Harmonic time slicing

$$\Box t = \nabla^{\alpha}\nabla_{\alpha} t = 0$$

This reduces to $g^{\alpha\beta}\Gamma^{0}_{\alpha\beta} = 0$, or

$$\boxed{\partial_t \alpha - \beta^k \partial_k \alpha = -\alpha^2 K}$$ Harmonic slicing

Shift still free, choose initial lapse and this determines lapse at $t>0$

In practice, a generalized version is more useful:

$$\boxed{\partial_t \alpha - \beta^k \partial_k \alpha = -\alpha^2 f(\alpha) K}$$ $f(\alpha)$ is some function >0

$f(\alpha) \to 1$ Harmonic slicing
$f(\alpha) \to 0$ geodesic slicing (for $\alpha = 1$ initially)
$f(\alpha) \to \infty$ maximal slicing

In practice, $f(\alpha) = \frac{2}{\alpha}$ has nice properties.

Called "1+log" slicing because for $\beta^i = 0$, $f(\alpha) = \frac{2}{\alpha} \Rightarrow \partial_t \alpha = -2\alpha K$

$\Rightarrow \alpha = 1 + \ln \gamma$

This condition, + variations, used in BSSN evolutions.

What about the shift?

Minimal Strain define strain $S = \int (\partial_t \gamma_{ij})(\partial_t \gamma_{ke}) \gamma^{ik} \gamma^{je} \gamma^{1/2} d^3x$

(remember $\partial_t \gamma_{ij} = -2\alpha K_{ij} + 2 D_{(i} \beta_{j)}$)

Minimize S by varying with respect to β^i

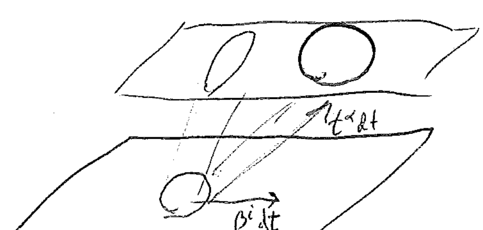
$$\Rightarrow D^i(D_i \beta_j + D_j \beta_i) = D^i(2\alpha K_{ij})$$ minimal strain shift.

linear Elliptic eqn for β^i

This shift minimizes global changes in γ_{ij} from one slice to the next. (Smarr-York 1978)

Related condition: Define $u_{ij} \equiv \partial_t \gamma_{ij} - \frac{1}{3} \gamma_{ij} \, Tr(\partial_t \gamma_{ij})$

Call this the distortion tensor. Describes change in shape (not volume) of pattern from $t=0$ to $t = \Delta t$



Define integrated distortion $\int u_{ij} u^{ij} \gamma^{1/2} d^3x$

Minimize by considering variations w/ respect to β^i

$\Rightarrow D^j u_{ij} = 0$

$$\Rightarrow \quad D_j D^j \beta^i + D_j D^i \beta^j - \frac{2}{3} D^i D_j \beta^j - 2 D_j (\alpha K^{ij}) + \frac{2}{3} D^i (\alpha K) = 0$$

Elliptic eqn for β^i Minimal Distortion shift

Can also be written

$$D^j D_j \beta_i + \frac{1}{3} D_i D_j \beta^j + {}^{(3)}R_{ij} \beta^j = D^j \left[2\alpha \left(K_{ij} - \frac{1}{3} \gamma_{ij} K \right) \right]$$

or

$$D^j D_j \beta^i + \frac{1}{3} D^i D_j \beta^j + {}^{(3)}R^i{}_j \beta^j = \gamma^{ik} D^j \left[2\alpha \left(K_{kj} - \frac{1}{3} \gamma_{kj} K \right) \right]$$

Good: minimizes coordinate shear

Bad: need to solve elliptic eqns. at each time step.

So let's try to find approx. to minimal distortion.

