Cosmology

"cosmological principle" – Universe is spatially homogeneous & isotropic on large scales, viewed by comoving observer.

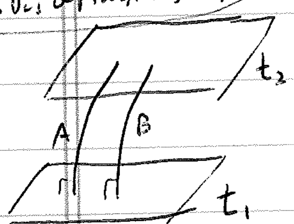
- slice spacetime into slices of const "cosmic time" t.
- comoving observers (galaxies) have 4-vel $u^a = n^a$ (normal to slice)
- Metric in 3+1 form $ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$

shift $\rightarrow 0$ (homogeneous/isotropic)

choose lapse $\rightarrow 1$ — t is proper time of comoving observer

$$\Rightarrow \quad ds^2 = -dt^2 + \gamma_{ij}\, dx^i dx^j$$

How does γ_{ij} depend on x^i, t?



Proper distance btwn A+B at t_1 is $\Delta\sigma(t_1) = (\gamma_{ij}\Delta x^i \Delta x^j)^{1/2}$

Isotropy: $\dfrac{\Delta\sigma(t_2)}{\Delta\sigma(t_1)}$ same for any B relative to A

Homogeneity: $\dfrac{\Delta\sigma(t_2)}{\Delta\sigma(t_1)}$ independent of x^i

$$\Rightarrow \quad \frac{\Delta\sigma(t_2)}{\Delta\sigma(t_1)} = \frac{a(t_2)}{a(t_1)} \quad \text{so} \quad \Delta\sigma(t) = a(t)\left[\bar{\gamma}_{ij}(x^i)\Delta x^i \Delta x^j\right]$$

$$\Rightarrow \quad ds^2 = -dt^2 + a^2(t)\,\bar{\gamma}_{ij}(x)\,dx^i dx^j$$

Form of $\bar{\gamma}_{ij}$: $ds^2 = -dt^2 + a^2(t)\left[A(r)dr^2 + B(r)\,d\Omega^2\right]$ (CP)

$d\theta^2 + \sin^2\theta\, d\varphi^2$

spher. symm.

choose $B(r) = r'^2$ + drop primes

$$\boxed{ds^2 = -dt^2 + a^2(t)\left[f(r)dr^2 + r^2 d\Omega^2\right]}$$

Find $f(r)$

Compute $^{(3)}R = \dfrac{3\left[r^2(1-f^{-2})\right]_{,r}}{2r^3}$ must be const (CP)

Integrate $\Rightarrow$ $r^2(1-f^{-2}) = Br^4 + C$ B, C consts

$f \to 1$ as $r \to 0$ (locally flat)

$\Rightarrow C = 0$

$\Rightarrow f^2 = (1-Br^2)^{-1}$

$$\Rightarrow ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Br^2} + r^2 d\Omega^2\right]$$

if $B \neq 0$ let $r' = B^{1/2} r$, drop primes, absorb factor of B into $a^2(t)$

$$\Rightarrow \boxed{ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \qquad k = 1, 0, -1}$$

alternatively, $r = \begin{cases} \sin\chi & k=1 \\ \chi & k=0 \\ \sinh\chi & k=-1 \end{cases}$

$\sin^2\chi,\ \chi^2,$ or $\sinh^2\chi$

$$\Rightarrow \boxed{ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + Q(\chi)\,d\Omega^2\right]}$$

Friedmann Robertson Walker metric (FRW)

$k=0$ Flat space (time still curved)

$k=1$ Closed universe 3-sphere embedded in 4d flat space

$$w^2+x^2+y^2+z^2=a^2$$

Pythagorean Theorem $a^2+b^2>c^2$

$$x=a\sin\chi\cos\theta$$
$$y=a\sin\chi\sin\theta\cos\varphi$$
$$z=a\sin\chi\sin\theta\sin\varphi$$
$$w=a\cos\chi$$

$k=-1$ open universe (saddle)

$$w^2-x^2-y^2-z^2=a^2$$

Pythag. Thm $a^2+b^2<c^2$

Our universe appears to be flat, $k=0$

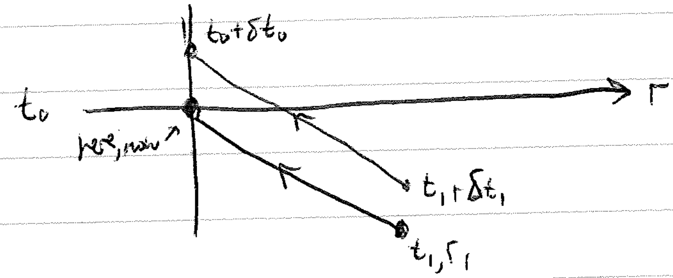
Expansion of universe:

