Ph 236b, Numerical Relativity

Until mid-1990s, all numerical relativity used ADM eqs. (and 3D simulations failed)
Alternative Formulations of Einstein's Eqs developed,
but not widely taken seriously until early-mid 2000s

What's wrong with ADM?

- Well-posedness issues
- Constraint growth issues

I. Well-posedness

Hadamard 1902: A problem is well-posed iff

- A solution exists
- The solution is unique
- The solution depends continuously on the initial + boundary data

Example 1 (Hadamard 1923)

$$\partial_t^2 u - \partial_x^2 u = 0$$

Initial Data $u|_{t=0} = 0$

$$\partial_t u|_{t=0} = \frac{\sin(2\pi n x)}{(2\pi n)^p}$$

B.C. $u=0$ at $x=0$, $x=1$ $\qquad p \geq 1$

Solution: $$u(x,t) = \frac{\sin(2\pi n t)\sin(2\pi n x)}{(2\pi n)^{p+1}}$$

as $n \to \infty$, initial data $\to 0$ and solution at late time $\to 0$

Well-posed

Example 2

$$\partial_t^2 u + \partial_x^2 u = 0$$

we changed only the sign

Same initial data + b.c.

$$u\big|_{t=0} = 0$$

$$\partial_t u\big|_{t=0} = \frac{\sin(2\pi n x)}{(2\pi n)^p}$$

$u = 0$ at $x = 0$ + at $x = 1$

Solution: $$u(x,t) = \frac{\sinh(2\pi n t)\sin(2\pi n x)}{(2\pi n)^{p+1}}$$

as $n \to \infty$, initial data still $\to 0$, but solution at later time $\to \infty$

In other words, given $\partial_t^2 u + \partial_x^2 u = 0$, plus <u>any</u> initial data and <u>any</u> boundary data, I can introduce a <u>small perturbation at $t=0$</u> (of the form $\frac{\sin(2\pi n x)}{(2\pi n)^p}$) that produces an <u>arbitrarily large</u> solution at a given <u>finite</u> time.

$\Rightarrow$ <u>Ill-posed</u>

Proof of well-posedness of wave eqn. $\partial_t^2 u - \partial_x^2 u = 0$ $\quad x \in [0,1]$

Define "Energy":

$$E = \frac{1}{2}\int_0^1 dx \left((\partial_t u)^2 + (\partial_x u)^2\right)$$

"Power in solution at some time"

$$E \geq 0$$

Note
$$\begin{aligned}
\partial_t E &= \int_0^1 dx \left(\partial_t^2 u\, \partial_t u + \partial_t \partial_x u\, \partial_x u\right) \\
&= \int_0^1 dx \left(\partial_x^2 u\, \partial_t u + \partial_t \partial_x u\, \partial_x u\right) \\
&= \int_0^1 dx \left(\partial_x[\partial_x u \partial_t u] - \cancel{\partial_x u \partial_x \partial_t u}^{\,0} + \cancel{\partial_t \partial_x u\, \partial_x u}^{\,0}\right) \\
&= \partial_x u\, \partial_t u \Big|_{x=0}^{x=1}
\end{aligned}$$

So $\partial_t E$ depends only on the flux of waves thru bdry.

For our B.C. $u = \partial_t u = 0$ so $\partial_t E = 0$.

Independent of initial data, E doesn't grow with time.

II. Hyperbolicity of PDEs

Consider 1st-order PDEs

(Any PDE system can be written as System of 1st order PDEs)

Example: Wave eqn

$$\partial_t^2\psi - \partial_x^2\psi = 0$$

define $\Pi \equiv -\partial_t\psi$

$\Phi \equiv \partial_x\psi$

$$\Rightarrow \quad \partial_t\psi = -\Pi$$
$$\partial_t\Pi + \partial_x\Phi = 0$$
$$\partial_t\Phi + \partial_x\Pi = 0$$

Wave Eqn in 1st order Form

(constraint: $\partial_x\psi - \Phi = 0$)

General Form (now u is a vector):

vector index

spatial index

$$\boxed{\partial_t u^a + \underbrace{A^{ia}{}_b}_{\text{no derivs of } u}\partial_i u^b = \underbrace{B^a(u)}_{\text{no derivs of } u}}$$

Wave eq. example $u^a = \begin{pmatrix}\psi\\ \Pi\\ \Phi\end{pmatrix}$ $\quad B^a = \begin{pmatrix}-\Pi\\ 0\\ 0\end{pmatrix}$

$$A^{xa}{}_b = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

(Def "Principal Part" of equation is part including only derivative terms)

Pick an arbitrary spatial unit vector n_i

Then $n_i A^{ia}{}_b$ is the characteristic matrix in direction n_i

Let $e^{\hat\alpha}{}_a$ be the $\hat\alpha$th (left) eigenvector of $n_i A^{ia}{}_b$, with eigenvalues $V_{(\hat\alpha)}$:

$$e^{\hat\alpha}{}_a\left(n_i A^{ia}{}_b\right) = V_{(\hat\alpha)} e^{\hat\alpha}{}_b \qquad \text{(no sum on } \hat\alpha)$$

Then $V_{(\hat{\alpha})}$ are called <u>characteristic speeds</u>

$u^{\hat{\alpha}} \equiv u^a e^{\hat{\alpha}}_{\ a}$ are called <u>characteristic fields</u>

(These are important for boundary conditions later...)

