Physics 236a assignment, Week 3:

(October 15, 2015. Due on October 22, 2015)

1. **Manipulations of differential forms** [10 points]

   (a) For 1-forms, we defined the wedge product as

   $$\tilde{u} \wedge \tilde{v} = \tilde{u} \otimes \tilde{v} - \tilde{v} \otimes \tilde{u}. \tag{1}$$

   If $\mathbf{\Omega}_1$ is a p-form and $\mathbf{\Omega}_2$ is a q-form, show that

   $$\mathbf{\Omega}_1 \wedge \mathbf{\Omega}_2 = (-1)^{pq} \mathbf{\Omega}_2 \wedge \mathbf{\Omega}_1. \tag{2}$$

   (b) In class, we saw that a key property of the exterior derivative $\tilde{d}$ is that

   $$\tilde{d}(\mathbf{\Omega}_1 \wedge \mathbf{\Omega}_2) = \tilde{d}\mathbf{\Omega}_1 \wedge \mathbf{\Omega}_2 + (-1)^p \, \mathbf{\Omega}_1 \wedge \tilde{d}\mathbf{\Omega}_2, \tag{3}$$

   where, oddly, the right-hand side has explicit dependence on p but not on q. Show that

   $$\tilde{d}(\mathbf{\Omega}_2 \wedge \mathbf{\Omega}_1) = \tilde{d}\mathbf{\Omega}_2 \wedge \mathbf{\Omega}_1 + (-1)^q \, \mathbf{\Omega}_2 \wedge \tilde{d}\mathbf{\Omega}_1, \tag{4}$$

   by starting with Eq. (3) and using Eq. (2) repeatedly.

2. **Electromagnetic invariants** [10 points]

   (a) Show that $\underset{\sim}{E} \cdot \underset{\sim}{B}$ and $B^2 - E^2$ are invariants, i.e. they are the same when measured in any frame.

   (b) Show that there are no independent invariant combinations of $\underset{\sim}{E}$ and $\underset{\sim}{B}$ other than $\underset{\sim}{E} \cdot \underset{\sim}{B}$ and $B^2 - E^2$.

3. **Antisymmetry of Faraday tensor** [10 points]

   Here we will prove that the Faraday tensor $F_{\alpha\beta}$ is antisymmetric.

(a) Show that if $A_{\alpha\beta}$ is any tensor that obeys

$$A_{\alpha\beta} u^\alpha u^\beta = 0 \tag{5}$$

for any timelike vector u^α, then $A_{\alpha\beta}$ is antisymmetric. (Hint, consider some simple families of timelike vectors, for example $\vec{e}_t + \epsilon \vec{e}_x$ where $0 < \epsilon < 1$.)

(b) Using the Lorentz force equation (and without using the explicit form of $F_{\alpha\beta}$ in terms of E and B fields), show that

$$F_{\alpha\beta} u^\alpha u^\beta = 0, \tag{6}$$

for any observer with 4-velocity $\vec{u}$. Therefore, from part 3a, $F_{\alpha\beta}$ must be antisymmetric.

4. **Stress-energy tensor of the EM field** [10 points]

The Maxwell stress-energy tensor is given by

$$T^{\alpha\beta} = \frac{1}{4\pi}\left(F^{\alpha\mu} F^{\beta}{}_{\mu} - \frac{1}{4} g^{\alpha\beta} F^{\mu\nu} F_{\mu\nu}\right). \tag{7}$$

Assume the Minkowski metric.

(a) Suppose a particular observer measures an electric field $\underset{\sim}{E}$ and a magnetic field $\underset{\sim}{B}$. Compute the energy density, momentum density, and 3d stress tensor measured by that observer, in terms of $\underset{\sim}{E}$ and $\underset{\sim}{B}$.

(b) Show that for Eq. (7),

$$T^{\alpha\beta}{}_{,\beta} = -F^{\alpha\mu} J_\mu, \tag{8}$$

where J^μ is the current density 4-vector.

(c) Suppose that we have a charged fluid, so that the total stress-energy is

$$T^{\alpha\beta} = T^{\alpha\beta}_{\rm EM} + T^{\alpha\beta}_{\rm fluid}, \tag{9}$$

where $T^{\alpha\beta}_{\rm EM}$ is the stress-energy tensor defined by Eq. (7), and $T^{\alpha\beta}_{\rm fluid}$ is the stress-energy tensor of the fluid. Write out the conservation equation

$$T^{\alpha\beta}{}_{,\beta} = 0 \tag{10}$$

in terms of $\underset{\sim}{E}$, $\underset{\sim}{B}$, the charge and current density ρ and $\underset{\sim}{J}$, and derivatives of $T^{\alpha\beta}_{\rm fluid}$. What is the physical interpretation of the time component of this equation (i.e. the $\alpha = 0$ part of Eq. (10))? What is the physical interpretation of the spatial components of this equation?

5. **Stress-energy tensor integrals** [10 points]

Assume the Minkowski metric, and use the relation $T^{\mu\nu}{}_{,\nu} = 0$ to prove the following results for a bounded system (a system for which $T^{\mu\nu} = 0$ outside a bounded region of space).

(a) $\dfrac{\partial}{\partial t} \displaystyle\int T^{0\alpha}\, d^3x = 0$ (conservation of energy and momentum)

(b) $\dfrac{\partial^2}{\partial t^2} \displaystyle\int T^{00} x^i x^j\, d^3x = 2 \int T^{ij}\, d^3x$ (tensor virial theorem).

(c) $\dfrac{\partial^2}{\partial t^2} \displaystyle\int T^{00} (x^i x_i)^2\, d^3x = 4 \int T^k{}_k x^i x_i\, d^3x + 8 \int T^{ij} x_i x_j d^3x.$

Here i, j, k indices go from 1 to 3, and x^i are the spatial Minkowski coordinates x, y, z.