Proper distance btwn 2 galaxies: $d_{prop}(t) = \int_0^{r_1} \sqrt{g_{rr}}\, dr$

$$= a(t) \int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}}$$

$$= a(t) \underbrace{\begin{cases} \sin^{-1}(r_1) & k=1 \\ r_1 & 0 \\ \sinh^{-1}(r_1) & -1 \end{cases}}_{\text{const.}}$$

Cosm. redshift:



We recieve photon from galaxy at t_1, r_1

$$ds^2 = 0 = -dt^2 + a^2(t)\frac{dr^2}{1-kr^2}$$

$$\Rightarrow \int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}}$$

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{a(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}}$$ RHSs are equal

Subtract: $-\int_{t_1}^{t_1+\delta t_1} \frac{dt}{a(t)} + \int_{t_0}^{t_0+\delta t_0} \frac{dt}{a(t)} = 0$

$\delta t \ll t \Rightarrow \frac{\delta t_1}{a(t_1)} = \frac{\delta t_0}{a(t_0)}$

or $\boxed{\frac{\nu_0}{\nu_1} = \frac{\delta t_1}{\delta t_0} = \frac{a(t_1)}{a(t_0)}}$ $\boxed{z \equiv \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{a(t_0)}{a(t_1)} - 1}$ Not SR Doppler shift

$a(t_0) > a(t_1)$ so $z > 0$

For small r_1, can reduce redshift to a velocity:

$$d_{proper} = a(t)\int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}} \sim a(t)\, r_1$$

$$\text{so} \quad v = \dot{d}_{proper} \sim \dot{a}(t)\, r_1 = d_{proper}\frac{\dot{a}}{a}$$

$$\text{so} \quad v = H(t)\, d_{proper} \quad \text{where} \quad H(t) \equiv \frac{\dot{a}(t)}{a(t)}$$

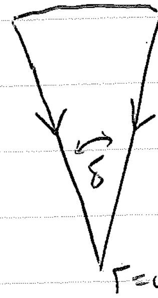
t_0 = present

$$H(t=t_0) \equiv H_0 \quad \text{called Hubble const.}$$

Distance measures

D = Physical size of object $t = t_1$, $r = r_1$

① Angular distance d_A



$r = 0,\ t = t_0$

$D = a(t_1) r_1 \delta \quad (\delta \ll 1)$

define $\boxed{d_A \equiv \frac{D}{\delta} = a(t_1) r_1}$

② Luminosity distance d_L

Object has luminosity L = energy per unit proper time

Observe Flux F

$$d_L \equiv \left(\frac{L}{4\pi F}\right)^{1/2} \qquad \text{inverse square}$$

In time δt_1, object emits energy $E = L\delta t_1$

$$E_{\text{observed}} = \underbrace{\frac{a(t_1)}{a_0}}_{\text{energy redshifted}} L\delta t_1$$

$$= \frac{a(t_1)^2}{a_0^2} L \delta t_0 \qquad \text{time redshifted}$$

$$\text{Flux}_{\text{obs}} = \frac{E_{\text{observed}}}{\delta t_0 \underbrace{(4\pi a_0^2 r_1^2)}_{\text{area of sphere}}} = \frac{a(t_1)^2}{4\pi a_0^4 r_1^2} L$$

So $\boxed{d_L = \frac{a_0^2}{a(t_1)} r_1} = \frac{a_0^2}{a(t_1)^2} d_A = (1+z)^2 d_A$

Distance - redshift relation

Strategy: - Find objects with known L (e.g. Type Ia SNe)

- Measure F on earth $\Rightarrow d_L$
- Measure z

get r_1 as fn of z

$ds^2 = 0$ for photon $\quad \int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_0^{r_1} \frac{dr}{(1-kr^2)^{1/2}}$

$\Rightarrow \quad a(t) \text{ vs } r_1$

$$\left(\frac{F}{L} = \frac{H_0^2}{16\pi} \frac{1}{(1+z)\left((1+z)^{1/2} - 1\right)^2} \quad \text{for flat universe with } P = 0 \right)$$

$\Rightarrow$ Measure H_0

Similarly can measure $q_0 = -\frac{\ddot{a} a}{\dot{a}^2}$ "deceleration parameter"

For small $t - t_0$,

$$a(t) = a_0\left(1 + H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 + \dots\right)$$

Have been measured $\quad H_0 = 70$ km/s/Mpc ± 10

$q_0 \sim -0.6$ negative! Universe accelerating!

