Physics 236a assignment, Week 1:

(October 1, 2015. Due on October 8, 2015)

1. **Proof of invariance of Δs^2.** [10 Points]

   We will prove that Δs^2 is invariant with respect to coordinate transformations between Lorentz frames, *without* assuming the explicit forms of Lorentz transformations. We will assume only that allowable coordinate transformations between Lorentz frames are linear, i.e. $\Delta x^{\bar{\mu}} = L^{\bar{\mu}}{}_{\nu} \Delta x^{\nu}$ where L is an unknown matrix of constant coefficients.

   (a) Argue that since the speed of light is the same in all reference frames, if $\Delta s^2 = \eta_{\alpha\beta}\Delta x^\alpha \Delta x^\beta$ is zero in one frame, then Δs^2 is zero in all frames.

   (b) Consider the quadratic form $Q = \eta_{\bar{\alpha}\bar{\beta}}\Delta x^{\bar{\alpha}}\Delta x^{\bar{\beta}}$, which is Δs^2 in the barred frame. Show that Q is also a quadratic form in Δx^α (with no bars), and that $Q = 0$ on the light cone in the x^α coordinate system.

   (c) Consider the intersection of the light cone with a surface of constant t. This intersection is a sphere, centered at the spatial origin. Argue that

   $$Q = c_1 \Delta t^2 + c_2(\Delta x^2 + \Delta y^2 + \Delta z^2), \tag{1}$$

   where c_1 and c_2 are constants that we will determine later.

   (d) Consider another surface, one with $\Delta y = \Delta z = 0$. By considering the intersection of the light cone with this surface, argue that $c_1 = -c_2$, so that

   $$Q = c_2 \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \tag{2}$$

   (e) Argue that c_2 should be 1.

2. **Derivation of Lorentz boost** [10 Points]

   We will derive the Lorentz transformation between two frames Σ and $\bar{\Sigma}$ with the axes of the two frames aligned, and $\bar{\Sigma}$ moving with respect to Σ with speed v along the x direction.

   (a) Suppose that the Lorentz transformation is linear:

   $$\begin{aligned} \bar{t} &= \alpha_{00} t + \alpha_{01} x \\ \bar{x} &= \alpha_{10} t + \alpha_{11} x \end{aligned} \tag{3}$$

   where each α is (so far) unknown. Show that $-\alpha_{10}/\alpha_{11} = v$.

   (b) Using a spacetime diagram, argue that $-\alpha_{01}/\alpha_{00} = v$, and therefore that

   $$\begin{aligned} \bar{t} &= \alpha_{00}(t - vx) \\ \bar{x} &= \alpha_{11}(x - vt) \end{aligned} \tag{4}$$

   (c) Using the invariance of δs^2, show that $\alpha_{00}^2 = \alpha_{11}^2 = 1/(1 - v^2)$. Argue that we should choose the positive square root: $\alpha_{00} = \alpha_{11} = 1/\sqrt{1 - v^2}$.

3. **Vector relations** [10 Points] Two vectors $\vec{A}$ and $\vec{B}$ are orthogonal if $\vec{A} \cdot \vec{B} = 0$.

   Assume the Minkowski metric $\eta_{\mu\nu}$, and

   (a) Show that if $\vec{A}$ and $\vec{B}$ are spacelike and orthogonal, then $\vec{A} + \vec{B}$ is spacelike.

   (b) Show that a null vector cannot be orthogonal to a timelike vector.

   (c) Show that the only non-spacelike vectors orthogonal to a (non-zero) null vector $\vec{k}$ are multiples of $\vec{k}$.

4. **Relative velocity formula** [10 Points]

   If two frames move with 3-velocities $\underset{\sim}{v}_1$, and $\underset{\sim}{v}_2$, show, that their relative velocity is given by

   $$v^2 = \frac{(\underset{\sim}{v}_1 - \underset{\sim}{v}_2)^2 - (\underset{\sim}{v}_1 \times \underset{\sim}{v}_2)^2}{(1 - \underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2)^2}. \tag{5}$$

   Do not use explicit Lorentz transformations. Instead, use invariants. (Hint: consider the 4-velocities $\vec{u}_1$ and $\vec{u}_2$ of the two frames).

5. **Trip to the center of the galaxy** [15 Points]

   (a) Assume a rocket ship has engines that give it a constant acceleration of $1g$ (relative to its instantaneous inertial frame). If the rocket starts from rest in the frame of the earth, how far from the earth will the rocket be in 30 years as measured on the earth? How far after 30 years as measured by a passenger in the rocket?

   (b) Compute the proper time for the passengers in the rocket ship to travel to the center of our galaxy, 30,000 light years. Assume that the rocket accelerate at $1g$ for the first half of the trip and decelerates at $1g$ for the second half. How much time will have passed on Earth during this trip?

   (c) Assume that the rocket carries its own fuel for the trip in part (b). What fraction of the rocket's initial mass can be payload (i.e. anything other than fuel)? Assume an ideal rocket engine that converts rest mass into radiation and ejects all the radiation out the back, perfectly collimated, with 100% efficiency.

6. **Center of Momentum frame** [10 Points]

(a) Consider a particle with 4-momentum $\vec{p}$ and an observer with 4-velocity $\vec{u}$. Show that in the observer's rest frame, the vector

$$\vec{p}_{(3)} = \vec{p} + (\vec{p} \cdot \vec{u})\vec{u} \tag{6}$$

has zero time component, and has spatial components equal to the 3-momentum of the particle as measured by the observer.

(b) Now assume that $\vec{p}$ is the total 4-momentum of a system of particles. Define the center-of-momentum frame to be that frame in which the total 3-momentum of the system of particles is zero. Argue that the 4-velocity $\vec{u}_{\rm cm}$ of the center-of-momentum frame satisfies

$$\vec{p} + (\vec{p} \cdot \vec{u}_{\rm cm})\vec{u}_{\rm cm} = 0 \tag{7}$$

and that this is true in any Lorentz frame.

(c) Show that if the total energy of a system of particles in the lab frame is $E_{\rm tot}$, and the total 3-momentum of the system of particles in the lab frame is $\underset{\sim}{p}_{\rm tot}$, then the 3-velocity of the center-of-momentum frame with respect to the lab frame is

$$\underset{\sim}{v}_{\rm cm} = \underset{\sim}{p}_{\rm tot}/E_{\rm tot}. \tag{8}$$