<u>Def</u> A 1st order PDE system is <u>weakly hyperbolic</u> if all eigenvalues $V_{(\hat{\alpha})}$ (in any arbitrary direction n_i) are <u>real</u>.

- For weakly hyperbolic systems, well-posedness depends on details of non-principal terms. Usually ill-posed.

<u>Def</u> A 1st order system of PDEs is <u>strongly hyperbolic</u> if all eigenvalues $V_{(\hat{\alpha})}$ are real, and there is a <u>complete set</u> of linearly independent eigenvectors, independent of the solution or of n_i.

- If a system is strongly hyperbolic:

(1) It is well-posed

(2) At a boundary with (outgoing) normal n_i,
- Boundary conditions <u>must</u> be imposed on all characteristic fields $u^{\hat{\alpha}}$ with $V_{(\hat{\alpha})} < 0$
- Boundary conditions <u>must not</u> be imposed on characteristic fields $u^{\hat{\alpha}}$ with $V_{(\hat{\alpha})} > 0$.

(failure to obey these kills well-posedness)

Def A 1st order PDE system is symmetric hyperbolic if there exists a positive-definite symmetric matrix S_{ab} (a "symmetrizer") such that $S_{ab} A^{ib}{}_{c}$ is symmetric on $a+c$ (for all i)

symmetric hyperbolic implies strongly hyperbolic

$\Rightarrow$ Well-posed

$\Rightarrow$ Well-defined prescription for BCs.

So what about ADM?

ADM is only weakly hyperbolic (Kidder, Scheel, Teukolsky 2001; Nagy, Ortiz, Reula 2004)

$\Rightarrow$ No well-posedness

$\Rightarrow$ No indication of which variables require boundary conditions (much less what these bcs should be).

Example: Scalar Waves, Flat space 1+1 D

$$\partial_t \psi = -\pi$$
$$\partial_t \pi + \partial_x \Phi = 0$$
$$\partial_t \Phi + \partial_x \pi = 0$$

$$\partial_t u^a + A^{ia}{}_b \partial_i u^b = B^a$$

$$u^a = \begin{pmatrix} \psi \\ \pi \\ \Phi \end{pmatrix} \qquad B^a = \begin{pmatrix} -\pi \\ 0 \\ 0 \end{pmatrix}$$

$$n_x A^{x\,a}{}_b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

<u>Char matrix</u>

Eigenvalues / vectors

$$V_{(0)} = 0 \qquad e^{0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$V_{(1)} = 1 \qquad e^{1} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$V_{(-1)} = -1 \qquad e^{-1} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

strongly hyperbolic

Char Fields
$$u^0 = \psi$$
$$u^{\pm} = \pi \pm \Phi$$

u^+ is right going solution $\pi + \Phi$, speed $+1$

u^- left " " $\pi - \Phi$, speed -1

need BC on u^-, not on u^+

III. Beyond the ADM Formulation

A. BSSN Formulation (Shibata/Nakamura 1995, Baumgarte/Shapiro 1999)

– Conformal Decomposition:

(1) Define $\bar{\gamma}_{ij} = e^{-4\phi}\gamma_{ij}$

and require $\det(\bar{\gamma}_{ij}) = 1 \quad \Rightarrow \quad \phi = \frac{1}{12}\ln\gamma$

This just separates $\det\gamma_{ij}$ from the rest of γ_{ij}

(2) Define
$$K_{ij} = e^{4\phi}\left(\bar{A}_{ij} + \frac{1}{3}\bar{\gamma}_{ij}K\right)$$

$Tr(K_{ij})$

$\bar{A}_{ij}$ trace-free

so instead of variables γ_{ij}, K_{ij} we now use $\bar{\gamma}_{ij}, \bar{A}_{ij}, \phi, K$

