Now look at $\mathcal{L}_t K_{ab}$

First $\mathcal{L}_{\alpha n} K_{ab} =$

$$= \alpha\left[n^c \nabla_c K_{ab} + 2 K_{c(a}\nabla_{b)} n^c\right] + 2 \overset{0}{\cancel{n^c}} K_{c(a}\nabla_{b)}\alpha$$

$$= -\alpha\left[n^c \nabla_c\left(\nabla_a n_b + n_a a_b\right) - K_{ca}\nabla_b n^c - K_{cb}\nabla_a n^c\right]$$

$$= -\alpha\left[\underbrace{n^c \nabla_c \nabla_a n_b}_{{}^{(4)}R_{beca} n^c n^e + n^c \nabla_a \nabla_c n_b} + \underbrace{n^c a_b \nabla_c n_a}_{a_b a_a} + n^c n_a \nabla_c a_b + K_{ca}\left(n_b a^c + K_b{}^c\right) + \nabla_a n^c\left(\nabla_c n_b + \overset{0}{\cancel{n_c a_b}}\right)\right]$$

$$= -\alpha\left[{}^{(4)}R_{beca} n^c n^e + a_a a_b + n_a n^c \nabla_c a_b + \underbrace{n^c \nabla_a \nabla_c n_b + \nabla_a n^c \nabla_c n^b}_{\underbrace{\nabla_a\left(n^c \nabla_c n_b\right)}_{\nabla_a a_b}} + K_{ca} K_b{}^c + \underbrace{K_{ca} n_b a^c}\right]$$

recall

$D_b V^c = \gamma^a{}_b \nabla_a V^c + K_{ba} V^a n^c$

so $K_{ca} n_b a^c$

$= D_a a_b - \gamma_a{}^c \nabla_c a_b$

$$\Rightarrow \mathcal{L}_{\alpha n} K_{ab} = -\alpha\left[{}^{(4)}R_{beca} n^c n^e + a_a a_b + \underbrace{\nabla_a a_b + n_a n^c \nabla_c a_b - \gamma_a{}^c \nabla_c a_b}_{0} + D_a a_b + K_{ca} K_b{}^c\right]$$

$$= -\alpha\left[{}^{(4)}R_{beca} n^c n^e + a_a a_b + D_a a_b + K_{ca} K_b{}^c\right]$$

$$\text{So } \mathcal{L}_{\alpha n} K_{ab} = -\alpha \left[a_a a_b + D_a a_b + K_{ca} K^c{}_b + \gamma_a{}^c \gamma_b{}^d n^e n^f \, {}^{(4)}R_{defc} \right]$$

$$\text{Now } D_a a_b = D_a (D_b \ln \alpha) = D_a \left(\frac{D_b \alpha}{\alpha} \right) = \frac{1}{\alpha} D_a D_b \alpha + D_b \alpha \left(-\frac{D_a \alpha}{\alpha^2} \right)$$

$$= \frac{1}{\alpha} D_a D_b \alpha - a_a a_b$$

$$\text{Also recall } {}^{(3)}R_{bd} = \gamma^{ge} \gamma_b{}^f \gamma_d{}^h \, {}^{(4)}R_{efgh} - K K_{bd} + K_{bc} K_d{}^c \qquad \left(\text{contraction of Gauss}\right)$$

$$\text{So } \gamma_a{}^c \gamma_b{}^d n^e n^f \, {}^{(4)}R_{defc} = -\gamma_a{}^c \gamma_b{}^d \left[\gamma^{ef} - g^{ef} \right] {}^{(4)}R_{edfc}$$

$$= -{}^{(3)}R_{ab} - K K_{ab} + K_{ac} K_b{}^c + \gamma_a{}^c \gamma_b{}^d \, \underbrace{{}^{(4)}R_{dc}}_{{}^{(4)}G_{dc} - \frac{1}{2} g_{dc} {}^{(4)}G}$$

$$= -{}^{(3)}R_{ab} - K K_{ab} + K_{ac} K_b{}^c + 8\pi S_{ab} - 4\pi \gamma_{ab} (S - \rho)$$

$$\text{where } S_{ab} \equiv \gamma_a{}^c \gamma_b{}^d T_{cd}$$

$$S \equiv S^a{}_a \qquad \text{note } T = G = g^{ab} T_{ab} 8\pi$$

$$= (\gamma^{ab} - n^a n^b) T_{ab} 8\pi$$

$$= (S - \rho) 8\pi$$

$$\Rightarrow \mathcal{L}_{\alpha n} K_{ab} = -D_a D_b \alpha - \alpha \left[-{}^{(3)}R_{ab} - K K_{ab} + 2 K_{ac} K_b{}^c + 8\pi S_{ab} - 4\pi \gamma_{ab} (S - \rho) \right]$$

