equation for $d^2\ell/d\tau^2$ in terms of the Riemann tensor.

(c) Assume that in the local Lorentz frame of the rod, the metric is given by a plane linearized gravitational wave propagating in the z direction
$$\bar{h}^{xx} = -\bar{h}^{yy} = h_+(t-z) \tag{5}$$
$$\bar{h}^{xy} = h_\times(t-z) \tag{6}$$
$$\bar{h}^{\alpha 0} = \bar{h}^{\alpha z} = 0, \tag{7}$$
for some functions $h_+(t-z)$ and $h_\times(t-z)$. Find all nonzero components of the Riemann tensor.

(d) Evaluate the acceleration you computed in part 2b in the local Lorentz frame of the rod, with the Riemann tensor given by the answer of part 2c. Assume that the rod is oriented in the direction $n^z = \cos\theta$, $n^x = \sin\theta\cos\phi$, $n^y = \sin\theta\sin\phi$. Solve the resulting differential equation for $\ell(\tau)$ to get
$$\ell = \ell_0 \left(1 + \frac{1}{2} h_+ \sin^2\theta \cos 2\phi + \frac{1}{2} h_\times \sin^2\theta \sin 2\phi\right), \tag{8}$$
where ℓ_0 is the equilibrium length of the rod. Assume that all metric perturbations are small.

3. **Riemann tensor in linearized theory** [10 points]

(a) Show that in linearized theory the components of the Riemann tensor are

$$R_{\alpha\mu\beta\nu} = \frac{1}{2}\left(h_{\alpha\nu,\mu\beta} + h_{\mu\beta,\nu\alpha} - h_{\mu\nu,\alpha\beta} - h_{\alpha\beta,\mu\nu}\right) \qquad (9)$$

(b) Show explicitly that this Riemann tensor is invariant under a gauge transformation.

4. **Lie Derivative and Killing vectors** [10 points]

   (a) Show that $\mathcal{L}_{\vec{u}}\mathcal{L}_{\vec{v}} - \mathcal{L}_{\vec{v}}\mathcal{L}_{\vec{u}} - \mathcal{L}_{[\vec{u},\vec{v}]} = 0$, where $\mathcal{L}$ denotes the Lie derivative.

   (b) Show that the commutator of two Killing vector fields is also a Killing vector field.

   (c) Show that the linear combination (with constant coefficients) of two Killing vectors is a Killing vector.

5. **Stationary and static solutions** [10 points] A spacetime is *stationary* if and only if it has a Killing vector field $\vec{\xi}$ that is timelike at infinity. For a stationary spacetime, it is customary to denote $\vec{\xi}$ as $\partial/\partial t$.

   There are two ways to define a *static* spacetime:

   - Stationary and invariant under time reversal $\partial/\partial t \to -\partial/\partial t$
   - Stationary and $\vec{\xi}$ is hypersurface orthogonal (recall from a previous problem set that hypersurface orthogonal means $\xi_\alpha = h f_{,\alpha}$ where h and f are scalars, or equivalently, $\xi_{[\mu;\nu}\xi_{\lambda]} = 0$).

   (a) Show that the first definition is equivalent to $g_{\alpha\beta,t} = 0$ and $g_{ti} = 0$ (Hint: what is $g_{t\alpha}$ in terms of $\vec{\xi}$?)

(b) Show that if $g_{ti} = 0$, then $\vec{\xi} = \partial/\partial t$ is hypersurface orthogonal.

(c) Show that if $\vec{\xi}$ is hypersurface orthogonal, then $g_{ti} = 0$.

Therefore both definitions of a static spacetime are equivalent.