# Physics 236a assignment, Week 7:

(November 12, 2015. Due on November 19, 2015)

1. **Einstein tensor** [10 points]

   (a) Given some time coordinate, show that the components of the Einstein tensor with at least one time index, $G^{\mu 0}$, involves only first time derivatives of the metric but not second time derivatives of the metric. (This means that the time components of Einstein's equation $G^{\mu\nu} = 8\pi T^{\mu\nu}$ are *constraint equations* like the Maxwell $\nabla \cdot E$ and $\nabla \cdot B$ equations, and constrain parts of the metric at one particular instant in time).

   (b) Show that the components of the Einstein tensor with all spatial indices $G^{ij}$ involve second time derivatives of the metric. (This means that the spatial components of Einstein's equation $G^{\mu\nu} = 8\pi T^{\mu\nu}$ are *evolution equations* like the Maxwell curl equations, and govern how the metric evolves in time).

2. **Euler equations** [10 points]

   The stress-energy tensor for a perfect fluid is

$$T_{\mu\nu} = (p + \rho) u_\mu u_\nu + p g_{\mu\nu}, \tag{1}$$

   where $\rho$ and $p$ are the energy density and pressure in the rest frame of the fluid, and $u_\mu$ is the 4-velocity of the fluid.

   Derive the Euler equation

$$(p + \rho) \nabla_{\vec{u}} \vec{u} = -\nabla p - \vec{u} \nabla_{\vec{u}} p. \tag{2}$$

3. **Raychaudhuri's Equation** [25 points]

(a) Suppose that $\vec{u}$ is a vector field describing a family of time-like world lines. Show that $\nabla \vec{u}$ can be decomposed as

$$u_{\alpha;\beta} = \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{3}\theta P_{\alpha\beta} - a_\alpha u_\beta, \tag{3}$$

where $\vec{a}$ is the 4-acceleration, $\theta$ is the *expansion* of the world lines

$$\theta = u^\alpha{}_{;\alpha}, \tag{4}$$

$\omega_{\alpha\beta}$ is the *rotation 2-form*

$$\omega_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha;\mu}P^\mu{}_\beta - u_{\beta;\mu}P^\mu{}_\alpha\right), \tag{5}$$

and $\sigma_{\alpha\beta}$ is the *shear tensor*

$$\sigma_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha;\mu}P^\mu{}_\beta + u_{\beta;\mu}P^\mu{}_\alpha\right) - \frac{1}{3}\theta P_{\alpha\beta}. \tag{6}$$

Here $P_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$ is the projection tensor that projects a vector onto the 3-surface perpendicular to $\vec{u}$.

(b) Derive the Raychaudhuri equation

$$\frac{d\theta}{d\tau} = a^\alpha{}_{;\alpha} + \omega_{\alpha\beta}\omega^{\alpha\beta} - \sigma_{\alpha\beta}\sigma^{\alpha\beta} - \frac{1}{3}\theta^2 - R_{\alpha\beta}u^\alpha u^\beta. \tag{7}$$

(c) A vector field $\vec{v}$ is called *hypersurface orthogonal* if there exists a family of 3-dimensional surfaces such that $\vec{v}$ is orthogonal to those surfaces. Show that if $\vec{v}$ is hypersurface orthogonal, then $v_{[\gamma}v_{\alpha;\beta]} = 0$. (Hint: assume the hypersurfaces are parameterized by $f$ =constant, where $f$ is some scalar function. Then $v_\alpha$ is proportional to $f_{;\alpha}$, or in other words, $v_\alpha = h f_{;\alpha}$, where $h$ is another scalar function.)

(d) Suppose that the worldlines are geodesics, and that $\vec{u}$ is hypersurface orthogonal. Suppose also that the *strong energy condition* holds:

$$T_{\mu\nu}u^\mu u^\nu \geq -\frac{1}{2}T \tag{8}$$

for all stress-energy tensors $T_{\mu\nu}$ and for all timelike $u^\mu$. Show that if $\theta$ is negative on any of the geodesics, then $\theta$ goes to $-\infty$ in a finite proper time. (Hint: use $u^\gamma u_{[\gamma}u_{\alpha;\beta]} = 0$ in Eq. (3) to derive a condition on $\omega_{\alpha\beta}$ for when $\vec{u}$ is hypersurface orthogonal.) This result is a key ingredient in theorems that predict the formation of singularities in GR.

4. **Necessity of Einstein tensor in field equations** [10 points]

Consider a modified form of the Einstein field equations

$$R_{\mu\nu} - a g_{\mu\nu} R = 8\pi T_{\mu\nu}, \tag{9}$$

where $a$ is some constant.

(a) For now, assume that $T^{\mu\nu}$ does *not* satisfy $T^{\mu\nu}{}_{;\nu} = 0$. Instead, derive an equation for $T^{\mu\nu}{}_{;\nu}$ using Eq. (9).

(b) Assume a perfect fluid with density $\rho$ and negligible pressure, and show that unless $a = 1/2$, the result of part 4a yields the incorrect Newtonian limit.

5. **Tidal forces** [10 points]

Examine the geodesic deviation equation in the Newtonian limit (small velocities, small $\Gamma^\gamma{}_{\alpha\beta}$), and compare with a purely Newtonian calculation of the acceleration of nearby test particles in a gravitational field. Show that in the Newtonian limit,

$$R_{j0k0} = \frac{\partial^2 \Phi}{\partial x^k \partial x^j}, \tag{10}$$

where $\Phi$ is the Newtonian gravitational potential.