Note: $\mathcal{L}_{\alpha n}(K_{ab}) = \mathcal{L}_{\alpha n}(K^c{}_b \gamma_{ac})$

$$= \gamma_{ac}\mathcal{L}_{\alpha n}(K^c{}_b) + K^c{}_b \underbrace{\mathcal{L}_{\alpha n}\gamma_{ac}}_{-2\alpha K_{ac}}$$

$\times\ g^{da}$

$$g^{da}\mathcal{L}_{\alpha n} K_{ab} = \mathcal{L}_{\alpha n} K^d{}_b - 2\alpha K^d{}_c K^c{}_b$$

$$\Rightarrow \boxed{\mathcal{L}_{\alpha n} K^a{}_b = -D^a D_b \alpha + \alpha\, {}^{(3)}R^a{}_b + \alpha K K^a{}_b - 8\pi\alpha S^a{}_b + 4\pi\alpha\gamma^a{}_b(S-\rho)}$$

$(n_i = 0)$

$n_a n^a = -1 = n_0 n^0$ so $n_0 = -\alpha$ $\boxed{n_a = (-\alpha, 0, 0, 0)}$

$$\boxed{\gamma_{ij} = g_{ij} + \cancel{n_i n_j}^{0} = g_{ij}}$$

$\gamma^{ab} n_b = 0 = \gamma^{a0} n_0$ so $\boxed{\gamma^{a0} = 0}$

$\Rightarrow \gamma^{ab} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{ij} \end{pmatrix}$ so γ^{ij} + γ_{ij} raise/lower spatial indices also $\gamma^0{}_a = g_{ab}\gamma^{b0} = 0$

$\boxed{\gamma^i{}_j = \delta^i{}_j}$ $(n_j = 0)$

Finally $g^{00} = \cancel{\gamma^{00}}^{0} - n^0 n^0 = -\frac{1}{\alpha^2}$

$g^{0i} = \cancel{\gamma^{0i}}^{0} - n^0 n^i = \frac{\cancel{+}\beta^i}{\alpha^2}$

$g^{ij} = \gamma^{ij} - n^i n^j = \gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2}$

$\Rightarrow g_{00} = \beta^i \beta_i - \alpha^2$
$g_{0i} = \beta_i$

So $\boxed{ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)}$

3+1 metric (ADM metric)

Arnowitt, Deser, Misner
1962

Now Finally, choose a basis:

$\text{so } \mathcal{L}_t = \frac{\partial}{\partial t}$

① Time vector $\vec{e}_0 = \frac{\partial}{\partial t} = t^a = \alpha n^a + \beta^a$

ith spatial basis vector

② Spatial vectors tangent to slices $\Omega_a (e_i)^a = 0$

③ Spatial vectors remain the same thru time $\mathcal{L}_t (e_i)^a = 0$

Check ②+③:

$$\mathcal{L}_t\left[(e_i)^a \Omega_a\right] = (e_i)^a \mathcal{L}_t \Omega_a + \Omega_a \cancel{\mathcal{L}_t (e_i)^a}$$

0 by property ②

$$= (e_i)^a \mathcal{L}_t \nabla_a t$$ using $\Omega_a = \nabla_a t$

$$= (e_i)^a \nabla_a \mathcal{L}_t t$$ Lie deriv commutes with cov deriv

$$= (e_i)^a \nabla_a \left[\underbrace{t^a \nabla_a t}_{1}\right]$$

$$= 0$$

so $(e_i)^a$ always tangent to slice,

Components: $0 = \Omega_a (e_i)^a = -\frac{n_a (e_i)^a}{\alpha}$ so $n_i \equiv n_a (e_i)^a = 0$