Recall BSSN variable $\bar{\Gamma}^i \equiv \bar{\gamma}^{jk} \bar{\Gamma}^i_{jk}$

Write down the condition $\partial_t \bar{\Gamma}^i = 0$

$$\Rightarrow \bar{\gamma}^{\ell j} \partial_j \partial_\ell \beta^i + \frac{1}{3} \bar{\gamma}^{\ell i} \partial_j \partial_\ell \beta^j + \beta^j \partial_j \bar{\Gamma}^i - \bar{\Gamma}^j \partial_j \beta^i + \frac{2}{3} \bar{\Gamma}^i \partial_j \beta^j$$

$$= 2 \bar{A}^{ij} \partial_j \alpha - 2\alpha \left[\bar{\Gamma}^i_{jk} \bar{A}^{kj} - \frac{2}{3} \bar{\gamma}^{ij} \partial_j K - 8\pi \bar{\gamma}^{ij} S_j + 6 \bar{A}^{ij} \partial_j \phi \right]$$

Similar to minimal distortion shift.

But $\partial_t \bar{\Gamma}^i = 0$ still elliptic. Hard to solve numerically

Trick: convert to hyperbolic eqn:

$$\boxed{\begin{aligned} \partial_t \beta^i &= \tfrac{3}{4} B^i \\ \partial_t B^i &= \partial_t \bar{\Gamma}^i - \eta B^i \end{aligned}}$$

def. of B^i

"Gamma driver"

where $\eta \sim \frac{1}{2M}$

can also replace $\partial_t \rightarrow \partial_t - \beta^k \partial_k$

Gamma Driver + variations used in BSSN evolutions.

B. Generalized Harmonic Formulation

1. Harmonic Coordinates (Einstein, 1912)

- Choose coordinates x^β such that $\underbrace{g_{\alpha\beta}\nabla_\gamma\nabla^\gamma x^\beta}_{-\Gamma_\alpha \;=\; -g^{\beta\gamma}\Gamma_{\alpha\beta\gamma}} = 0$

- Note that Ricci $^{(4\text{-d})}$ can be written:

$$R_{\alpha\beta} = -\frac{1}{2} g^{\gamma\delta}\partial_\gamma\partial_\delta g_{\alpha\beta} + \nabla_{(\alpha}\Gamma_{\beta)} + g^{\gamma\delta}g^{\rho\sigma}\left(\partial g_{\gamma\alpha}\partial_\sigma g_{\delta\beta} - \Gamma_{\alpha\gamma\rho}\Gamma_{\beta\delta\sigma}\right)$$

here $\nabla_\alpha\Gamma_\beta \equiv \partial_\alpha\Gamma_\beta - g^{\gamma\sigma}\Gamma_{\gamma\alpha\beta}\Gamma_\sigma$

So in harmonic coordinates $(\Gamma_\alpha = 0)$, Einstein's Eqs. become

$$g^{\gamma\sigma}\partial_\gamma\partial_\sigma g_{\alpha\beta} = 2 g^{\gamma\sigma}g^{\rho\epsilon}\left(\partial_\rho g_{\gamma\alpha}\partial_\epsilon g_{\sigma\beta} - \Gamma_{\alpha\gamma\rho}\Gamma_{\beta\sigma\epsilon}\right) - 16\pi T_{\alpha\beta} + 8\pi g_{\alpha\beta} T$$

only 2nd derivs are here.

stress-energy

Wave eqn! Symmetric hyperbolic! Well-posedness proven 1952 Choquet-Bruhat

Constraint: $\Gamma_\alpha = 0$

2. <u>Generalized</u> harmonic formulation: (Friedrich 1985, Pretorius 2005, Gundlach 2005)

Choose $g_{\alpha\beta}\nabla_\gamma\nabla^\gamma x^\beta = \underbrace{H_\alpha(t, x^i, g_{\alpha\beta})}_{\text{whatever you want}}$

