Physics 236a assignment, Week 2:

(October 8, 2015. Due on October 15, 2015)

1. **Equation of motion for a spin in a magnetic field.** [10 points]

   We will obtain the relativistic generalization of the nonrelativistic (i.e. rest frame) equation of motion for the spin $\underset{\sim}{s}$ of a particle of mass m, charge q, in a uniform $\underset{\sim}{B}$ field:
   $$\frac{d\underset{\sim}{s}}{dt} = \frac{gq}{2m}\underset{\sim}{s} \times \underset{\sim}{B}, \tag{1}$$
   where g is the Landé g factor, and $\times$ is the usual 3-vector cross product.

   First, construct a spin 4-vector $\vec{S}$ which, by definition, reduces to $\vec{S} = (0, \underset{\sim}{s})$ in the particle rest frame.

   Next, write
   $$\frac{dS^\alpha}{d\tau} = A_1 F^\alpha{}_\beta S^\beta + A_2 u^\alpha, \tag{2}$$
   where $F^\alpha{}_\beta$ are components of the electromagnetic tensor, $\vec{u}$ is the 4-velocity of the particle, and A_1 and A_2 are scalars to be determined.

   (a) Evaluate $\vec{S} \cdot \vec{u}$ and $d/d\tau(\vec{S} \cdot \vec{u})$.

   (b) Determine A_1 and A_2.

   (c) Determine the time rate of change of the helicity $\underset{\sim}{s} \cdot \underset{\sim}{v}$ in a pure $\vec{B}$ field (i.e. in a frame where the electric field $\underset{\sim}{E} = 0$. (Hint, let $\vec{S} = (s^0, \underset{\sim}{S})$ and expand $\vec{S} \cdot \vec{u}$ to evaluate the helicity.) Comment on the result for a particle with $g = 2$.

2. **Projection tensor.** [10 points] Given a particle with 4-velocity $\vec{u}$, define the projection tensor $\mathbf{P}$ by
   $$P_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta. \tag{3}$$

(a) For an arbitrary vector $\vec{v}$, define $\vec{v}_\perp = \mathbf{P}(\vec{v})$, and show that $\vec{v}_\perp$ is orthogonal to $\vec{u}$.

(b) Show that $\mathbf{P}(\vec{v}_\perp) = \vec{v}_\perp$. Thus $\mathbf{P}$ is the unique projection tensor that projects projects an arbitrary vector $\vec{v}$ into a 3-surface orthogonal to $\vec{u}$.

(c) Given an arbitrary but non-null vector $\vec{q}$, construct the projection tensor that projects an arbitrary vector $\vec{v}$ into a 3-surface orthogonal to $\vec{q}$.

(d) Given a null vector $\vec{k}$, show that there is no *unique* projection tensor that projects an arbitrary vector $\vec{v}$ into a 3-surface orthogonal to $\vec{k}$.

3. **Curvilinear coordinates in Euclidean space** [10 points] Work in 3-dimensional Euclidean space. Suppose we have a Cartesian coordinate system, with coordinates (x, y, z), the usual coordinate basis vectors $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$, and the usual coordinate basis 1-forms $(\tilde{d}x, \tilde{d}y, \tilde{d}z)$.

   (a) First define three vector fields $\vec{e}_r$, $\vec{e}_\theta$, and $\vec{e}_\phi$ as follows: Suppose that the entire space is filled with curves having $\theta = \text{const}$, $\phi = \text{const}$ and parameterized by r, where (r, θ, ϕ) are the usual spherical coordinates. For each point in space (except the origin), there is one and only one such curve passing through it. Define $\vec{e}_r$ as the tangent vector to that curve, and write down the expansion of $\vec{e}_r$ in terms of $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ and (r, θ, ϕ). Similarly, define $\vec{e}_\theta$ as tangent vectors of curves parameterized by θ and having $r = \text{const}$ and $\phi = \text{const}$, and define $\vec{e}_\phi$ as tangent vectors of curves parameterized by ϕ and having $r = \text{const}$ and $\theta = \text{const}$. Write down $\vec{e}_\theta$ and $\vec{e}_\phi$ in terms of $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ and (r, θ, ϕ).

   (b) Consider r as a function of x, y, z, and write down the gradient $\tilde{d}r$ in terms of the basis one-forms $(\tilde{d}x, \tilde{d}y, \tilde{d}z)$. Similarly, consider θ and ϕ as functions of x, y, z, and write down the gradients $\tilde{d}\phi$ and $\tilde{d}\theta$ in terms of $(\tilde{d}x, \tilde{d}y, \tilde{d}z)$.

(c) Show that $(\tilde{d}r, \tilde{d}\theta, \tilde{d}\phi)$ are basis one-forms that are dual to the basis vectors $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$.

(d) The metric components in the Cartesian basis are the components of the Kroneker delta. Write down the components of the metric in the spherical coordinate basis of part 3c. Also write down the full expression for the metric tensor $\mathbf{g}$ as a combination of tensor products of $\tilde{d}r$, $\tilde{d}\theta$, and $\tilde{d}\phi$. Note that even though this is still a simple Euclidean space, the flat metric looks complicated when expressed in terms of a complicated basis.