2011 Nobel prize

Dynamics

Let $T_{ab} = (\rho + P) n_a n_b + P g_{ab}$ perfect Fluid

P, ρ Function of t only

$$G_{ab} - \Lambda g_{ab} = 8\pi T_{ab}$$

cosmological const.

Can treat Λ as effective T_{ab} $\Rightarrow$ $T_{ab}^{(\Lambda)} = \frac{\Lambda}{8\pi} g_{ab}$

$\rho = \Lambda/8\pi$

$P = -\Lambda/8\pi$

So let's drop Λ, include it in ρ, P.

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \Rightarrow \boxed{\begin{aligned} \left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} &= \frac{8}{3}\pi\rho \quad (1) \\ \frac{\ddot a}{a} &= -\tfrac{4}{3}\pi(3P+\rho) \quad (2) \end{aligned}}$$

Friedmann Eqs.

$$T^{\mu\nu}{}_{;\nu} = 0 \Rightarrow \boxed{\dot\rho = \frac{-3\dot a}{a}(P+\rho)} \quad (3)$$

can get from (1)+(2)

Often use shortcuts $H(t) \equiv \frac{\dot a}{a}$

$$q(t) \equiv \frac{-\ddot a a}{\dot a^2}$$

Limiting cases

ⓐ "Matter dominated" $\rho \gg P$

③ $\dot{\rho} = -\frac{3\dot{a}}{a}(\rho + \cancel{P}^{\approx 0}) \Rightarrow \boxed{\rho(t) = \rho_0 \frac{a_0^3}{a(t)^3}}$

① (Flat) $\Rightarrow \dot{a}^2 = \frac{\text{const}}{a} \Rightarrow \boxed{a \sim t^{2/3}}$

ⓑ "Radiation dominated" $P = \frac{1}{3}\rho$

③ $\Rightarrow \dot{\rho} = -\frac{4\dot{a}}{a}\rho \Rightarrow \boxed{\rho(t) = \rho_0 a_0^4 / a(t)^3}$

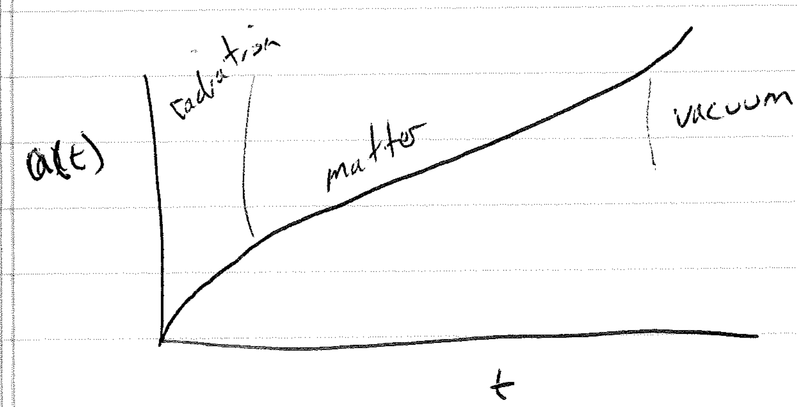
① (Flat) $\Rightarrow \dot{a}^2 = \text{const}/a^2 \Rightarrow \boxed{a \sim t^{1/2}}$

ⓒ "Vacuum dominated"