Then Einstein's eqs. become

<u>Ev.</u> $$g^{\gamma\sigma}\partial_\gamma\partial_\sigma g_{\alpha\beta} = -2\nabla_{(\alpha}H_{\beta)} + 2g^{\gamma\sigma}g^{\delta\epsilon}\left(\partial_\delta g_{\gamma\alpha}\partial_\epsilon g_{\sigma\beta} - \Gamma_{\alpha\gamma\delta}\Gamma_{\beta\sigma\epsilon}\right) - 16\pi T_{\alpha\beta} + 8\pi g_{\alpha\beta}T$$

<u>Constr</u>: $\Gamma_\alpha = -H_\alpha$

Now you have freedom to choose coordinates, and symmetric hyperbolicity.

(As long as H_α doesn't depend on <u>derivs</u> of metric)

- Char fields, speeds, independent of choice of H_α

- This is a different way of choosing coordinates than ADM/BSSN

ADM/BSSN: Choose lapse α
Choose shift β^i

GH: Choose H_α

Relation between them:

$$\partial_t\alpha - \beta^k\partial_k\alpha = -\alpha\left(H_t - \beta^i H_i + \alpha K\right)$$
$$\partial_t\beta^i - \beta^k\partial_k\beta^i = \alpha\,\gamma^{ij}\left[\alpha\left(H_j + \gamma^{k\ell}\Gamma_{jk\ell}\right) - \partial_j\alpha\right]$$

So choice of H_α determines evolution of lapse & shift.

Expect any gauge approachable via choice of H_α

GH Constraints

The GH constraints are $\quad C_\alpha = H_\alpha + \Gamma_\alpha = 0$

How does this relate to Hamil / Moment. constraints?

1st note that GH eqs. can be written in the form

$$R_{\alpha\beta} - \nabla_{(\alpha} C_{\beta)} = 8\pi T_{\alpha\beta} - 4\pi g_{\alpha\beta} T$$

also note that the Hamil + Momentum constraints can be combined into one quantity

$$\mathcal{M}_\alpha \equiv \left(R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R - 8\pi T_{\alpha\beta}\right) t^\beta$$

where t^β is the normal to the spatial slice.

Therefore $\quad \mathcal{M}_\alpha = \left(\nabla_{(\alpha} C_{\beta)} + 16\pi T g_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} \nabla^\gamma C_\gamma\right) t^\beta$

So Hamil / Momen constraints are projections of derivs of C_α

So Specifying C_α, $\partial_t C_\alpha$ on some surface is equivalent to specifying Hamil / Mom constraints on that surface.

Propagation of constraints

Question: If C_α is satisfied at $t=0$, is it satisfied later?

To answer this, again write GH system as

$$R_{\alpha\beta} - \nabla_{(\alpha} C_{\beta)} = 8\pi T_{\alpha\beta} - 4\pi g_{\alpha\beta} T$$

or equivalently $$G_{\alpha\beta} - \nabla_{(\alpha} C_{\beta)} + \frac{1}{2} g_{\alpha\beta} \nabla^\gamma C_\gamma = 8\pi T_{\alpha\beta} \qquad \star$$

Take divergence of $\star$, use Bianchi identity, commute derivs

$$\Rightarrow \nabla^\beta \nabla_\beta C_\alpha + R_{\alpha\beta} C^\beta = 0$$

$$\Rightarrow \nabla^\beta \nabla_\beta C_\alpha + C^\beta \left[\nabla_{(\alpha} C_{\beta)} + 8\pi T_{\alpha\beta} - 4\pi g_{\alpha\beta} T \right] = 0$$

$\Rightarrow$ C_α obeys wave eqn with source.

$$\boxed{\text{if } \left. \begin{aligned} C_\alpha &= 0 \\ \partial_t C_\alpha &= 0 \end{aligned} \right|_{t=0} \quad \text{then} \quad C_\alpha = 0 \quad \text{for } t > 0}$$

Constraints preserved

Constraint Damping

(Gundlach 2005)
(Pretorius 2005)

Problem: If the constraints C_α are small but nonzero, they can grow.

