Physics 236a assignment, Week 4:

(October 22, 2015. Due on October 29, 2015)

1. **First Law** [10 points]

   For this problem, assume special relativity (or equivalently, assume that you are in a local Lorentz frame).

   The first law of thermodynamics for a relativistic fluid can be written
   $$d(\rho V) = -P\,dV + T\,dS, \tag{1}$$
   where ρ is the energy density (not mass density in relativity!), V is the volume, P is the pressure, and S is the entropy. (Here dS, dV, etc are not 1-forms; they are the usual differential changes in entropy and volume that should be familiar from thermodynamics.)

   (a) If n is the baryon number density (i.e. number of baryons per unit volume), and $s = S/(nV)$ is the entropy per baryon, show that the first law can be written
   $$d\rho = \frac{\rho + P}{n}\,dn + nT\,ds. \tag{2}$$

   (b) For a perfect fluid, $T_{\mu\nu} = (P + \rho)u_\mu u_\nu + Pg_{\mu\nu}$. Use the conservation of stress-energy and the conservation of baryons $(nu^\alpha)_{;\alpha} = 0$ to show that the flow of a perfect fluid is isentropic (i.e. $ds/dt = 0$ in the fluid rest frame).

2. **Tangent vectors** [10 points]

   In 3-dimensional space with Cartesian coordinates (x, y, z), define three curves:
   $$x^\alpha(\lambda) = (\lambda, 1, \lambda) \tag{3}$$
   $$x^\alpha(\xi) = (\sin\xi, \cos\xi, \xi) \tag{4}$$
   $$x^\alpha(\rho) = (\sinh\rho, \cosh\rho, \rho + \rho^3), \tag{5}$$

and let $f(x^\alpha) = x^2 - y^2 + z^2$. Note that all three curves pass through the point $P_0 = (0, 1, 0)$.

(a) Compute $d/d\lambda(f)$, $d/d\xi(f)$, and $d/d\rho(f)$ at P_0.

(b) Compute the components of the vectors $d/d\lambda$, $d/d\xi$, and $d/d\rho$ at P_0, in terms of the coordinate basis vectors $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$.

3. **Vector commutators** [10 points]

(a) Prove the Jacobi identity for arbitary vector fields $\vec{A}$,$\vec{B}$,$\vec{C}$:

$$[\vec{A}, [\vec{B}, \vec{C}]] + [\vec{B}, [\vec{C}, \vec{A}]] + [\vec{C}, [\vec{A}, \vec{B}]] = 0. \tag{6}$$

(b) Draw a picture that illustrates this identity.

(c) Express this identity as an identity involving the commutation coefficients $c_{\alpha\beta}{}^{\gamma}$.

4. **Index gymnastics** [15 points]

Prove the following identities:

(a) $g^{\alpha\beta}{}_{,\gamma} = -g^{\nu\alpha} g^{\mu\beta} g_{\mu\nu,\gamma}$

(b) $g_{,\alpha} = g g^{\mu\nu} g_{\mu\nu,\alpha}$, where g without indices is the determinant of the matrix $g_{\mu\nu}$.

Hint: Consider a square matrix C that is a function of some parameter ϵ. Taylor expand $C(\epsilon) = A + \epsilon B$ about $\epsilon = 0$, and then use the identities

$$\begin{aligned} \det(1 + \epsilon M) &= 1 + \epsilon\, \mathrm{tr} M + O(\epsilon^2) \\ \det(AB) &= \det A \, \det B \end{aligned} \tag{7}$$

to write out $\det C$ to first order in ϵ, and then to derive a formula for the derivative of $\det C$ in terms of C^{-1} and the derivative of C.

5. **Return to spherical polar coordinates** [10 points]

   Consider 3-dimensional Euclidean space. In Cartesian coordinates (x, y, z) the metric is $g_{ij} = \delta_{ij}$. We define the usual spherical coordinates by

   $$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta. \tag{8}$$

   We also define the orthonormal basis vectors

   $$\vec{e}_{\hat{r}} = \frac{\partial}{\partial r}, \qquad \vec{e}_{\hat{\theta}} = \frac{1}{r}\frac{\partial}{\partial \theta}, \qquad \vec{e}_{\hat{\phi}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}. \tag{9}$$

   (a) What are the basis one-forms $(\tilde{\omega}^{\hat{r}}, \tilde{\omega}^{\hat{\theta}}, \tilde{\omega}^{\hat{\phi}})$ dual to the orthonormal basis vectors?

   (b) Compute all the commutation coefficients $c_{\hat{\imath}\hat{\jmath}}{}^{\hat{k}}$. Explain why $c_{\hat{\imath}\hat{\jmath}}{}^{\hat{k}}$ is not a tensor.

6. **Connection coefficients are not tensor components** [10 points]

   Define the connection coefficients $\Gamma^{\gamma}{}_{\alpha\beta}$ in the usual way in terms of the covariant derivative of a basis vector along another basis vector:

   $$\nabla_{\vec{e}_\beta} \vec{e}_\alpha = \Gamma^{\gamma}{}_{\alpha\beta} \vec{e}_\gamma \tag{10}$$

   Show that the coefficients $\Gamma^{\gamma}{}_{\alpha\beta}$ do *not* transform like tensor components under a change of basis.