Also $\boxed{\beta^a = (0, \beta^i)}$ because $n_a \beta^a = 0 = n_0 \beta^0$

$\boxed{t^a = (1,0,0,0)}$ by construction

so since $t^a = \alpha n^a + \beta^a$, $\boxed{n^a = \frac{1}{\alpha}(1, -\beta^i)}$

Evolution eqs become:

From $\mathcal{L}_\beta \gamma_{ab}$

$$\boxed{\begin{aligned} \partial_t \gamma_{ij} &= -2\alpha K_{ij} + 2D_{(i}\beta_{j)} \\ \partial_t K_{ij} &= -D_i D_j \alpha + \alpha\left[{}^{(3)}R_{ij} + K K_{ij} - 2K_{im}K^m{}_j \right. \\ &\quad \left. - 8\pi S_{ij} - 4\pi(\rho - S)\gamma_{ij}\right] \\ &\quad + \beta^m D_m K_{ij} + 2K_{m(i}D_{j)}\beta^m \end{aligned}}$$

last terms from $\mathcal{L}_\beta K_{ab}$

"ADM evolution eqs." ADM 1962, York 1979

Plus constraints:

$$\boxed{\begin{aligned} &{}^{(3)}R + K^2 - K_{ij}K^{ij} - 16\pi\rho = 0 && \text{Hamil constraint} \\ &D_i K^i{}_j - D_j K = 8\pi S_j && \text{Mom. constraint} \end{aligned}}$$

12 evolved variables: γ_{ij}, K_{ij}

12 evolution equations

4 constraints on γ_{ij}, K_{ij}

4 Free variables: α, β^i choose however you want

$\rightarrow$ Gauge freedom

Ph 236b, Numerical Relativity

Until mid-1990s, all numerical relativity used ADM eqs. (and 3D simulations failed)
Alternative Formulations of Einstein's Eqs developed,
but not widely taken seriously until early-mid 2000s

What's wrong with ADM?

- Well-posedness issues
- Constraint growth issues

I. Well-posedness

Hadamard 1902: A problem is well-posed iff

- A solution exists
- The solution is unique
- The solution depends continuously on the initial + boundary data

Example 1 (Hadamard 1923)

$$\partial_t^2 u - \partial_x^2 u = 0$$

Initial Data $u|_{t=0} = 0$

$\partial_t u|_{t=0} = \dfrac{\sin(2\pi n x)}{(2\pi n)^p}$

B.C. $u=0$ at $x=0$, $x=1$ $\qquad p \geq 1$

Solution: $$u(x,t) = \frac{\sin(2\pi n t)\sin(2\pi n x)}{(2\pi n)^{p+1}}$$

as $n \to \infty$, initial data $\to 0$ and solution at late time $\to 0$
Well-posed

Example 2

$$\partial_t^2 u + \partial_x^2 u = 0$$

we changed only the sign

Same initial data + b.c.

$$u\big|_{t=0} = 0$$

$$\partial_t u\big|_{t=0} = \frac{\sin(2\pi n x)}{(2\pi n)^p}$$

$u = 0$ at $x = 0$ + at $x = 1$

Solution: $$u(x,t) = \frac{\sinh(2\pi n t)\sin(2\pi n x)}{(2\pi n)^{p+1}}$$

as $n \to \infty$, initial data still $\to 0$, but solution at later time $\to \infty$

In other words, given $\partial_t^2 u + \partial_x^2 u = 0$, plus any initial data and any boundary data, I can introduce a small perturbation at $t=0$ (of the form $\frac{\sin(2\pi n x)}{(2\pi n)^p}$) that produces an arbitrarily large solution at a given finite time.

