Physics 236a assignment, Week 6:

(November 5, 2015. Due on November 12, 2015)

1. **A 2-sphere** [10 points]

   Consider a sphere with fixed radius r. This is a 2-dimensional manifold, and in spherical coordinates θ, ϕ, the metric is

   $$ds^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1}$$

   (a) Compute all the connection coefficients $\Gamma^{\gamma}{}_{\alpha\beta}$.

   (b) Write down the geodesic equation, and show that geodesics are great circles.

   (c) Compute all components of the Riemann tensor.

2. **Gravitational redshift** [10 points]

   The metric outside the sun is to a very good approximation

   $$ds^2 = -(1 - 2M/r)\, dt^2 + (1 + 2M/r)(dx^2 + dy^2 + dz^2), \tag{2}$$

   where M is the mass of the sun, and $r^2 = x^2 + y^2 + z^2$. We choose units such that $G = c = 1$. Consider a photon traveling along a geodesic near the sun. The geodesic equation for a photon is $\nabla_{\vec{p}}\vec{p} = 0$, where $\vec{p} = d/d\lambda$ is the photon 4-momentum and is also the tangent to the photon's world line. Here λ is an affine parameter.

   (a) Use the geodesic equation to write out $dp_\alpha/d\lambda$ (note the lower index) in terms of the connection coefficients (don't evaluate the connection coefficients yet).

   (b) Show that for the metric in Eq. (2), p_0 is constant along the photon's worldline.

   (c) Is p^0 also constant along the photon's worldline?

(d) An atom is at rest (meaning that its coordinates x, y, z remain constant) at the surface of the sun. If the atom has 4-velocity $\vec{u}_e$, what is the component u_e^0?

(e) A photon is emitted by the atom in part 2d, and the wavelength of the photon in the rest frame of the atom is λ_e. The same photon is received by an observer at rest (again, meaning x, y, z of the observer remain fixed) far from the sun, and is measured by this observer to have wavelength λ_r. The redshift of the photon is defined by

$$z = \frac{\lambda_r - \lambda_e}{\lambda_e}. \tag{3}$$

Show that $z = M/R + \mathcal{O}((M/R)^2)$ where R is the radius of the sun, so that $z \sim 2 \times 10^{-6}$.

3. **Commutation of 2nd derivatives** [10 points]

Given the relations for commuting 2nd covariant derivatives for vectors and one forms,

$$v^\alpha{}_{;\mu\nu} - v^\alpha{}_{;\nu\mu} = R^\alpha{}_{\lambda\nu\mu} v^\lambda, \tag{4}$$

$$\omega_{\alpha;\mu\nu} - \omega_{\alpha;\nu\mu} = -R^\lambda{}_{\alpha\nu\mu}\omega_\lambda, \tag{5}$$

show that $T^\alpha{}_{\beta;\mu\nu} - T^\alpha{}_{\beta;\nu\mu} = R^\alpha{}_{\lambda\nu\mu}T^\lambda{}_\beta - R^\lambda{}_{\beta\nu\mu}T^\alpha{}_\lambda$ for the rank 2 tensor T.

4. **Coriolis force** [15 points]

(This problem looks really long, but most of it is explanation.) Assume an observer with 4-velocity $\vec{u}$ is accelerating and also rotating. Rotation means that he does not Fermi-Walker transport his orthonormal tetrad; instead the rule for transport of his tetrad is

$$\nabla_{\vec{u}}\vec{e}_\alpha = -\vec{e}_\beta \Omega^\beta{}_\alpha, \tag{6}$$

where

$$\Omega^{\mu\nu} = a^\mu u^\nu - u^\mu a^\nu + u_\lambda \omega_\beta \epsilon^{\lambda\beta\mu\nu}. \tag{7}$$

The last term of Eq. (7) was called $\omega^{\mu\nu}$ in class, and was shown to be antisymmetric, but here we write this last term in terms of a quantity ω^β, which can be interpreted as a rotation vector. If this last term were zero, then Eqs. (6) and (7) would reduce to Fermi-Walker transport. Note that as in class, the observer chooses his timelike basis vector to be his 4-velocity, $\vec{e}_0 = \vec{u}$, and he chooses all his basis vectors to be orthonormal $\vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta}$.