– Define new variable $\bar{\Gamma}^i \equiv \bar{\gamma}^{jk}\bar{\Gamma}^i_{jk} = -\partial_j\bar{\gamma}^{ij}$

connection associated with $\bar{\gamma}_{ij}$

Then Ricci tensor becomes

$$R_{ij} = R^{\phi}_{ij} + \bar{R}_{ij}$$

where
$$\bar{R}_{ij} = -\frac{1}{2}\bar{\gamma}^{lm}\partial_m\partial_l\bar{\gamma}_{ij} + \bar{\gamma}_{k(i}\partial_{j)}\bar{\Gamma}^k + \bar{\Gamma}^k\bar{\Gamma}_{(ij)k} + \bar{\gamma}^{lm}\left(2\bar{\Gamma}^k_{l(i}\bar{\Gamma}_{j)km} + \bar{\Gamma}^k_{im}\bar{\Gamma}_{klj}\right)$$

$$R^{\phi}_{ij} = -2\left(\bar{D}_i\bar{D}_j\ln\phi + \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\bar{D}_m\ln\phi\right) + 4\left(\bar{D}_i\ln\phi\,\bar{D}_j\ln\phi - \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\ln\phi\,\bar{D}_m\ln\phi\right)$$

Note that all explicit 2nd derivs of γ_{ij} occur in the Laplace-like term

$$\gamma^{\ell m}\partial_m\partial_\ell\bar{\gamma}_{ij}$$

– So we have variables $\bar{\gamma}_{ij}, \bar{A}_{ij}, \phi, K, \bar{\Gamma}^i$ satisfying Evolution Eqs:

$$\partial_t\phi - \beta^i\partial_i\phi = \frac{1}{6}\partial_i\beta^i - \frac{1}{6}\alpha K$$

$$\partial_t\bar{\gamma}_{ij} - \beta^k\partial_k\bar{\gamma}_{ij} = -2\alpha\bar{A}_{ij} + 2\bar{\gamma}_{k(i}\partial_{j)}\beta^k - \frac{2}{3}\bar{\gamma}_{ij}\partial_k\beta^k$$

$$\partial_t K - \beta^i\partial_i K = -\gamma^{ij}D_iD_j\alpha + \alpha\left(\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}K^2\right) + 4\pi\alpha\left(\rho_{ADM} + S\right)$$

$$\partial_t\bar{A}_{ij} - \beta^k\partial_k\bar{A}_{ij} = e^{-4\phi}\left(-(D_iD_j\alpha)^{TF} + \alpha\left(R_{ij}^{TF} - 8\pi S_{ij}^{TF}\right)\right)$$
$$+\alpha\left(K\bar{A}_{ij} - 2\bar{A}_{i\ell}\bar{A}^\ell_j\right) + 2\bar{A}_{k(i}\partial_{j)}\beta^k - \frac{2}{3}A_{ij}\partial_k\beta^k$$

$$\partial_t\bar{\Gamma}^i - \beta^k\partial_k\bar{\Gamma}^i = -2\bar{A}^{ij}\partial_j\alpha + 2\alpha\left(\bar{\Gamma}^i_{jk}\bar{A}^{kj} - \frac{2}{3}\bar{\gamma}^{ij}\partial_j K - 8\pi\bar{\gamma}^{ij}S_j + 6\bar{A}^{ij}\partial_j\phi\right)$$
$$-\bar{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\bar{\Gamma}^i\partial_j\beta^j + \frac{1}{3}\bar{\gamma}^{\ell i}\partial_\ell\partial_j\beta^j + \gamma^{\ell j}\partial_j\partial_\ell\beta^i$$

(Here TF means trace free, i.e. $R_{ij}^{TF} = R_{ij} - \frac{1}{3}\gamma_{ij}R$

and $\bar{A}^{ij} \equiv \bar{\gamma}^{ik}\bar{\gamma}^{j\ell}\bar{A}_{k\ell}$)

Constraints

$$\mathcal{H} = \bar{\gamma}^{ij}\bar{D}_i\bar{D}_j e^\phi - \frac{e^\phi}{8}\bar{R} + \frac{e^{5\phi}}{8}\bar{A}_{ij}\bar{A}^{ij} - \frac{e^{5\phi}}{12}K^2 + 2\pi e^{5\phi}\rho_{ADM}$$

$$\mathcal{M}^i = \bar{D}_j\left(e^{6\phi}\bar{A}^{ij}\right) - \frac{2}{3}e^{6\phi}D^iK - 8\pi e^{6\phi}S^i$$

$\partial_j\bar{\gamma}^{ij} + \bar{\Gamma}^i = 0$ $\quad$ $\det\bar{\gamma}_{ij} = 1$ $\quad$ $\bar{\gamma}^{ij}\bar{A}_{ij} = 0$

Hyperbolicity:

- BSSN is Strongly Hyperbolic for <u>Fixed</u> α, β^i (Nagy, Ortiz, Reula 2004) (Sarbach, Calabrese, Pullin, Tiglio 2002)

  Hyperbolicity depends on gauge choice!
  Char speeds / Fields depend on gauge choice!