$\Rightarrow$ Ill-posed

$x \in [0,1]$

Proof of well-posedness of wave eqn. $\partial_t^2 u - \partial_x^2 u = 0$

Define "Energy":

$$E = \frac{1}{2}\int_0^1 dx\, \left((\partial_t u)^2 + (\partial_x u)^2\right)$$

"Power in solution at some time"

$$E \geq 0$$

Note

$$\begin{aligned}
\partial_t E &= \int_0^1 dx\, \left(\partial_t^2 u\, \partial_t u + \partial_t \partial_x u\, \partial_x u\right) \\
&= \int_0^1 dx\, \left(\partial_x^2 u\, \partial_t u + \partial_t \partial_x u\, \partial_x u\right) \\
&= \int_0^1 dx\, \left(\partial_x[\partial_x u\, \partial_t u] \cancelto{0}{- \partial_x u\, \partial_x \partial_t u + \partial_t \partial_x u\, \partial_x u}\right) \\
&= \partial_x u\, \partial_t u \Big|_{x=0}^{x=1}
\end{aligned}$$

So $\partial_t E$ depends only on the flux of waves thru bdry.

For our B.C. $u = \partial_t u = 0$ so $\partial_t E = 0$.

Independent of initial data, E doesn't grow with time.

II. Hyperbolicity of PDEs

Consider 1st-order PDEs

(Any PDE system can be written as system of 1st order PDEs)

Example: wave eqn

$$\partial_t^2\psi - \partial_x^2\psi = 0$$

define $\Pi \equiv -\partial_t\psi$

$\Phi \equiv \partial_x\psi$

$$\Rightarrow \quad \partial_t\psi = -\Pi$$
$$\partial_t\Pi + \partial_x\Phi = 0$$
$$\partial_t\Phi + \partial_x\Pi = 0$$

Wave Eqn in 1st order form

(constraint: $\partial_x\psi - \Phi = 0$)

General Form (now u is a vector):

vector index

spatial index

$$\boxed{\partial_t u^a + \underbrace{A^{ia}{}_b}_{\text{no derivs of } u} \partial_i u^b = \underbrace{B^a(u)}_{\text{no derivs of } u}}$$

Wave eq. example $u^a = \begin{pmatrix}\psi\\ \Pi\\ \Phi\end{pmatrix}$ $B^a = \begin{pmatrix}-\Pi\\ 0\\ 0\end{pmatrix}$

$$A^{xa}{}_b = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

(**Def** "Principal Part" of equation is part including only derivative terms)

Pick an arbitrary spatial unit vector n_i

Then $n_i A^{ia}{}_b$ is the characteristic matrix in direction n_i

Let $e^{\hat\alpha}{}_a$ be the $\hat\alpha$ th (left) eigenvector of $n_i A^{ia}{}_b$, with eigenvalues $V_{(\hat\alpha)}$:

$$e^{\hat\alpha}{}_a \left(n_i A^{ia}{}_b\right) = V_{(\hat\alpha)}\, e^{\hat\alpha}{}_b \qquad \text{(no sum on } \hat\alpha\text{)}$$

Then $V_{(\hat{\alpha})}$ are called <u>characteristic speeds</u>

$u^{\hat{\alpha}} \equiv u^a e^{\hat{\alpha}}_{\ a}$ are called <u>characteristic fields</u>

(These are important for boundary conditions Later...)

<u>Def</u> A 1st order PDE system is <u>weakly hyperbolic</u> if all eigenvalues $V_{(\hat{\alpha})}$ (in any arbitrary direction n_i) are <u>real</u>.

- For weakly hyperbolic systems, well-posedness depends on details of non-principal terms. Usually ill-posed.

<u>Def</u> A 1st order system of PDEs is <u>strongly hyperbolic</u> if all eigenvalues $V_{(\hat{\alpha})}$ are real, and there is a <u>complete set</u> of linearly independent eigenvectors, independent of the solution or of n_i.

- If a system is strongly hyperbolic:

  (1) It is well-posed

  (2) At a boundary with (outgoing) normal n_i,
  - Boundary conditions <u>must</u> be imposed on all characteristic fields $u^{\hat{\alpha}}$ with $V_{(\hat{\alpha})} < 0$
  - Boundary conditions <u>must not</u> be imposed on characteristic fields $u^{\hat{\alpha}}$ with $V_{(\hat{\alpha})} > 0$.

  (failure to obey these kills well-posedness)

Def A 1st order PDE system is <u>symmetric hyperbolic</u> if there exists a positive-definite symmetric matrix S_{ab} (a "symmetrizer") such that $S_{ab} A^{ib}{}_{c}$ is symmetric on $a+c$ (for all i)

symmetric hyperbolic implies strongly hyperbolic

$\Rightarrow$ Well-posed

$\Rightarrow$ Well-defined prescription for BCs.