Solution: Constraint damping: Add a new term to the GH eqns:

$$g^{\gamma\sigma}\partial_\gamma\partial_\sigma g_{\alpha\beta} = -2\nabla_{(\alpha}H_{\beta)} + 2g^{\gamma\sigma}g^{\delta\varepsilon}\left(\partial_\delta g_{\gamma\alpha}\partial_\varepsilon g_{\sigma\beta} - \Gamma_{\alpha\delta\gamma}\Gamma_{\beta\sigma\varepsilon}\right) - 16\pi T_{\alpha\beta} + 8\pi g_{\alpha\beta}T$$

$$\underbrace{+2\gamma_0\left[t_{(\alpha}C_{\beta)} - \frac{1}{2}g_{\alpha\beta}t^\gamma C_\gamma\right]}_{\text{New term}}$$

γ_0 is a new parameter
t_α is unit normal to slice

How does this work?

- Write out "evolution system for constraints", i.e. an equation for evolution of C_α. Same as before but with new term:

$$\Rightarrow \nabla^\beta\nabla_\beta C_\alpha + C^\beta\left[\nabla_{(\alpha}C_{\beta)} + 8\pi T_{\alpha\beta} - 4\pi g_{\alpha\beta}T\right] \underbrace{-2\gamma_0\nabla^\beta\left[t_{(\beta}C_{\alpha)}\right]}_{\text{damping term}} - \frac{1}{2}\gamma_0 t_\alpha C^\beta C_\beta = 0$$

$\sim \gamma_0 t_\beta \nabla^\beta C_\alpha \sim \partial_t C_\alpha$

Damped Harmonic gauge (in GH system)

(Lindblom/Szilagyi: 2009
Choptuik/Pretorius: 2009)

$$\nabla^\gamma \nabla_\gamma X^i = \mu t^\gamma \partial_\gamma X^i$$

Damped wave eq for X^i

$(t^\alpha = (\frac{1}{\alpha}, -\frac{\beta^i}{\alpha}))$
$(\gamma_{0i} = \beta_i)$

can write as $\nabla^\gamma \nabla_\gamma X^i = \mu t^i$

$$= \mu \frac{\beta^i}{\alpha}$$

recall $\nabla^\alpha \nabla_\alpha X^i = H^i$ so $\boxed{H_\alpha = -\mu \gamma_{\alpha i} \beta^i / \alpha}$

works quite well. But how to choose $t^\alpha H_\alpha$?

But cannot choose damped time coord!

t should increase monotonically, not damp away!

well, note that $t^\alpha H_\alpha = t^\alpha \partial_\alpha \log\left(\frac{\gamma^{1/2}}{\alpha}\right) - \frac{1}{\alpha} \partial_k \beta^k$

So try to damp variations in $\frac{\gamma^{1/2}}{\alpha}$. Set $t^\alpha H_\alpha = -\eta \log\left(\frac{\gamma^{1/2}}{\alpha}\right)$

Then $t^\alpha \partial_\alpha \log\left(\frac{\gamma^{1/2}}{\alpha}\right) + \eta \log\left(\frac{\gamma^{1/2}}{\alpha}\right) = \frac{1}{\alpha} \partial_k \beta^k$

So growth of $\frac{\gamma^{1/2}}{\alpha}$ damped

So Finally

$$\boxed{H_\alpha = \eta \log\left(\frac{\gamma^{1/2}}{\alpha}\right) t_\alpha - \mu \frac{1}{\alpha} \gamma_{\alpha i} \beta^i}$$

we set $\mu = \eta = \mu_0 \left[\log\left(\frac{\gamma^{1/2}}{\alpha}\right)\right]^2$

for even stronger suppression of gauge singularities