(e) The basis of of part 3c is a *coordinate basis*. Show that by rescaling $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi)$ at each point, we can define a new non-coordinate basis, $(\vec{e}_{\hat{r}}, \vec{e}_{\hat{\theta}}, \vec{e}_{\hat{\phi}})$, such that the metric components in this basis become the Kroneker delta. A basis in which the metric components are the Kroneker delta in a Euclidean geometry or the Minkowski metric in a Lorentzian geometry is called an *orthonormal basis*.

4. **Index notation.** [10 points]

(a) Let $A_{\mu\nu}$ be an antisymmetric tensor and let $S_{\mu\nu}$ be a symmetric tensor. Show that $A_{\mu\nu}S^{\mu\nu} = 0$.

(b) Let $A_{\mu\nu}$ be an antisymmetric tensor and let $S_{\mu\nu}$ be a symmetric tensor. Show that for an arbitrary tensor $V_{\mu\nu}$,

$$V^{\mu\nu}A_{\mu\nu} = \frac{1}{2}\left(V^{\mu\nu} - V^{\nu\mu}\right)A_{\mu\nu} \tag{4}$$

and

$$V^{\mu\nu}S_{\mu\nu} = \frac{1}{2}\left(V^{\mu\nu} + V^{\nu\mu}\right)S_{\mu\nu} \tag{5}$$

(c) Given that basis vectors and basis one-forms transform from the x^{μ} frame to the $x^{\bar{\mu}}$ frame according to

$$\vec{e}_{\bar{\mu}} = \Lambda^{\mu}{}_{\bar{\mu}} \vec{e}_{\mu}, \qquad \tilde{\omega}^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_{\mu} \tilde{\omega}^{\mu}, \tag{6}$$

show that the two transformation matrices $\Lambda^{\mu}{}_{\bar{\mu}}$ and $\Lambda^{\bar{\mu}}{}_{\mu}$ are inverses of each other (when multiplied in either order), *i.e.*, $\Lambda^{\bar{\mu}}{}_{\mu} \Lambda^{\mu}{}_{\bar{\nu}} = \delta^{\bar{\mu}}_{\bar{\nu}}$ and $\Lambda^{\beta}{}_{\bar{\mu}} \Lambda^{\bar{\mu}}{}_{\alpha} = \delta^{\beta}_{\alpha}$. Also show that components of a tensor transform like

$$T^{\bar{\alpha}}{}_{\bar{\beta}}{}^{\bar{\gamma}} = \Lambda^{\bar{\alpha}}{}_{\alpha} \Lambda^{\beta}{}_{\bar{\beta}} \Lambda^{\bar{\gamma}}{}_{\gamma} T^{\alpha}{}_{\beta}{}^{\gamma}. \tag{7}$$

(d) In class, we defined $g^{\alpha\beta}$ as the matrix inverse of $g_{\alpha\beta}$:

$$g_{\alpha\beta} g^{\beta\gamma} = \delta^{\gamma}_{\alpha}. \tag{8}$$

Show that $g^{\alpha\beta}$ can be obtained by "raising indices" from $g_{\alpha\beta}$, and that $g_{\alpha\beta}$ can be obtained by "lowering indices" from $g^{\alpha\beta}$. Also show that $g^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta}$.

5. **Levi-civita tensor.** [15 points]

(a) One can construct several permuation tensors from the Levi-Civita tensor:

$$\begin{aligned} \delta^{\alpha\beta\gamma}{}_{\mu\nu\lambda} &\equiv -\epsilon^{\alpha\beta\gamma\rho} \epsilon_{\mu\nu\lambda\rho}, \\ \delta^{\alpha\beta}{}_{\mu\nu} &\equiv -\frac{1}{2} \epsilon^{\alpha\beta\gamma\rho} \epsilon_{\mu\nu\gamma\rho}. \end{aligned} \tag{9}$$

Show that

$$\delta^{\alpha\beta\gamma}{}_{\mu\nu\lambda} = \begin{cases} +1 \text{ if } \alpha\beta\gamma \text{ is an even permutation of } \mu\nu\lambda \\ \text{-1 if } \alpha\beta\gamma \text{ is an odd permutation of } \mu\nu\lambda \\ 0 \text{ otherwise} \end{cases} \tag{10}$$

and

$$\delta^{\alpha\beta}{}_{\mu\nu} = \begin{cases} +1 \text{ if } \alpha\beta \text{ is an even permutation of } \mu\nu \\ \text{-1 if } \alpha\beta \text{ is an odd permutation of } \mu\nu \\ 0 \text{ otherwise} \end{cases} \tag{11}$$

(b) The dual ${}^\star T$ of an antisymmetric tensor T is defined for 1, 2, and 3-index antisymmetric tensors by

$$ {}^\star J_{\alpha\beta\gamma} = J^\mu \epsilon_{\mu\alpha\beta\gamma}, \quad {}^\star F_{\alpha\beta} = \frac{1}{2!} F^{\mu\nu} \epsilon_{\mu\nu\alpha\beta}, \quad {}^\star B_\alpha = \frac{1}{3!} B^{\mu\nu\gamma} \epsilon_{\mu\nu\gamma\alpha}. \tag{12} $$

Show that when you apply the dual operation twice,

$$ {}^{\star\star} J_{\alpha\beta\gamma} = J_{\alpha\beta\gamma}, \quad {}^{\star\star} F_{\alpha\beta} = -F_{\alpha\beta}, \quad {}^{\star\star} B_\alpha = B_\alpha. \tag{13} $$