So what about ADM?

ADM is only weakly hyperbolic (Kidder, Scheel, Teukolsky 2001; Nagy, Ortiz, Reula 2004)

$\Rightarrow$ No well-posedness

$\Rightarrow$ No indication of which variables require boundary conditions (much less what these bcs should be).

Example: Scalar Waves, Flat space 1+1 D

$$\partial_t \psi = -\pi$$
$$\partial_t \pi + \partial_x \Phi = 0$$
$$\partial_t \Phi + \partial_x \pi = 0$$

$$\partial_t u^a + A^{ia}{}_b \partial_i u^b = B^a$$

$$u^a = \begin{pmatrix} \psi \\ \pi \\ \Phi \end{pmatrix} \qquad B^a = \begin{pmatrix} -\pi \\ 0 \\ 0 \end{pmatrix}$$

$$n_x A^{xa}{}_b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Char matrix

Eigenvalues / vectors

$$V_{(0)} = 0 \qquad e^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$V_{(1)} = 1 \qquad e^1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$V_{(-1)} = -1 \qquad e^{-1} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

strongly hyperbolic

Char Fields
$$u^0 = \psi$$
$$u^\pm = \pi \pm \Phi$$

u^+ is rightgoing solution $\pi + \Phi$, speed $+1$

u^- left " " $\pi - \Phi$, speed -1

need BC on u^-, not on u^+

III. Beyond the ADM Formulation

A. BSSN Formulation (Shibata/Nakamura 1995, Baumgarte/Shapiro 1999)

– Conformal Decomposition:

(1) Define $\bar{\gamma}_{ij} = e^{-4\phi}\gamma_{ij}$

and require $\det(\bar{\gamma}_{ij}) = 1 \quad \Rightarrow \quad \phi = \frac{1}{12}\ln\gamma$

This just separates $\det\gamma_{ij}$ from the rest of γ_{ij}

(2) Define $$K_{ij} = e^{4\phi}\big(\bar{A}_{ij} + \tfrac{1}{3}\bar{\gamma}_{ij}\underbrace{K}_{Tr(K_{ij})}\big)$$ $\bar{A}_{ij}$ trace-free

so instead of variables γ_{ij}, K_{ij} we now use $\bar{\gamma}_{ij}, \bar{A}_{ij}, \phi, K$

– Define new variable $\bar{\Gamma}^i \equiv \bar{\gamma}^{jk}\bar{\Gamma}^i_{jk} = -\partial_j\bar{\gamma}^{ij}$

connection associated with $\bar{\gamma}_{ij}$

Then Ricci tensor becomes

$$R_{ij} = R^{\phi}_{ij} + \bar{R}_{ij}$$

where
$$\bar{R}_{ij} = -\tfrac{1}{2}\bar{\gamma}^{lm}\partial_m\partial_l\bar{\gamma}_{ij} + \bar{\gamma}_{k(i}\partial_{j)}\bar{\Gamma}^k + \bar{\Gamma}^k\bar{\Gamma}_{(ij)k} + \bar{\gamma}^{lm}\left(2\bar{\Gamma}^k_{l(i}\bar{\Gamma}_{j)km} + \bar{\Gamma}^k_{im}\bar{\Gamma}_{klj}\right)$$

$$R^{\phi}_{ij} = -2\left(\bar{D}_i\bar{D}_j\ln\phi + \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\bar{D}_m\ln\phi\right) + 4\left(\bar{D}_i\ln\phi\,\bar{D}_j\ln\phi - \bar{\gamma}_{ij}\bar{\gamma}^{lm}\bar{D}_l\ln\phi\,\bar{D}_m\ln\phi\right)$$

Note that all explicit 2nd derivs of γ_{ij} occur in the Laplace-like term
$$\gamma^{\ell m}\partial_m\partial_\ell\bar{\gamma}_{ij}$$

