Physics 236a assignment, Week 5:

(October 29, 2015. Due on November 5, 2015)

1. **Spherical polar coordinates yet again** [20 points]

   Consider 3-dimensional Euclidean space. In Cartesian coordinates (x, y, z) the metric is $g_{ij} = \delta_{ij}$. We define the usual spherical coordinates by

   $$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta. \tag{1}$$

   (a) In an earlier assignment, you computed the metric components in the coordinate basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$. Now compute all the connection coefficients $\Gamma^i{}_{jk}$ associated with the coordinate basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$.

   (b) Compute all the connection coefficients $\Gamma^{\hat{\imath}}{}_{\hat{\jmath}\hat{k}}$ associated with the orthonormal basis

   $$\vec{e}_{\hat{r}} = \frac{\partial}{\partial r}, \qquad \vec{e}_{\hat{\theta}} = \frac{1}{r}\frac{\partial}{\partial\theta}, \qquad \vec{e}_{\hat{\phi}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}. \tag{2}$$

   (c) For an arbitrary vector $\vec{A}$, we define the divergence of $\vec{A}$ as

   $$\operatorname{div} A = \nabla_k A^k = A^k{}_{;k}. \tag{3}$$

   Show that the divergence of $\vec{A}$ in Cartesian coordinates is just $\partial_k A^k$. Compute the divergence of $\vec{A}$ in the coordinate basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$, (i.e. in terms of partial derivatives of A^i, where i is (r, θ, ϕ), along r, θ, ϕ directions.) Compute the divergence of $\vec{A}$ in the orthonormal basis $(\vec{e}_{\hat{r}}, \vec{e}_{\hat{\theta}}, \vec{e}_{\hat{\phi}})$, (i.e. in terms of partial derivatives of $A^{\hat{\imath}}$, where $\hat{\imath}$ is $(\hat{r}, \hat{\theta}, \hat{\phi})$, along r, θ, ϕ directions.)

(d) Just like we did in 4-d space, we define the 3-dimensional Levi-Civita tensor to be totally antisymmetric, and we set the sign convention so that

$$\epsilon_{123} = +\sqrt{\det g}, \tag{4}$$

where g is the 3-dimensional metric. What are the components of ϵ^{ijk} in the coordinate basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$?

(e) We define the curl of a vector as

$$(\mathrm{curl}\, A)^i = \epsilon^{ijk}\nabla_j A_k = \epsilon^{ijk} A_{k;j}. \tag{5}$$

What is the curl of $\vec{A}$ in terms of $\vec{A}$'s components in the coordinate basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$? What is the curl of $\vec{A}$ in terms of $\vec{A}$'s components in the orthornormal basis $(\vec{e}_{\hat{r}}, \vec{e}_{\hat{\theta}}, \vec{e}_{\hat{\phi}})$?

2. **Index gymnastics** [20 points] Prove the following identities:

(a) $g^{\alpha\beta}{}_{,\gamma} = -\Gamma^{\alpha}{}_{\mu\gamma} g^{\mu\beta} - \Gamma^{\beta}{}_{\mu\gamma} g^{\mu\alpha}$

(b) $\Gamma^{\alpha}{}_{\alpha\beta} = (\log((-g)^{1/2}))_{,\beta}$ in a coordinate basis. (Hint: use the identity $g_{,\alpha} = g g^{\mu\nu} g_{\mu\nu,\alpha}$ from the previous homework set).

(c) $g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu} = -(-g)^{-1/2}(g^{\alpha\beta}(-g)^{1/2})_{,\beta}$ in a coordinate basis.

(d) $A^{\alpha}{}_{;\alpha} = (-g)^{-1/2}((-g)^{1/2}A^{\alpha})_{,\alpha}$ in a coordinate basis.

(e) $\epsilon_{\alpha\beta\gamma\delta;\mu} = 0$

3. **Parallel transport of 1-forms** [10 points]

The geodesic equation $\nabla_{\vec{u}}\vec{u} = 0$ written in components takes the form

$$\frac{du^{\mu}}{d\lambda} + \Gamma^{\mu}{}_{\alpha\beta} u^{\alpha} u^{\beta} = 0. \tag{6}$$

(a) Suppose $\tilde{u}$ is a 1-form. Write out in components the equation $\nabla_{\vec{u}}\tilde{u} = 0$, i.e. the parallel transport equation for the one-form $\tilde{u}$ along the vector $\vec{u}$.

(b) If $\tilde{u}$ is the one-form corresponding to the vector $\vec{u}$, show that your answer in part 3a is consistent with Eq. (6).

4. **Properties of Geodesics** [10 points]

   (a) Show that if a geodesic is timelike at one point, it is timelike everywhere, and similarly for spacelike and null geodesics.

   (b) Consider a curve parameterized by its proper length s (not an arbitrary parameter). Let $y = g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$, where x^μ are coordinates, and the dot means d/ds. We saw in class that the geodesic equation is equivalent to the condition that

$$\delta \int y^{1/2}\, ds = 0. \tag{7}$$

   Show that

$$\delta \int F(y)\, ds = 0 \tag{8}$$

   gives the same geodesics as Eq. (7), for any monotonic function $F(y)$.

5. **Covariant derivative practice via chain rule** [10 points]

   Let $S^{\alpha\beta}{}_\gamma$ and $M_\beta{}^\gamma$ be components of tensors. Then $S^{\alpha\beta}{}_\gamma M_\beta{}^\gamma$ are the components of a vector. The chain rule for the divergence of this vector is

$$(S^{\alpha\beta}{}_\gamma M_\beta{}^\gamma)_{;\alpha} = S^{\alpha\beta}{}_{\gamma;\alpha} M_\beta{}^\gamma + S^{\alpha\beta}{}_\gamma M_\beta{}^\gamma{}_{;\alpha}. \tag{9}$$

   Verify that the chain rule holds by expanding the left and the right sides in terms of directional derivatives and connection coefficients, and showing that both sides are equal.