(a) Some of the connection coefficients on the observer's world line can be computed from Eqs. (6) and (7). Show that

$$\Gamma^i{}_{00} = a^i \qquad \Gamma^0{}_{00} = 0 \tag{8}$$

$$\Gamma^0{}_{i0} = a_i \qquad \Gamma^i{}_{j0} = -\omega^k \epsilon_{0k}{}^i{}_j. \tag{9}$$

(b) Suppose the observer chooses coordinates *near* his world line as follows: at each point P on his world line, he sends out spatial geodesics in a pattern like a sea urchin. If one of these spatial geodesics has tangent $\vec{n}$ at P, then he labels points along that spatial geodesic using coordinates

$$x^0 = \tau \quad \text{(his proper time)} \tag{10}$$

$$x^i = sn^i \quad \text{where } s \text{ is proper distance along geodesic.} \tag{11}$$

In a small enough neighborhood around P (small so that geodesics don't cross each other), all points can be labeled in this way. Show that the geodesic equation $d^2x^\alpha/ds^2 = \ldots$ for these geodesics implies that at P, $\Gamma^0{}_{jk} = \Gamma^i{}_{jk} = 0$.

(c) Now suppose that the observer watches the motion of a *freely falling* particle that passes through the observer's origin. Suppose that the ordinary 3-velocity of the particle, as it passes the observer's origin, is

$$\underset{\sim}{v} = (dx^i/dx^0)\vec{e}_i. \tag{12}$$

Show that the ordinary acceleration of the particle, as measured by the observer, is

$$\frac{d^2x^i}{d\tau^2}\vec{e}_i = -\underset{\sim}{a} - 2\underset{\sim}{\omega} \times \underset{\sim}{v} + 2(\underset{\sim}{a} \cdot \underset{\sim}{v})\underset{\sim}{v}. \tag{13}$$

The first term is just (minus) the acceleration of the observer as expected, the second term is the Coriolis acceleration (and thus justifies the interpretation of ω_β in Eq. (7)), and the third term is a special relativistic correction.

5. **Gravitational torque on an extended body** [15 points] A nonspherical extended spinning body moving along a geodesic will experience a torque in a nonuniform external gravitational field. We will derive an expression for this torque, and the rate of change of the spin of the body.

   (a) Work in a local Lorentz frame comoving with the center of mass of the body. In this frame, compute the acceleration of a mass element at position x^i, relative to the mass element at the origin, caused by geodesic deviation.

   (b) By integrating over mass elements, compute the total torque on the body in terms of ρ, the energy density of the body. Assume that the Riemann tensor is constant over the body. Show that this torque is given by

   $$\tau_i = -\epsilon_{i\ell j} t^{\ell k} R^j{}_{0k0}, \tag{14}$$

   where

   $$t^{\ell k} = \int \rho \left(x^\ell x^k - \frac{1}{3} r^2 \delta^{\ell k} \right) d^3x \tag{15}$$

   is the reduced quadrupole moment tensor.

   (c) If S^i are components of the spin vector of the body in the local Lorentz frame, the above results allow us to write

   $$dS^i/dt = -\epsilon^i{}_{\ell j} t^{\ell k} R^j{}_{0k0}. \tag{16}$$

Define a spin 4-vector S^μ such that in the LLF, the spatial components are just S^i and the time component is zero. In other words, if $\vec{u}$ is the 4-velocity of the center of mass of the body, then $S^\mu u_\mu = 0$. Similarly define the 4-d quadrupole moment tensor $t^{\alpha\beta}$ so that in the LLF its spatial components are given by Eq. (15) and its time components are zero, so that $t^{\alpha\beta}u_\beta = 0$. Rewrite Eq. (16) in terms of 4-dimensional quantities in the LLF, and get

$$\frac{DS^\nu}{d\tau} = \epsilon^{\nu\beta\alpha\mu} u_\mu u^\sigma u^\lambda t_{\beta\eta} R^\eta{}_{\sigma\alpha\lambda}. \tag{17}$$

This is a tensor equation, so if true in the LLF, it will be true in any frame. This is the effect responsible for the "precession of the equinoxes", discovered by Hipparchus in 150 B.C., in which the Earth's axis precesses relative to distant stars with a period of about 26,000 years.