- So we have variables $\bar{\gamma}_{ij}, \bar{A}_{ij}, \phi, K, \bar{\Gamma}^i$ satisfying Evolution Eqs:

$$\partial_t\phi - \beta^i\partial_i\phi = \frac{1}{6}\partial_i\beta^i - \frac{1}{6}\alpha K$$

$$\partial_t\bar{\gamma}_{ij} - \beta^k\partial_k\bar{\gamma}_{ij} = -2\alpha\bar{A}_{ij} + 2\bar{\gamma}_{k(i}\partial_{j)}\beta^k - \frac{2}{3}\bar{\gamma}_{ij}\partial_k\beta^k$$

$$\partial_t K - \beta^i\partial_i K = -\gamma^{ij}D_iD_j\alpha + \alpha\left(\bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3}K^2\right) + 4\pi\alpha\left(\rho_{ADM} + S\right)$$

$$\partial_t\bar{A}_{ij} - \beta^k\partial_k\bar{A}_{ij} = e^{-4\phi}\left(-(D_iD_j\alpha)^{TF} + \alpha\left(R_{ij}^{TF} - 8\pi S_{ij}^{TF}\right)\right)$$
$$+\alpha\left(K\bar{A}_{ij} - 2\bar{A}_{i\ell}\bar{A}^{\ell}_{\ j}\right) + 2\bar{A}_{k(i}\partial_{j)}\beta^k - \frac{2}{3}A_{ij}\partial_k\beta^k$$

$$\partial_t\bar{\Gamma}^i - \beta^k\partial_k\bar{\Gamma}^i = -2\bar{A}^{ij}\partial_j\alpha + 2\alpha\left(\bar{\Gamma}^i_{jk}\bar{A}^{kj} - \frac{2}{3}\bar{\gamma}^{ij}\partial_j K - 8\pi\bar{\gamma}^{ij}S_j + 6\bar{A}^{ij}\partial_j\phi\right)$$
$$-\bar{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\bar{\Gamma}^i\partial_j\beta^j + \frac{1}{3}\bar{\gamma}^{\ell i}\partial_\ell\partial_j\beta^j + \gamma^{\ell j}\partial_j\partial_\ell\beta^i$$

(Here TF means trace free, i.e. $R_{ij}^{TF} = R_{ij} - \frac{1}{3}\gamma_{ij}R$
and $\bar{A}^{ij} \equiv \bar{\gamma}^{ik}\bar{\gamma}^{j\ell}\bar{A}_{k\ell}$)

Constraints

$$\mathcal{H} = \bar{\gamma}^{ij}\bar{D}_i\bar{D}_j e^{\phi} - \frac{e^{\phi}}{8}\bar{R} + \frac{e^{5\phi}}{8}\bar{A}_{ij}\bar{A}^{ij} - \frac{e^{5\phi}}{12}K^2 + 2\pi e^{5\phi}\rho_{ADM}$$

$$\mathcal{M}^i = \bar{D}_j\left(e^{6\phi}\bar{A}^{ij}\right) - \frac{2}{3}e^{6\phi}D^iK - 8\pi e^{6\phi}S^i$$

$\partial_j\bar{\gamma}^{ij} + \bar{\Gamma}^i = 0$ $\quad\quad$ $\det\bar{\gamma}_{ij} = 1$ $\quad$ $\bar{\gamma}^{ij}\bar{A}_{ij} = 0$

Hyperbolicity:

- BSSN is Strongly Hyperbolic for <u>Fixed</u> α, β^i (Nagy, Ortiz, Reula 2004) (Sarbach, Calabrese, Pullin, Tiglio 2002)

  Hyperbolicity depends on gauge choice!
  Char speeds / Fields depend on gauge choice!

- BSSN with dynamical gauge conditions (will talk about later...) is strongly hyperbolic if the shift is not too large (Gundlach, Martin-Garcia 2006)

- Hyperbolicity of BSSN still being researched...

- BSSN currently the most widely used Formulation in NR

Gauge Conditions

(ADM/BSSN) Einstein's Eqs do not determine lapse α, shift β^i.

Must choose these somehow!