- BSSN with dynamical gauge conditions (will talk about later...) is strongly hyperbolic if the shift is not too large (Gundlach, Martin-Garcia 2006)

- Hyperbolicity of BSSN still being researched...

- BSSN currently the most widely used Formulation in NR

Gauge Conditions

(ADM/BSSN) Einstein's Eqs do not determine lapse α, shift β^i.

Must choose these somehow!

— Simplest choice: geodesic slicing $\alpha = 1 \quad \beta^i = 0$

coord observers are free-falling.

problem: singularities form. to see why:

$$\partial_t K = \alpha\left(\bar{A}^{ij}\bar{A}_{ij} + \frac{1}{3}K^2\right) = \alpha K^{ij}K_{ij}$$ in vacuum geodesic slicing

$\partial_t K \geq 0$ so $K \to \infty$ at late times except for trivial flat solution $K=0$ $\bar{A}_{ij}=0$

+ notice $\partial_t \ln \gamma^{1/2} = -K$ (Geodesic slicing)

So if $K \to \infty$, $\gamma \to 0$ Volume element vanishes.

This is coord singularity.

— Try again. How about Maximal Slicing $K = 0$

Why is this a gauge condition?

From ADM $$\partial_t K - \beta^i \partial_i K = -\gamma^{ij} D_i D_j \alpha + \alpha\left(K_{ij}K^{ij} + 4\pi(\rho_{ADM} + S)\right)$$

So if you demand $K = \partial_t K = 0$ $\Rightarrow$ $$\boxed{\gamma^{ij} D_i D_j \alpha = \alpha\left(K_{ij}K^{ij} + 4\pi(\rho_{ADM} + S)\right)}$$

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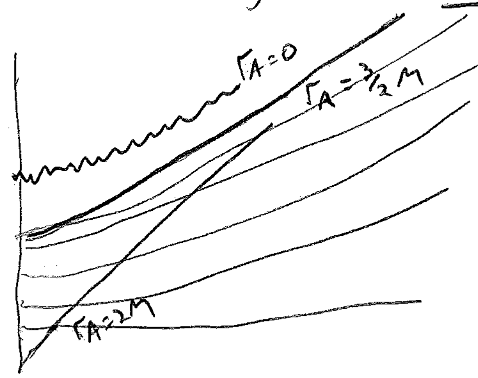
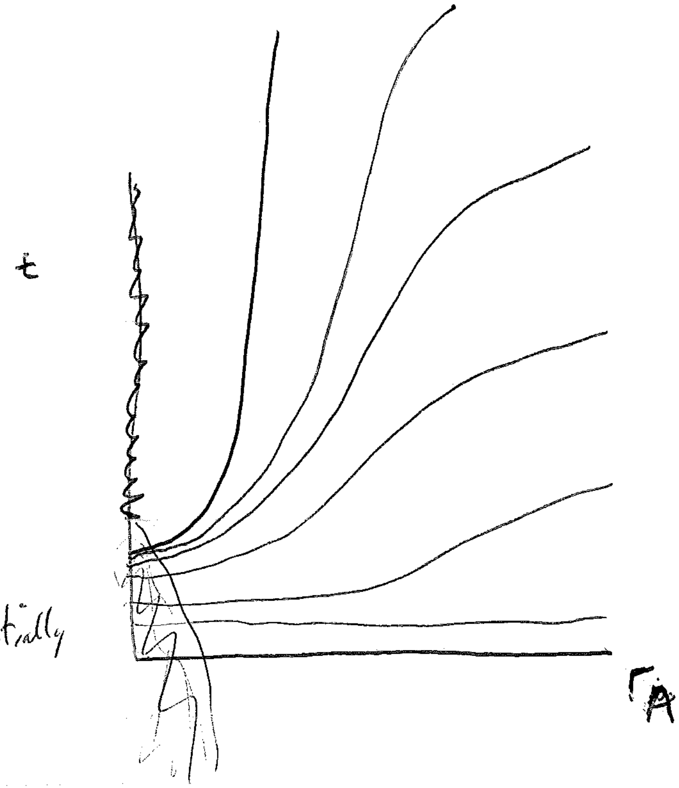
Maximal slicing gives a condition on α. leaves β^i free.

Properties of maximal slicing: singularity avoidance



Schwarzchild BH.
Maximal slices stop at limiting surface $r_A = \frac{3}{2}M$ inside the horizon.

"Collapse of the lapse"



Time advances quickly at large radii, slowly at small.

Inside BH, $\alpha \to 0$ exponentially

"Grid stretching" each slice gets stretched, metric coefficient γ_{rr} blows up at late times.

etc.

- Maximal slicing good for analytic proofs.
- Disadvantage: must solve elliptic eqn. (but can get around: K-driver $\partial_t \alpha = -\varepsilon(\partial_t K + cK)$)
-