$P = -\rho$

③ $\Rightarrow \dot{\rho} = 0$ $\boxed{\rho(t) = \rho_0}$

① Flat $\frac{\dot{a}}{a} = \text{const} = H$ $\boxed{a \sim e^{H(t-t_0)}}$



Let ρ_c = critical density $\equiv \dfrac{3H_0^2}{8\pi}$

$$① \Rightarrow \quad H_0^2 + \frac{k}{a_0^2} = \frac{8\pi}{3}\rho_0$$

if $\rho_0 > \rho_c$ $\quad k > 0$

if $\rho_0 < \rho_c$ $\quad k < 0$

if $\rho_0 = \rho_c$ $\quad k = 0$

Let $\Omega_0 \equiv \rho_0/\rho_c$

let $\Omega_m \equiv \dfrac{\rho_{m_0}}{\rho_c}$ — MATTER

$\Omega_r \equiv \dfrac{\rho_{r_0}}{\rho_c}$ — radiation

$\Omega_\Lambda \equiv \dfrac{\rho_\Lambda}{\rho_c}$ — cosm. const

Then $\quad ① \Rightarrow \quad 1 + \dfrac{k}{a_0^2}\dfrac{3}{8\pi} = \Omega_0$

$$= \Omega_m + \Omega_r + \Omega_\Lambda$$

CMB Measurements (Planck) show
2015

$\Omega_\Lambda = 0.69 \pm .006$

$\Omega_m = 0.309 \pm .006$ — most is dark matter, only 5% of Ω_0 is baryons

Ω_r = (small at present) $\sim 8\times10^{-5}$

$H_0 = 67.74 \pm 0.46$ km/s/Mpc

$\Omega_0 = 1.002 \pm 0.005$

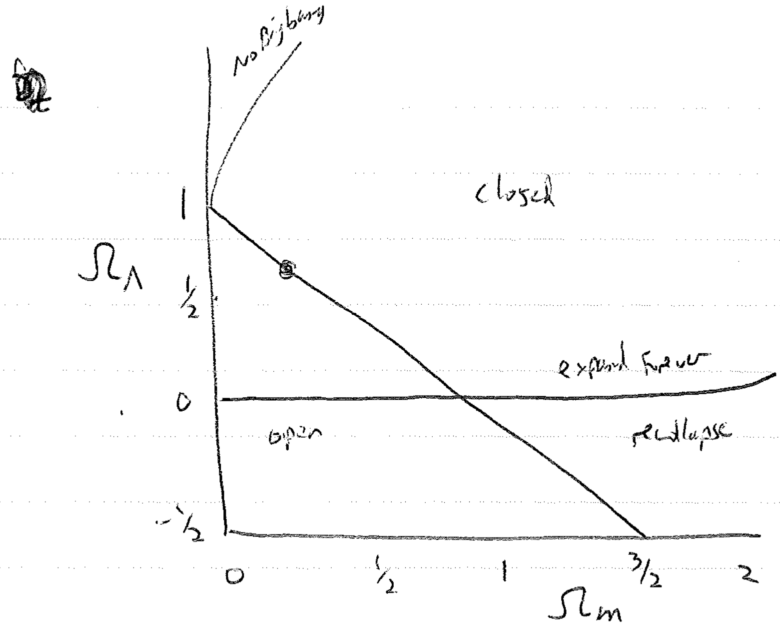
$t_0 = 13.8 \pm .02$ Gyr

Flat universe, Λ dominated at present

Also $\quad q = \dfrac{-\ddot{a}}{a}\left(\dfrac{a^2}{\dot{a}^2}\right) = \dfrac{1}{H^2}\dfrac{4\pi}{3}(3P+\rho) \quad \Rightarrow \quad q_0 = \dfrac{1}{\rho_c}\dfrac{1}{2}(3P+\rho)$

$$= \frac{1}{2}\sum \Omega_i\left(1+3P/\rho\right) \sim \frac{1}{2}\Omega_m - \Omega_\Lambda$$

$$= -0.5$$



No Big bang
closed
1
Ω_Λ
1/2
expand forever
0
open
recollapse
-1/2
0
1/2
1
3/2
2
Ω_m

Olbers' Paradox (H.W. Olbers, 1758-1840) (Thomas Digges, c.1546-1595 Kepler, etc.)

Why is the night sky dark?

Static, ∞, eternal universe:

Energy Flux from 1 star $\sim \frac{1}{r^2}$

$N_{stars} \sim r^3$

$\Rightarrow$ total energy flux $\sim r$

Sky should be bright!

Soln

Single star w/ luminosity $L = \frac{\text{Energy}}{\text{Proper time}}$

Total energy in universe now from that star is $\int_0^{t_0} L \frac{a(t)}{a_0} dt$

energy is redshifted

Matter dominated: $a(t) = a_0\left(\frac{3}{2}H_0 t\right)^{2/3}$ where $t_0 = \frac{2}{3}\frac{1}{H_0}$

So $E = \int_0^{t_0} L\frac{a(t)}{a_0} = \frac{2}{5}\frac{L}{H_0}$

Energy/volume $= n_0 \frac{2}{5}\frac{L}{H_0}$

stars per unit volume now

Flux is isotropic

$$\boxed{F = \frac{1}{4\pi} n_0 \frac{2}{5}\frac{L}{H_0}}$$

Finite