- Simplest choice: geodesic slicing $\alpha = 1 \quad \beta^i = 0$

  coord observers are free-falling.

problem: singularities form. to see why:

$$\partial_t K = \alpha\left(\tilde{A}^{ij}\tilde{A}_{ij} + \frac{1}{3}K^2\right) = \alpha K^{ij}K_{ij}$$ in vacuum geodesic slicing

$$\partial_t K \geq 0$$ so $K \to \infty$ at late times except for trivial flat solution $K=0 \quad \tilde{A}_{ij} = 0$

+ notice $\partial_t \ln \gamma^{1/2} = -K$ (Geodesic slicing)

So if $K \to \infty$, $\gamma \to 0$ Volume element vanishes.

This is coord singularity.

- Try again. How about Maximal Slicing $K = 0$

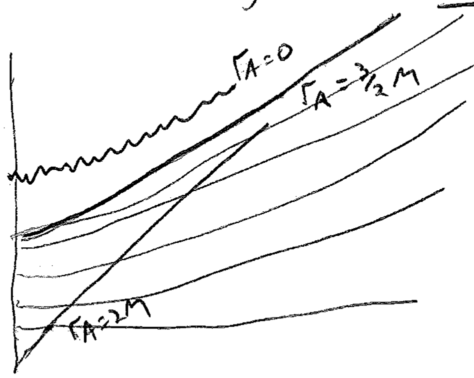
  Why is this a gauge condition?

From ADM $$\partial_t K - \beta^i \partial_i K = -\gamma^{ij} D_i D_j \alpha + \alpha\left(K_{ij}K^{ij} + 4\pi(\rho_{ADM} + S)\right)$$

So if you demand $K = \partial_t K = 0$ $\Rightarrow$ $$\boxed{\gamma^{ij} D_i D_j \alpha = \alpha\left(K_{ij}K^{ij} + 4\pi(\rho_{ADM} + S)\right)}$$

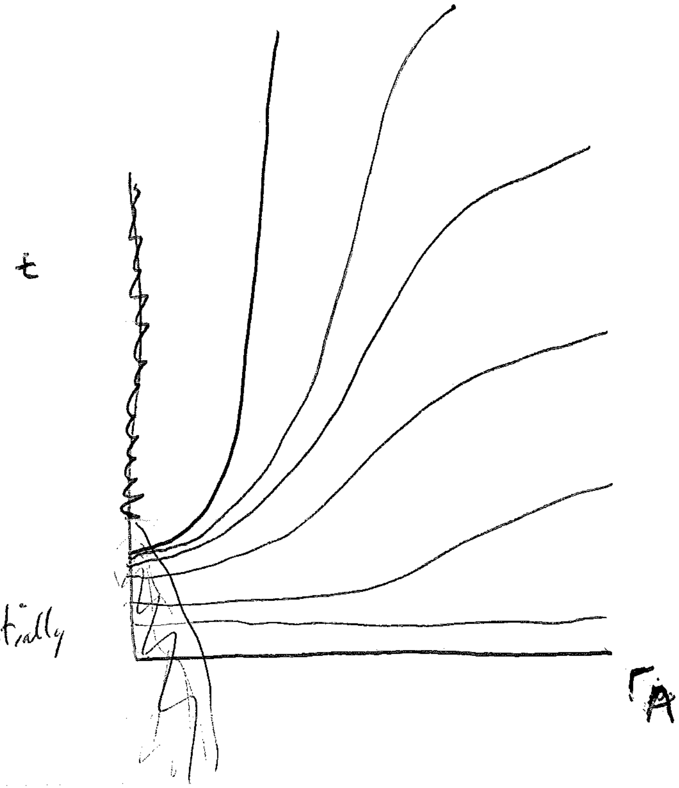
Maximal slicing gives a condition on α. leaves β^i free.

Properties of maximal slicing: singularity avoidance



Schwarzchild BH.
Maximal slices stop at limiting surface $r_A = \frac{3}{2}M$ inside the horizon.

"Collapse of the lapse"



Time advances quickly at large radii, slowly at small.

Inside BH, $\alpha \to 0$ exponentially

"Grid stretching" each slice gets stretched, metric coefficient γ_{rr} blows up at late times.

etc.

- Maximal slicing good for analytic proofs.
- Disadvantage: must solve elliptic eqn. (but can get around: K-driver $\partial_t \alpha = -\varepsilon(\partial_t K + cK)$)
-